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ON TWO VARIABLE FUNCTIONAL INEQUALITY AND RELATED FUNCTIONAL EQUATION

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*Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday*

Abstract. In this paper we show that solutions for a large class of functional inequalities can be obtained from solutions of the corresponding functional equations.

1. Introduction

Let $I \subset \mathbb{R}$ be a nonempty and open interval. We say that a function $M : I^2 \to I$ is a *mean* on I if

$$\min\{x,y\} \leqslant M(x,y) \leqslant \max\{x,y\}$$

for all $x, y \in I$. If, moreover, these inequalities are sharp for all $x \neq y$, then M is said to be a *strict mean*.

Let $m : I^2 \to I$ and $n : I^2 \to I$ be some continuous strict means. In [4] the author considered the following inequality

$$f\big(n(x,y)\big) + f\big(m(x,y)\big) \leqslant f(x) + f(y), \quad x,y \in I, \tag{A}$$

and the corresponding functional equation

$$\varphi\big(n(x,y)\big) + \varphi\big(m(x,y)\big) = \varphi(x) + \varphi(y), \quad x,y \in I, \tag{B}$$

with unknown functions $f : I \to \mathbb{R}$ and $\varphi : I \to \mathbb{R}$, respectively. He proved that: *if there exists a continuous strictly increasing solution* $\varphi : I \to R$ *of* (B)*, then a continuous function* $f : I \to \mathbb{R}$ *satisfies* (A) *if and only if* $f \circ \varphi^{-1}$ *is a convex function on* $\varphi(I)$.

In this paper we extend that result for a large class of functional inequalities.

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2. Main result

Let $\lambda : I^2 \to (0,1)$ be a fixed function. For a given $f : I \to \mathbb{R}$ we adopt the following symbol

$$\left(T_f^{\lambda}\right)(x,y) := \lambda(x,y)f(x) + (1-\lambda(x,y))f(y).$$

For the given means $m,n : I^2 \to \mathbb{R}$ we consider the following inequality

$$\left(T_f^{\lambda}\right)\left(m(x,y),n(x,y)\right) \leqslant \left(T_f^{\lambda}\right)(x,y), \quad x,y \in I, \tag{I}$$

and the related functional equation

$$\left(T_\varphi^{\lambda}\right)\left(m(x,y),n(x,y)\right) = \left(T_\varphi^{\lambda}\right)(x,y), \quad x,y \in I, \tag{II}$$

with unknown functions $f : I \to \mathbb{R}$ and $\varphi : I \to \mathbb{R}$, respectively.

It easy to see, that if λ is a constant function, equals identically one-half, then (I) becomes (A) and (II) becomes (B).

THEOREM 1. *Let $\lambda : I^2 \to (0,1)$ be a function, $n,m : I^2 \to I$ be continuous means and one of them is strict. If there exists a continuous, one-to-one solution $\varphi : I \to \mathbb{R}$ of* (II)*, then a lower semicontinuous function $f : I \to \mathbb{R}$ satisfies* (I) *if and only if $f \circ \varphi^{-1}$ is a convex function on $\varphi(I)$.*

Proof. Assume that a function φ is continuous, one-to-one and satisfies (II) and a function f is lower semi-continuous satisfying (I). Take arbitrary $x,y \in I$ and define sequence (x_k,y_k) in the following way

$$\begin{aligned} x_1 &:= x & y_1 &:= y, \\ x_{k+1} &:= m(x_k,y_k) & y_{k+1} &:= n(x_k,y_k). \end{aligned} \tag{1}$$

Such a sequence is said to be the *Gauss-iteration* determined by the pair (m,n) with the initial values $(x,y) \in I^2$ (cf. [2]). Under our assumptions the Gauss-iteration (1) is convergent, i.e.

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k$$

for any value $(x,y) \in I^2$ and it determines an other mean. Such a mean is said to be *Gauss-composition* of (m,n), we denote it by $m \otimes n$ and define it as a common limit of the sequences (x_k) and (y_k), i.e.

$$m \otimes n(x,y) := \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k, \tag{2}$$

for all $x,y \in I$ (cf. [2], p.164,165).

From definition (1), inequality (I) and equation (II) we get

$$\left(T_f^{\lambda}\right)(x_k,y_k) \leqslant \left(T_f^{\lambda}\right)(x,y) \quad \text{and} \quad \left(T_\varphi^{\lambda}\right)(x_k,y_k) = \left(T_\varphi^{\lambda}\right)(x,y), \qquad k \in \mathbb{N}. \tag{3}$$

Making use of the lower semicontinuity of f, the continuity of φ and (2), we conclude that for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ we have

$$\begin{aligned} & f(m\otimes n(x,y)) - \varepsilon < f(x_k) \\ \text{and} \quad & \\ & f(m\otimes n(x,y)) - \varepsilon < f(y_k) \end{aligned} \tag{4}$$

as well as

$$\begin{aligned} & \varphi(m\otimes n(x,y)) - \varepsilon < \varphi(x_k) < \phi(m\otimes n(x,y)) + \varepsilon \\ \text{and} \quad & \\ & \varphi(m\otimes n(x,y)) - \varepsilon < \varphi(y_k) < \varphi(m\otimes n(x,y)) + \varepsilon. \end{aligned} \tag{5}$$

