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EULER HARMONIC IDENTITIES AND MOMENTS OF REAL BOREL MEASURES

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. Generalizations of Euler identities involving μ-harmonic sequences of functions and moments of real Borel measure μ are established. Some Ostrowski and Euler-Grüss type inequalities for functions of various classes are proved.

1. Introduction

For $a,b\in\mathbb{R},\ a<b$, let $C[a,b]$ be the Banach space of all continuous functions $f:[a,b]\to\mathbb{R}$ with the max norm, and $M[a,b]$ the Banach space of all real Borel measures on $[a,b]$ with the total variation norm.

Introduce the sequence of functions $\check{K}_n:[a,b]\times[a,b]\to\mathbb{R},\ n\geqslant 1$, by

$$\check{K}_n(x,t)=\begin{cases}\check{\mu}_n(t), & a\leqslant t\leqslant x\\ \check{\mu}_n(t)+\frac{(-1)^n}{(n-1)!}e_{n-1}(t,\mu), & x<t\leqslant b\end{cases}$$

for $a\leqslant x<b$, while for $x=b$

$$\check{K}_n(b,t)=\begin{cases}\check{\mu}_n(t), & a\leqslant t<b\\ 0, & t=b\end{cases}.$$

Here $\check{\mu}_n:[a,b]\to\mathbb{R},\ n\geqslant 1$ denotes the general distribution function of $\mu\in M[a,b]$ defined by

$$\check{\mu}_n(t)=\frac{1}{(n-1)!}\int_{[a,t]}(t-s)^{n-1}d\mu(s),$$

and

$$e_n(x,\mu)=\int_{[a,b]}(s-x)^n d\mu(s),\ n\geqslant 0,\ x\in[a,b]$$

is the n-th x-centered moment of μ.

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Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. In the recent paper $[1]$ the following identity has been proved:

$$\int_{[a,b]} f(t)d\mu(t)+\check{S}_n(x)=\check{R}_n(x) \tag{1.1}$$

for every $x\in[a,b]$, where

$$\check{S}_n(x)=-\sum_{k=0}^{n-1}\tfrac{1}{k!}f^{(k)}(x)e_k(x,\mu),$$

and

$$\check{R}_n(x)=(-1)^n\int_{[a,b]}\check{K}_n(x,t)df^{(n-1)}(t).$$

This identity has been used in $[1]$ to prove some generalizations of weighted Ostrowski inequality. See also $[5]$.

The aim of this paper is to generalize formula (1.1), by replacing the sequence $(\check{K}_n,\ n\geqslant 1)$ with a more general sequence, and using it to prove some further generalizations of inequalities of Ostrowski type and Euler-Grüss type inequalities.

2. Some integral identities

Let $\mu\in M[a,b]$. A sequence of functions $P_n:[a,b]\to\mathbb{R},\ n\geqslant 1$, is called μ-*harmonic sequence of functions* on $[a,b]$ if

$$P_n'(t)=P_{n-1}(t),\ t\in[a,b],\ n\geqslant 2,$$

and

$$P_1(t)=c+\check{\mu}_1(t),\ t\in[a,b]$$

for some $c\in\mathbf{R}$.

The sequence $(\check{\mu}_n,\ n\geqslant 1)$ is an example of μ-harmonic sequence of functions on $[a,b]$. The notion of a μ-harmonic sequence of functions has been introduced in [4].

A sequence of polynomials $(Q_n,\ n\geqslant 1)$ is called *semiharmonic sequence of polynomials* if

$$Q_n'(t)=Q_{n-1}(t),\ t\in[a,b],\ n\geqslant 2,$$

and

$$Q_1(t)=c,\ t\in[a,b]$$

for some $c\in\mathbb{R}$. If $(Q_n,\ n\geqslant 1)$ is a semiharmonic sequence of polynomials and $\mu\in M[a,b]$, then the sequence $P_n=Q_n+\check{\mu}_n,\ n\geqslant 1$ is a μ-harmonic sequence of functions on $[a,b]$. Obviously, the converse also holds.

