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PROPERTIES OF THE INTERMEDIATE POINT FROM THE TAYLOR'S THEOREM

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. If $I \subseteq \mathbb{R}$ is an interval, $a \in I$ and $f : I \to \mathbb{R}$ is $n \geqslant 1$ times differentiable on I, then, in view of Taylor's theorem, there exists a function $\overline{c} : I \to I$ such that, for each $x \in I$,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(\overline{c}(x))}{n!}(x-a)^n.$$

In this paper we study the behaviour of the derivatives $\overline{c}^{(p)}$ and $\overline{\theta}^{(p)}$ of the functions $\overline{c}$ and $\overline{\theta}$, respectively, when x approaches a, where $\overline{\theta} : I \to]0,1[$ is defined by $\overline{\theta}(x) = (c(x) - a)/(x-a)$, if $x \in I \setminus \{a\}$ and $\overline{\theta}(a) = 1/(n+1)$.

Taylor's theorem (or Taylor's formula) is one of the most important theorems in calculus. Taylor's theorem is usually presented in the following form:

THEOREM 1. (Taylor's theorem) *Let I be an interval in $\mathbb{R}$, $a \in I$ and $f : I \to \mathbb{R}$ be a function. If the function f is $n \geqslant 1$ times differentiable on I, then for each $x \in I \setminus \{a\}$ there exists a real number c_x from the interval with the extremities x and a such that*

$$f(x) = (T_{n-1}f)(x) + \frac{f^{(n)}(c_x)}{n!}(x-a)^n, \tag{1}$$

where $T_{n-1}f : \mathbb{R} \to \mathbb{R}$ is the polynomial function defined by

$$(T_{n-1}f)(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k, \text{ for all } x \in \mathbb{R}.$$

If $f^{(n)}$ is injective on I, then the number c_x is unique. In this case, we can define the function $c : I \setminus \{a\} \to I \setminus \{a\}$ by

$$c(x) = c_x, \quad \text{for all } x \in I \setminus \{a\}. \tag{2}$$

The function c has the property that

$$f(x) = (T_{n-1}f)(x) + \frac{f^{(n)}(c(x))}{n!}(x-a)^n, \tag{3}$$

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for all $x \in I \setminus \{a\}$.

If the function $f^{(n)}$ is not injective on I, then, for some $x \in I \setminus \{a\}$, there exist several real numbers c_x from the interval with the extremities x and a such that (1) holds. If, for each $x \in I \setminus \{a\}$, we choose one c_x from the interval with the extremities x and a which satisfies (1), then we can also define the function $c : I \setminus \{a\} \to I \setminus \{a\}$ by formula (2). This function c satisfies (3), too.

If $x \in I \setminus \{a\}$ tends to a, because $|c(x) - a| \leqslant |x - a|$, we have

$$\lim_{x \to a} c(x) = a.$$

Then the function $\overline{c} : I \to I$ defined by

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in I \setminus \{a\} \\ a & \text{if } x = a, \end{cases} \tag{4}$$

is continuous at $x = a$.

In the last decades there was some interest in the behavior of the function $\overline{c}$ when x approaches a. Azpeitia [5] (see, also [11], [6], [7]) proved that if f is $n+p$, $(p \in \mathbb{N})$ times differentiable on I, $f^{(n+p)}$ is continuous at $x = a$, and

$$f^{(n+1)}(a) = \cdots = f^{(n+p-1)}(a) = 0, \quad f^{(n+p)}(a) \neq 0,$$

then the function $\overline{c}$ is differentiable at $x = a$ and

$$\overline{c}^{(1)}(a) = \binom{n+p}{p}^{-1/p}.$$

In the special case when $n = 1$ (mean value theorem) we obtain

$$\overline{c}^{(1)}(a) = \frac{1}{(p+1)^{1/p}}.$$

In the particular case when $p = 1$ (i.e. $f^{(n+1)}(a) \neq 0$) we have

$$\overline{c}^{(1)}(a) = \frac{1}{n+1}.$$

These results were generalized by D. I. Duca and O. Pop [8] for Cauchy mean value theorem. T. Trif [16] gave the asymptotic behavior of the intermediate points in the Cauchy-Taylor mean value theorem, a generalization due to I. Pawlikowska [12] of Flett's mean value theorem, and a Cauchy version of Pawlikowska's mean value theorem.