Let $(P_n,\ n \geqslant 1)$ be a μ-harmonic sequence of functions on $[a,b]$. Define function $K_n : [a,b] \times [a,b] \to \mathbb{R},\ n \geqslant 1$ by

$$K_n(x,t) = \begin{cases} P_n(t), & a \leqslant t \leqslant x \\ P_n(t) + \frac{(-1)^n}{(n-1)!} e_{n-1}(t,\mu), & x < t \leqslant b \end{cases}$$

for $a \leqslant x < b$, while for $x = b$

$$K_n(b,t) = \begin{cases} P_n(t), & a \leqslant t < b \\ P_n(b) - \check{\mu}_n(b), & t = b \end{cases}.$$

It is easy to see that for $n \geqslant 1$,

$$K_n(x,a) = P_n(a),\ \ K_n(x,b) = P_n(b) - \check{\mu}_n(b)$$

for every $x \in [a,b]$, and that $K_n(x,\cdot),\ n \geqslant 2$ is continuous on $[a,b] \backslash \{x\}$, having a jump of

$$\frac{(-1)^n}{(n-1)!} e_{n-1}(x,\mu)$$

at x. Further, $K_n(x,\cdot),\ n \geqslant 2$ is differentiable on $[a,b] \backslash \{x\}$ and

$$K'_{n+1}(x,\cdot) = K_n(x,\cdot).$$

LEMMA 1. *For $n \geqslant 2,\ x \in [a,b]$, and $f \in C[a,b]$, we have*

$$\int_{[a,b]} f(t) dK_n(x,t) = \int_a^b f(t) K_{n-1}(x,t) dt + \frac{(-1)^n}{(n-1)!} e_{n-1}(x,\mu) f(x),$$

while for $n = 1$,

$$\int_{[a,b]} f(t) dK_1(x,t) = \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\}) f(a) - \mu([a,b]) f(x).$$

Proof. For $n \geqslant 2$, the function $K_n(x,\cdot)$ is differentiable on $[a,b] \setminus \{x\}$ and its derivative is equal to $K_{n-1}(x,\cdot)$. It has a jump of $\frac{(-1)^n}{(n-1)!} e_{n-1}(x,\mu)$ at x, which gives the first formula. Further, $K_1(x,\cdot)$ has a jump of $-\check{\mu}_1(b)$ at x, and by [2, Lemma 2.2.] we have

$$\begin{aligned} \int_{[a,b]} f(t) dK_1(x,t) &= \int_{[a,b]} f(t) d\check{\mu}_1(t) - \check{\mu}_1(b) f(x) \\ &= \int_{[a,b]} f(t) d\mu(t) - \check{\mu}_1(a) f(a) - \check{\mu}_1(b) f(x) \\ &= \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\}) f(a) - \mu([a,b]) f(x), \end{aligned}$$

which proves the second formula. $\square$

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n\geqslant 1$. Then for every semiharmonic sequence of polynomials $(Q_n,\ n\geqslant 1)$ we have*

$$\int_{[a,b]} f(t)d\mu(t)+S_n(x)=R_n(x) \tag{2.1}$$

for every $x\in[a,b]$, where

$$R_n(x)=(-1)^n\int_{[a,b]}K_n(x,t)\,df^{(n-1)}(t)$$

and

$$S_n(x)=\sum_{k=1}^{n}(-1)^k[Q_k(b)f^{(k-1)}(b)-Q_k(a)f^{(k-1)}(a)]-\sum_{k=0}^{n-1}\tfrac{1}{k!}f^{(k)}(x)e_k(x,\mu).$$

Proof. Integrating by parts, for $k\geqslant 1$, we have

$$R_k(x)=(-1)^kK_k(x,t)\,f^{(k-1)}(t)\Big|_a^b-(-1)^k\int_{[a,b]} f^{(k-1)}(t)dK_k(x,t)\,. \tag{2.2}$$

Since $K_k(x,a)=P_k(a)$ and $K_k(x,b)=P_k(b)-\check{\mu}_k(b)=Q_k(b)$, by the first formula of Lemma 1, for $k\geqslant 2$ we have

$$\begin{aligned}R_k(x) &= (-1)^kQ_k(b)f^{(k-1)}(b)-(-1)^kQ_k(a)f^{(k-1)}(a)\\ &\quad+(-1)^{k-1}\int_{[a,b]} f^{(k-1)}(t)dK_k(x,t)\\ &=(-1)^kQ_k(b)f^{(k-1)}(b)-(-1)^kQ_k(a)f^{(k-1)}(a)\\ &\quad+(-1)^{k-1}\int_a^b f^{(k-1)}(t)K_{k-1}(x,t)\,dt\\ &\quad+(-1)^{k-1}\tfrac{(-1)^k}{(k-1)!}e_{k-1}(x,\mu)f^{(k-1)}(x)\\ &=(-1)^kQ_k(b)f^{(k-1)}(b)-(-1)^kQ_k(a)f^{(k-1)}(a)\\ &\quad-\tfrac{1}{(k-1)!}f^{(k-1)}(x)e_{k-1}(x,\mu)+R_{k-1}(x).\end{aligned} \tag{2.3}$$

By the second formula of Lemma 1, for $k=1$, (2.2) becomes

$$\begin{aligned}R_1(x) &= -Q_1(b)f(b)+P_1(a)f(a)+\int_{[a,b]} f(t)dK_1(x,t)\\ &=-Q_1(b)f(b)+Q_1(a)f(a)+\mu(\{a\})f(a)\end{aligned}$$

$$
\begin{aligned}
&+\int_{[a,b]} f(t)d\mu(t)-\mu(\{a\})f(a)-\mu([a,b])f(x)\\
&=-Q_1(b)f(b)+Q_1(a)f(a)\\
&+\int_{[a,b]} f(t)d\mu(t)-\mu([a,b])f(x). \qquad (2.4)
\end{aligned}
$$

From (2.3) and (2.4), by iteration, it follows that

$$
\begin{aligned}
R_n(x) &= \sum_{k=2}^{n}(-1)^k[Q_k(b)f^{(k-1)}(b)-Q_k(a)f^{(k-1)}(a)]\\
&-\sum_{k=2}^{n}\tfrac{1}{(k-1)!}f^{(k-1)}(x)e_{k-1}(x,\mu)+R_1(x)\\
&=\sum_{k=1}^{n}(-1)^k[Q_k(b)f^{(k-1)}(b)-Q_k(a)f^{(k-1)}(a)]\\
&-\sum_{k=2}^{n}\tfrac{1}{(k-1)!}f^{(k-1)}(x)e_{k-1}(x,\mu)-\mu([a,b])f(x)+\int_{[a,b]} f(t)d\mu(t)\\
&=\sum_{k=1}^{n}(-1)^k[Q_k(b)f^{(k-1)}(b)-Q_k(a)f^{(k-1)}(a)]\\
&-\sum_{k=0}^{n-1}\tfrac{1}{k!}f^{(k)}(x)e_k(x,\mu)+\int_{[a,b]} f(t)d\mu(t),
\end{aligned}
$$

which proves our assertion. □

REMARK 1. In the special case $P_n=\check{\mu}_n,\ n\geqslant 1$, i.e. $Q_n=0,\ n\geqslant 1$, the sequence $(K_n,\ n\geqslant 1)$ reduces to the sequence $(\check{K}_n,\ n\geqslant 1)$ from the Introduction, $S_n(x)$ reduces to $\check{S}_n(x)$, and identity (2.1) becomes identity (1.1).

3. Generalizations of weighted Ostrowski inequality

In this section we shall use the same notations as above.

THEOREM 2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n\geqslant 1$. Then*

$$
\left|\int_{[a,b]} f(t)d\mu(t)+S_n(x)\right|\leqslant L\int_a^b |K_n(x,t)|\,dt
$$

for every $x\in[a,b]$.