One of the purposes of this paper is to establish under which circumstances the function $\overline{c}$ is p times differentiable at the point $x = a$ and to compute its derivative $\overline{c}^{(p)}(a)$. Does the derivative of the function $\overline{c}$ at the point $x = a$ depend upon the function f? Under which circumstances is the function $\overline{c}$ unique; if there exist several functions $\overline{c}$ which satisfy (3), does the derivative of the function $\overline{c}$ at $x = a$ depend upon the function $\overline{c}$ we choose?

Since for $x \in I \setminus \{a\}$,

$$\frac{\overline{c}(x) - \overline{c}(a)}{x-a} = \frac{c(x) - a}{x-a},$$

if we denote by

$$\theta(x) = \frac{c(x) - a}{x-a},$$

then $\theta(x) \in]0,1[$, $c(x) = a + (x-a)\theta(x)$ and hence

$$f(x) = (T_{n-1}f)(x) + \frac{f^{(n)}(a + (x-a)\theta(x))}{n!}(x-a)^n.$$

Obviously, the function $\overline{c} : I \to I$ defined by (4) is differentiable at $x = a$ if and only if the function $\theta : I \setminus \{a\} \to]0,1[$ defined by

$$\theta(x) = \frac{\overline{c}(x) - \overline{c}(a)}{x-a} = \frac{c(x) - a}{x-a}, \text{ for all } x \in I \setminus \{a\}$$

has limit at the point $x = a$. Moreover, if the function $\overline{c}$ is differentiable at $x = a$, then

$$\overline{c}^{(1)}(a) = \lim_{x \to a} \theta(x).$$

The function $\theta : I \setminus \{a\} \to]0,1[$ has a simple geometric interpretation: for $x \in I \setminus \{a\}$, the number $\theta(x) \in]0,1[$ is the ratio between the length of the interval with the extremities a and $c(x)$ and the length of the interval with the extremities a and x.

In this paper we study the behaviour of the derivatives $c^{(p)}$ and $\theta^{(p)}$ of the functions c and θ when x approaches a.

We shall recall two known results, which can be found in [10].

THEOREM 2. *Let I, J be two intervals of real numbers and $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ two functions such that $f(I) \subset J$. If f is n times differentiable on I, and g is n times differentiable on J, then the function $g \circ f : I \to \mathbb{R}$ is also n times differentiable on I and the following holds for every $x \in I$:*

$$(g \circ f)^{(n)}(x) = \sum_{m=1}^{n} \left(g^{(m)} \circ f\right)(x) \tag{5}$$

$$\times \sum_{\substack{i_1 + 2i_2 + \ldots + ni_n = n \\ i_1 + i_2 + \ldots + i_n = m}} \frac{n!}{i_1! i_2! \cdot \ldots \cdot i_n!} \left(\frac{f^{(1)}(x)}{1!}\right)^{i_1} \left(\frac{f^{(2)}(x)}{2!}\right)^{i_2} \cdot \ldots \cdot \left(\frac{f^{(n)}(x)}{n!}\right)^{i_n}.$$

THEOREM 3. *Let $I, J \subseteq \mathbb{R}$ be two intervals and $f : I \to J$ a bijective function. If f is n times differentiable on I and $f'(x) \neq 0$, for all $x \in I$, then the function $f^{-1} : J \to I$*

is n times differentiable on J and, for each $y \in J$:

$$(f^{-1})^{(n)}(y) = \sum_{\substack{i_2+2i_3+\ldots+(n-1)i_n=n-1 \\ i_1+i_2+\ldots+i_n=n-1}} \frac{(-1)^{n-1+i_1}(2n-2-i_1)!}{i_2!i_3!\cdot\ldots\cdot i_n!} \times \frac{1}{\left(f^{(1)}(x)\right)^{2n-1}} \left(\frac{f^{(1)}(x)}{1!}\right)^{i_1} \left(\frac{f^{(2)}(x)}{2!}\right)^{i_2} \cdot\ldots\cdot \left(\frac{f^{(n)}(x)}{n!}\right)^{i_n} \tag{6}$$

where $x = f^{-1}(y)$.

In what follows we need the following theorem:

THEOREM 4. *Let $n,\ p \geqslant 1$ be two integer numbers, $I \subseteq \mathbb{R}$ be an interval, a an interior point of I and $f : I \to \mathbb{R}$ be a function that satisfies the conditions:*
(i) the function f is $n+p$ times differentiable on I,
(ii) the function $f^{(n+p)} : I \to \mathbb{R}$ is continuous at a.
Then there exists

$$\lim_{x\to a}\left(\frac{f(x)-(T_{n-1}f)(x)}{(x-a)^n}\right)^{(p)}$$

and

$$\lim_{x\to a}\left(\frac{f(x)-(T_{n-1}f)(x)}{(x-a)^n}\right)^{(p)} = \frac{p!}{(p+n)!}f^{(p+n)}(a). \tag{7}$$

Proof. Taking Leibniz's rule into account, for each $x \in I$, we have that

$$\begin{aligned}
&\left(\frac{f(x)-(T_{n-1}f)(x)}{(x-a)^n}\right)^{(p)} \\
&= \left[\left(f(x)-(T_{n-1}f)(x)\right)\frac{1}{(x-a)^n}\right]^{(p)} \\
&= \sum_{k=0}^{p}\binom{p}{k}(f-T_{n-1}f)^{(p-k)}(x)\left(\frac{1}{(x-a)^n}\right)^{(k)} \\
&= \sum_{k=0}^{p}\binom{p}{k}(f-T_{n-1}f)^{(p-k)}(x)\frac{(-1)^k(n+k-1)!}{(n-1)!(x-a)^{n+k}} \\
&= \frac{1}{(x-a)^{n+p}}\sum_{k=0}^{p}(-1)^k\frac{(n+k-1)!}{(n-1)!}\binom{p}{k}(f-T_{n-1}f)^{(p-k)}(x)(x-a)^{p-k}.
\end{aligned}$$

Taking this to the limit, applying l'Hôpital's rule n times and rearranging, yields:

$$\begin{aligned}
&\lim_{x\to a}\left(\frac{f(x)-(T_{n-1}f)(x)}{(x-a)^n}\right)^{(p)} \\
&= \lim_{x\to a}\frac{p!}{(n+p)!(x-a)^p}\sum_{k=0}^{p}(-1)^k a_{p-k}f^{(p+n-k)}(x)(x-a)^{p-k},
\end{aligned}$$

where

$$a_{p-k} = \sum_{j=0}^{k} (-1)^j \frac{(n+k-j-1)!}{(n-1)!} \frac{(p-k+j)!}{(p-k)!} \binom{p}{k-j} \binom{n}{j},$$

for all $k \in \{0,...,p\}$. For each $k \in \{1,...,p\}$, we have

$$a_{p-k} = \frac{p!n}{(p-k)!} \sum_{j=0}^{k} (-1)^j \frac{(n+k-j-1)!}{j!\,(n-j)!\,(k-j)!} = 0,$$

because

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}.$$

Then we deduce that

$$\lim_{x\to a} \left(\frac{f(x) - (T_{n-1}f)(x)}{(x-a)^n} \right)^{(p)} = \lim_{x\to a} \frac{p!}{(n+p)!\,(x-a)^p} f^{(p+n)}(x)\,(x-a)^p$$
$$= \frac{p!}{(p+n)!} f^{(p+n)}(a).$$

REMARK 5. Theorem 4 remains true if the point a is an extremity of the interval I.