Proof. If $\varphi : [a,b] \to \mathbb{R}$ is L-Lipschitzian on $[a,b]$, then for any integrable function $g : [a,b] \to \mathbb{R}$ we have $|\int_{[a,b]} g(t)d\varphi(t)| \leqslant L\int_a^b |g(t)|dt$. Using this estimate and Theorem 1 we get

$$|R_n(x)| = \Big|\int_{[a,b]} K_n(x,t)\, df^{(n-1)}(t)\Big| \leqslant L\int_a^b |K_n(x,t)|\,dt,$$

which proves our assertion. □

COROLLARY 1. *If f is L-Lipschitzian on $[a,b]$, then for every $x \in [a,b]$, $c \in \mathbb{R}$ and $\mu \in M[a,b]$ we have*

$$\begin{aligned}
&\Big|\int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(x) + c[f(a)-f(b)]\Big| \\
&\leqslant L[\int_a^x |c+\mu([a,t])|\,dt + \int_x^b |c-\mu((t,b])|\,dt] \\
&\leqslant L(b-a)(|c| + \|\mu\|).
\end{aligned}$$

Proof. Put $n=1$ in the theorem above and note that $P_1(t) = c + \check{\mu}_1(t) = c + \mu([a,t])$, $t \in [a,b]$, for some $c \in \mathbb{R}$, and $S_1(x) = -\mu([a,b])f(x) + c[f(a)-f(b)]$, while

$$\int_a^b |K_1(x,t)|\,dt = \int_a^x |c+\mu([a,t])|\,dt + \int_x^b |c-\mu((t,b])|\,dt \leqslant (|c|+\|\mu\|)(b-a). \quad \square$$

THEOREM 3. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then*

$$\Big|\int_{[a,b]} f(t)d\mu(t) + S_n(x)\Big| \leqslant \max_{t\in[a,b]} |K_n(x,t)|\, V_a^b(f^{(n-1)})$$

for every $x \in [a,b]$, where $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$.

Proof. If $F : [a,b] \to \mathbb{R}$ is bounded and the Stieltjes integral $\int_{[a,b]} F(t)df^{(n-1)}(t)$ exists, then

$$\Big|\int_{[a,b]} F(t)df^{(n-1)}(t)\Big| \leqslant \max_{t\in[a,b]} |F(t)| \cdot V_a^b(f^{(n-1)}).$$

By applying this estimation to formula (2.1) we have

$$|R_n(x)| = \Big|\int_{[a,b]} K_n(x,t)\, df^{(n-1)}(t)\Big| \leqslant \max_{t\in[a,b]} |K_n(x,t)|\, V_a^b(f^{(n-1)}),$$

which proves our assertion. □

COROLLARY 2. *If f is a continuous function of bounded variation on $[a,b]$, then for every $x \in [a,b]$, $c \in \mathbb{R}$ and $\mu \in M[a,b]$ we have*

$$\begin{aligned}
&\Big| \int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(x) + c\,[f(a) - f(b)] \Big| \\
&\leqslant \max\{\max_{a\leqslant t\leqslant x} |c + \mu\,([a,t])|\,, \max_{x<t\leqslant b} |c - \mu\,((t,b])|\} V_a^b(f) \\
&\leqslant (|c| + \|\mu\|)\, V_a^b(f).
\end{aligned}$$

Proof. Put $n = 1$ in the theorem above and note that

$$\max_{t\in[a,b]} |K_1\,(x,t)| = \max\{\max_{a\leqslant t\leqslant x} |c + \mu\,([a,t])|\,, \max_{x<t\leqslant b} |c - \mu\,((t,b])|\} \leqslant |c| + \|\mu\|\,. \quad \square$$

REMARK 2. In the special case, when $Q_n = 0,\ n \geqslant 1$, i.e. $P_n = \check{\mu}_n,\ n \geqslant 1$, Theorems 2 and 3, according to Remark 1, imply [1, Theorems 2 and 3], respectively.

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is integrable for some $n \geqslant 1$. Then*

$$\Big| \int_{[a,b]} f(t)d\mu(t) + S_n(x) \Big| \leqslant \max_{t\in[a,b]} |K_n(x,t)|\, \|f^{(n)}\|_1$$

for every $x \in [a,b]$.