THEOREM 6. *Let $n,\ p \geqslant 1$ be two integer numbers, $I \subseteq \mathbb{R}$ be an interval, a an interior point of I and $f : I \to \mathbb{R}$ a function satisfying the conditions:*

(i) the function f is $n+p$ times differentiable on I,

(ii) the function $f^{(n+p)}$ is continuous on I,

(iii) $f^{(n+1)}(a) \neq 0$.

Then the following statements are true:

1^0 There exists a real number $\delta > 0$, such that $]a-\delta, a+\delta[\subseteq I$, for which the function $c :]a-\delta, a+\delta[\setminus\{a\} \to]a-\delta, a+\delta[\setminus\{a\}$, which satisfies

$$f(x) = (T_{n-1}f)(x) + \frac{f^{(n)}(c(x))}{n!}(x-a)^n, \tag{8}$$

for all $x \in]a-\delta, a+\delta[\setminus\{a\}$, is unique.

2^0 The function $\overline{c} :]a-\delta, a+\delta[\to]a-\delta, a+\delta[$ defined by

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in]a-\delta, a+\delta[\setminus\{a\} \\ a, & \text{if } x = a, \end{cases} \tag{9}$$

is p times differentiable on $]a-\delta, a+\delta[$ and

$$\overline{c}^{(p)}(x) = \sum_{m=1}^{p} \left(\left(\varphi^{-1}\right)^{(m)} \circ g \right)(x)$$
$$\times \sum_{\substack{i_1+2i_2+...+pi_p=p \\ i_1+i_2+...+i_p=m}} \frac{p!}{i_1!i_2!\cdot\ldots\cdot i_p!} \left(\frac{g^{(1)}(x)}{1!} \right)^{i_1} \left(\frac{g^{(2)}(x)}{2!} \right)^{i_2} \cdot\ldots\cdot \left(\frac{g^{(p)}(x)}{p!} \right)^{i_p},$$

for all $x \in]a-\delta, a+\delta[\setminus\{a\}$ *and*

$$\overline{c}^{(p)}(a) = \sum_{m=1}^{p} \left(\left(\varphi^{-1}\right)^{(m)} \circ g \right)(a) \times \sum_{\substack{i_1+2i_2+\ldots+pi_p=p \\ i_1+i_2+\ldots+i_p=m}} \frac{p!}{i_1! i_2! \cdot \ldots \cdot i_p!} F_{i_1,i_2,\ldots,i_p}(a), \qquad (10)$$

where, for each $i_1, \ldots, i_p \in \{0,1,\ldots,p\}$,

$$F_{i_1,i_2,\ldots,i_p}(a) \qquad (11)$$

$$= \left(\frac{n!}{(n+1)!} f^{(n+1)}(a) \right)^{i_1} \left(\frac{n!}{(n+2)!} f^{(n+2)}(a) \right)^{i_2} \cdot \ldots \cdot \left(\frac{n!}{(n+p)!} f^{(n+p)}(a) \right)^{i_p}$$

and $g :]a-\delta, a+\delta[\to \mathbb{R}$ *is the function defined by*

$$g(x) = \begin{cases} n! \dfrac{f(x) - (T_{n-1}f)(x)}{(x-a)^n}, & \text{if } x \in]a-\delta, a+\delta[\setminus\{a\} \\ f^{(n)}(a), & \text{if } x = a. \end{cases} \qquad (12)$$

Moreover,

$$\overline{c}^{(p)}(x) = \sum_{m=1}^{p} \sum_{\substack{j_2+2j_3+\ldots+(m-1)j_m=m-1 \\ j_1+j_2+\ldots+j_m=m-1}} \frac{(-1)^{m-1+j_1}(2m-2-j_1)!}{j_2! j_3! \cdot \ldots \cdot j_m!}$$

$$\times \frac{1}{\left(\varphi^{(1)}(x)\right)^{2m-1}} \left(\frac{\varphi^{(1)}(x)}{1!} \right)^{j_1} \left(\frac{\varphi^{(2)}(x)}{2!} \right)^{j_2} \cdots \left(\frac{\varphi^{(m)}(x)}{m!} \right)^{j_m}$$

$$\times \sum_{\substack{i_1+2i_2+\ldots+pi_p=p \\ i_1+i_2+\ldots+i_p=m}} \frac{p!}{i_1! i_2! \cdot \ldots \cdot i_p!} \left(\frac{g^{(1)}(x)}{1!} \right)^{i_1} \left(\frac{g^{(2)}(x)}{2!} \right)^{i_2} \cdot \ldots \cdot \left(\frac{g^{(p)}(x)}{p!} \right)^{i_p},$$

for all $x \in]a-\delta, a+\delta[\setminus\{a\}$ *and*

$$\overline{c}^{(p)}(a) = \sum_{m=1}^{p} \sum_{\substack{j_2+2j_3+\ldots+(m-1)j_m=m-1 \\ j_1+j_2+\ldots+j_m=m-1}} \frac{(-1)^{m-1+j_1}(2m-2-j_1)!}{j_2! j_3! \cdot \ldots \cdot j_m! \left[f^{(n+1)}(a)\right]^{2m-1}} \qquad (13)$$

$$\times \left(\frac{f^{(n+1)}(a)}{1!} \right)^{j_1} \left(\frac{f^{(n+2)}(a)}{2!} \right)^{j_2} \cdot \ldots \cdot \left(\frac{f^{(n+m)}(a)}{m!} \right)^{j_m}$$

$$\times \sum_{\substack{i_1+2i_2+\ldots+pi_p=p \\ i_1+i_2+\ldots+i_p=m}} \frac{p!}{i_1! i_2! \cdot \ldots \cdot i_p!} F_{i_1,i_2,\ldots,i_p}(a)$$

where $F_{i_1,i_2,\ldots,i_p}(a)$ $(i_1, \ldots, i_p \in \{0,1,\ldots,p\})$ *are given by* (11).

3^0 *The function* $\theta :]a-\delta, a+\delta[\setminus\{a\} \to]0,1[$ *for which*

$$f(x) - (T_{n-1}f)(x) = \frac{(x-a)^n}{n!} f^{(n)}(a + (x-a)\theta(x)), \qquad (14)$$

for all $x \in]a-\delta, a+\delta[\setminus \{a\}$, *is unique.*

4^0 *The function* $\overline{\theta} :]a-\delta, a+\delta[\to]0,1[$ *defined by*

$$\overline{\theta}(x) = \begin{cases} \theta(x), & x \in]a-\delta, a+\delta[\setminus \{a\} \\ \dfrac{1}{n+1}, & x = a \end{cases}$$

is $p-1$ *times differentiable on* $]a-\delta, a+\delta[$ *and*

$$\overline{\theta}^{(p-1)}(a) = \frac{1}{p}\overline{c}^{(p)}(a). \tag{15}$$

Proof. 1^0 Assume that $f^{(n+1)}(a) > 0$. Since a is an interior point of I, then there exists a real number $\delta > 0$ such that $]a-\delta, a+\delta[\subseteq I$ and $f^{(n+1)}(x) > 0$, for all $x \in]a-\delta, a+\delta[$.

It follows that the function $f^{(n)}$ is increasing on $]a-\delta, a+\delta[$ and therefore injective on $]a-\delta, a+\delta[$. Then the function $\varphi :]a-\delta, a+\delta[\to f^{(n)}(]a-\delta, a+\delta[)$ defined by

$$\varphi(x) = f^{(n)}(x), \text{ for all } x \in]a-\delta, a+\delta[,$$

is bijective and hence the function c is unique. If $f^{(n+1)}(a) < 0$, the proof is similar.