Proof. Note that in this case $V_a^b(f^{(n-1)}) = \int_a^b |f^{(n)}(t)|dt = \|f^{(n)}\|_1$, and apply Theorem 3. $\square$

THEOREM 5. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$ for some $n \geqslant 1$. Then*

$$\Big| \int_{[a,b]} f(t)d\mu(t) + S_n(x) \Big| \leqslant \int_a^b |K_n(x,t)|\, dt \cdot \|f^{(n)}\|_\infty$$

for every $x \in [a,b]$.

Proof. In this case $f^{(n-1)}$ is L-Lipschitzian with $L = \|f^{(n)}\|_\infty$. $\square$

THEOREM 6. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$ for some $n \geqslant 1$ and $1 < p < \infty$. Then*

$$\Big| \int_{[a,b]} f(t)d\mu(t) + S_n(x) \Big| \leqslant \|K_n\,(x,\cdot)\,\|_q \|f^{(n)}\|_p$$

for every $x \in [a,b]$, where $1/p + 1/q = 1$.

Proof. By applying the Hölder inequality we have

$$\Big|\int_{[a,b]} f(t)d\mu(t)+S_n(x)\Big| \leqslant \int_a^b |K_n(x,t)|\,|f^{(n)}(t)|dt \leqslant \Big(\int_a^b |K_n(x,t)|^q\,dt\Big)^{1/q}\|f^{(n)}\|_p,$$

which proves our assertion. □

4. Some Grüss-type inequalities

In this section we use the identity obtained in Theorem 1 to prove some general Euler-Grüss type inequalities which hold for a class of functions f possessing derivatives $f^{(n)}$ in $L_\infty[a,b]$ for some $n \geqslant 1$. For such functions we can always assume that

$$m_n \leqslant f^{(n)}(t) \leqslant M_n,\ t\in[a,b],\ a.e.$$

for some real constants m_n and M_n.

REMARK 3. Since

$$K_n(x,t) = Q_n(t) + \check{K}_n(x,t),\ a\leqslant t\leqslant b,\ a\leqslant x\leqslant b,$$

by using [1, Remark 4] we have

$$\begin{aligned}\int_a^b |K_n(x,t)|\,dt &\leqslant \int_a^b |Q_n(t)|\,dt + \int_a^b \big|\check{K}_n(x,t)\big|\,dt\\ &\leqslant \int_a^b |Q_n(t)|\,dt + \tfrac{1}{n!}\int_{[a,b]} |t-x|^n\,d|\mu|(t)\\ &\leqslant \int_a^b |Q_n(t)|\,dt + \tfrac{1}{n!}\,\|\mu\|\,[\tfrac{b-a}{2}+|x-\tfrac{a+b}{2}|]^n.\end{aligned}$$

Measure $\mu\in M[a,b]$ is called *balanced* if $\mu([a,b])=0$.

THEOREM 7. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_\infty[a,b]$, for some $n\geqslant 1$. If $x\in[a,b]$ and $(P_k,\ k\geqslant 1)$ are such that $Q_{n+1}(b)-Q_{n+1}(a)+\frac{(-1)^n}{n!}e_n(x,\mu)=0$, then*

$$\begin{aligned}\Big|\int_{[a,b]} f(t)d\mu(t)+S_n(x)\Big| &\leqslant \tfrac{1}{2}(M_n-m_n)\int_a^b |K_n(x,t)|\,dt\\ &\leqslant \tfrac{1}{2}(M_n-m_n)\Big[\int_a^b |Q_n(t)|\,dt + \tfrac{1}{n!}\int_{[a,b]} |t-x|^n\,d|\mu|(t)\Big]\\ &\leqslant \tfrac{1}{2}(M_n-m_n)\Big[\int_a^b |Q_n(t)|\,dt + \tfrac{1}{n!}\,\|\mu\|\,[\tfrac{b-a}{2}+|x-\tfrac{a+b}{2}|]^n\Big].\end{aligned}$$