2^0 Taking 1^0 into account, (8) yields that c has the following expression

$$c(x) = \left(\varphi^{-1}\right)\left(n!\frac{f(x) - (T_{n-1}f)(x)}{(x-a)^n}\right), \text{ for all } x \in]a-\delta, a+\delta[\setminus \{a\}. \tag{16}$$

According to (16) and (12), relation (9) becomes

$$\overline{c}(x) = \begin{cases} \left(\varphi^{-1} \circ g\right)(x), & \text{if } x \in]a-\delta, a+\delta[\setminus \{a\} \\ a, & \text{if } x = a. \end{cases}$$

From (i) and the definition of g we have that the function $\overline{c}$ is p times differentiable on $]a-\delta, a+\delta[$ and heeding (5) and (6), for all $x \in]a-\delta, a+\delta[\setminus \{a\}$ we have that

$$\begin{aligned}
\overline{c}^{(p)}(x) &= \sum_{m=1}^{p} \left(\left(\varphi^{-1}\right)^{(m)} \circ g\right)(x) \\
&\quad \times \sum_{\substack{i_1+2i_2+\ldots+pi_p=p \\ i_1+i_2+\ldots+i_p=m}} \frac{p!}{i_1!i_2!\cdot\ldots\cdot i_p!}\left(\frac{g^{(1)}(x)}{1!}\right)^{i_1}\left(\frac{g^{(2)}(x)}{2!}\right)^{i_2}\cdot\ldots\cdot\left(\frac{g^{(p)}(x)}{p!}\right)^{i_p} \\
&= \sum_{m=1}^{p} \sum_{\substack{j_2+2j_3+\ldots+(m-1)j_m=m-1 \\ j_1+j_2+\ldots+j_m=m-1}} \frac{(-1)^{m-1+j_1}(2m-2-j_1)!}{j_2!j_3!\cdot\ldots\cdot j_m!} \\
&\quad \times \frac{1}{\left(\varphi^{(1)}(x)\right)^{2m-1}}\left(\frac{\varphi^{(1)}(x)}{1!}\right)^{j_1}\left(\frac{\varphi^{(2)}(x)}{2!}\right)^{j_2}\cdots\left(\frac{\varphi^{(m)}(x)}{m!}\right)^{j_m} \\
&\quad \times \sum_{\substack{i_1+2i_2+\ldots+pi_p=p \\ i_1+i_2+\ldots+i_p=m}} \frac{p!}{i_1!i_2!\cdot\ldots\cdot i_p!}\left(\frac{g^{(1)}(x)}{1!}\right)^{i_1}\left(\frac{g^{(2)}(x)}{2!}\right)^{i_2}\cdot\ldots\cdot\left(\frac{g^{(p)}(x)}{p!}\right)^{i_p}.
\end{aligned}$$

Taking it to the limit and recalling (7), one obtains (10) and (13).

3^0 The uniqueness of the function θ follows immediately by the uniqueness of the function c.

4^0 Obviously

$$\lim_{x\to a}\theta(x)=\lim_{x\to a}\frac{c(x)-c(a)}{x-a}=\lim_{x\to a}\frac{\overline{c}(x)-\overline{c}(a)}{x-a}=\overline{c}^{(1)}(a)=\frac{1}{n+1},$$

hence the function $\overline{\theta}$ is continuous at $x=a$. Moreover, from (14) and (9) it follows that the function $\overline{\theta}$ is $p-1$ times differentiable on $]a-\delta,a+\delta[$ and

$$\overline{\theta}^{(p-1)}(x)=\left(\frac{\overline{c}(x)-\overline{c}(a)}{x-a}\right)^{(p-1)},$$

for all $x\in]a-\delta,a+\delta[\setminus\{a\}$.

Considering (7) for the function $f=\overline{c}$ and $n=1$, one obtains (15).

REMARK 7. Theorem 6 remains true if the point a is an extremity of the interval I.

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