Proof. Define measure ν_n by $d\nu_n(t) = (-1)^n K_n(x,t)\,dt$. Then

$$\begin{aligned}\nu_n([a,b]) &= (-1)^n \int_a^b K_n(x,t)\,dt \\ &= (-1)^n \Big[\int_a^x K_n(x,t)\,dt + \int_x^b K_n(x,t)\,dt\Big] \\ &= (-1)^n \Big[\int_a^x P_n(t)\,dt + \int_x^b [P_n(t) + \tfrac{(-1)^n}{(n-1)!} e_{n-1}(t,\mu)]dt\Big] \\ &= (-1)^n [Q_{n+1}(b) - Q_{n+1}(a) + \tfrac{(-1)^n}{n!} e_n(x,\mu)],\end{aligned}$$

which means that ν_n is a balanced measure since, by our condition, $\nu_n([a,b]) = 0$. Further, $\|\nu_n\| = \int_a^b |K_n(x,t)|\,dt$. Therefore, according to [3, Theorem 1] we have

$$|R_n(x)| = \Big|\int_a^b K_n(x,t) f^{(n)}(t) dt\Big| \leqslant \tfrac{1}{2}(M_n - m_n)\,\|\nu_n\| = \tfrac{1}{2}(M_n - m_n)\int_a^b |K_n(x,t)|\,dt,$$

which proves our assertion, by using the remark above. □

COROLLARY 3. *For $f \in L_\infty[a,b]$ let $x \in [a,b]$, $c \in \mathbb{R}$ and $\mu \in M[a,b]$ be such that $e_1(x,\mu) = c(b-a)$. Then*

$$\begin{aligned}&\Big|\int_{[a,b]} f(t) d\mu(t) - \mu([a,b]) f(x) + c\,[f(a) - f(b)]\Big| \\ &\leqslant \tfrac{1}{2}(M_1 - m_1)\Big[\int_a^x |c + \mu([a,t])|\,dt + \int_x^b |c - \mu((t,b])|\,dt\Big] \\ &\leqslant \tfrac{1}{2}(M_1 - m_1)(b-a)\,(|c| + \|\mu\|).\end{aligned}$$

Proof. Put $n = 1$ in the theorem above and note that $Q_2(t) = c_1 + c(t-a)$ for some c_1, $c \in \mathbb{R}$, and $Q_2(b) - Q_2(a) - e_1(x,\mu) = c(b-a) - e_1(x,\mu) = 0$. □

In the following corollary $m_n(\mu)$, $n \geqslant 0$ denotes the n-th moment of $\mu \in M[a,b]$.

COROLLARY 4. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$ for some $n \geqslant 1$. If $\mu \in M[a,b]$ and $(P_k,\ k \geqslant 1)$ are such that $Q_{n+1}(b) = Q_{n+1}(a)$ and $m_k(\mu) = 0$ for $k = 0,\ldots,n$, then*

$$\begin{aligned}&\Big|\int_{[a,b]} f(t) d\mu(t) + \sum_{k=1}^{n} (-1)^k [Q_k(b) f^{(k-1)}(b) - Q_k(a) f^{(k-1)}(a)]\Big| \\ &\leqslant \tfrac{1}{2}(M_n - m_n)\int_a^b |K_n(x,t)|\,dt\end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above and note that in this case we have $e_k(x,\mu) = 0$ for $k = 0,\dots,n$ and for every $x \in [a,b]$, while

$$S_n(x) = \sum_{k=1}^{n} (-1)^k [Q_k(b) f^{(k-1)}(b) - Q_k(a) f^{(k-1)}(a)]. \quad \square$$

COROLLARY 5. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n)} \in L_\infty[a,b]$ *for some* $n \geqslant 1$. *If* $\check{\mu}_k(b) = 0$ *for* $k = 1,\dots,n+1$, *and* $Q_{n+1}(b) = Q_{n+1}(a)$, *then*

$$\Big| \int_{[a,b]} f(t) d\mu(t) + \sum_{k=1}^{n} (-1)^k [Q_k(b) f^{(k-1)}(b) - Q_k(a) f^{(k-1)}(a)] \Big|$$
$$\leqslant \tfrac{1}{2}(M_n - m_n) \int_a^b |K_n(x,t)| \, dt$$

for every $x \in [a,b]$.

Proof. Note that in this case we have $m_0(\mu) = m_1(\mu) = \cdots = m_n(\mu) = 0$, by using [3, Theorem 4]. Now apply Corollary 4. $\square$

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