Multiplying inequalities (4) by $\lambda(x_k,y_k)$ and $1-\lambda(x_k,y_k)$, respectively, adding them and using inequality in (3) and the definition of T_f^λ we obtain

$$f(m\otimes n(x,y)) - \varepsilon < \left(T_f^\lambda\right)(x_k,y_k) \leqslant \left(T_f^\lambda\right)(x,y).$$

Similarly, multiplying inequalities (5) by $\lambda(x_k,y_k)$ and $1-\lambda(x_k,y_k)$, respectively, adding them and using equality in (3) we conclude that

$$\varphi(m\otimes n(x,y)) = \lim_{k\to\infty}(T_\varphi^\lambda)(x_k,y_k) = \left(T_\varphi^\lambda\right)(x,y).$$

Thus

$$f(m\otimes n(x,y)) \leqslant \lambda(x,y)f(x) + (1-\lambda(x,y))f(y) \tag{6}$$

and

$$\varphi(m\otimes n(x,y)) = \lambda(x,y)\varphi(x) + (1-\lambda(x,y))\varphi(y). \tag{7}$$

Function φ is one-to-one, then from (7) we have

$$m\otimes n(x,y) = \varphi^{-1}\left(\lambda(x,y)\varphi(x) + (1-\lambda(x,y))\varphi(y)\right)$$

and taking into account (6) we obtain

$$f\left(\varphi^{-1}\left(\lambda(x,y)\varphi(x) + (1-\lambda(x,y))\varphi(y)\right)\right) \leqslant \lambda(x,y)f(x) + (1-\lambda(x,y))f(y).$$

Putting in the last inequality $s := \varphi(x)$, $t := \varphi(y)$, $\mu(s,t) := \lambda(\varphi^{-1}(s),\varphi^{-1}(t))$ and $g := f\circ\varphi^{-1}$ we get

$$g\left((\mu(s,t)s + (1-\mu(s,t))t)\right) \leqslant \mu(s,t)g(s) + (1-\mu(s,t))g(t).$$

It means that the function g is $\mu - convex$ on $\varphi(I)$. Moreover, g is lower semicontinuous, then it must be convex (cf. [1]).

Now assume that φ satisfies (II) and that $f\circ\varphi^{-1}$ is a convex function on $\varphi(I)$. Take arbitrary $x,y\in I$. Because m,n are means and function φ, as a continuous and one-to-one, is monotone, we deduce that

$$\begin{aligned} &\min\{\varphi(x),\varphi(y)\}\leqslant\varphi(m(x,y))\leqslant\max\{\varphi(x),\varphi(y)\}\\ \text{and}\quad&\\ &\min\{\varphi(x),\varphi(y)\}\leqslant\varphi(n(x,y))\leqslant\max\{\varphi(x),\varphi(y)\}. \end{aligned}\tag{8}$$

From convexity of the function $f\circ\varphi^{-1}$ we conclude that

$$\alpha\big(f\circ\varphi^{-1}\big)(u)+(1-\alpha)\big(f\circ\varphi^{-1}\big)(v)\leqslant\beta\big(f\circ\varphi^{-1}\big)(w)+(1-\beta)\big(f\circ\varphi^{-1}\big)(z),\tag{9}$$

as long as

$$\alpha u+(1-\alpha)v=\beta w+(1-\beta)z,$$

for some $u,v,w,z\in\varphi(I)$ such that u and v are between w and z, and $\alpha,\beta\in[0,1]$. Finally, on account of (II), (8) and (9) we get (I), which was to be proved. □

On the base of the following lemma we will be able to modify some assumptions of Theorem 1.

LEMMA 1. *Let $\lambda: I^2\to(0,1)$ be a function, $n,m: I^2\to I$ be continuous strict means and $\varphi: I\to\mathbb{R}$ is a nonconstant and continuous solution of equation* (II)*, then φ is one-to-one.*

Proof. Let $m\otimes n: I^2\to I$ be a Gauss-composition of (m,n). In our assumptions $m\otimes n$ is continuous strict mean (cf. [2], p.164,165). Suppose that there exist $a,b\in I$, $a\neq b$, such that $\varphi(a)=\varphi(b)$ and put

$$J:=\{x\in I:\varphi(x)=\varphi(a)\}.$$

By the continuity of φ, the set J is closed in I. Note that J is an interval. In the opposite case we could find $a_1,b_1\in J$ $(a_1<b_1)$ such, that

$$\varphi(x)\neq\varphi(a),\qquad\qquad x\in(a_1,b_1).\tag{10}$$

Function φ satisfies equation (II), thus setting in it $x=a_1$, $y=b_1$ we get

$$\varphi\big(m\otimes n(a_1,b_1)\big)=\lambda(a_1,b_1)\varphi(a_1)+(1-\lambda(a_1,b_1))\varphi(b_1)=\varphi(a).$$

But in view of (10) it is impossible, because $a_1<m\otimes n(a_1,b_1)<b_1$. It means that J is an interval. Now if $J\neq I$, then $\sup J<\sup I$ or $\inf J<\inf I$. Suppose that $\sup J<\sup I$ (if $\inf J<\inf I$ the proof is similar and we omit it). Take arbitrary $x\in \operatorname{int} J$ and $y\in I\setminus J$ $(\sup J<y)$ such, that $m\otimes n(x,y)\in J$ (it is possible by strictnesses and continuity of $m\otimes n$). Finally from (II) we obtain

$$\begin{aligned}\varphi(a)=\varphi\big(m\otimes n(x,y)\big)&=\lambda(x,y)\varphi(x)+(1-\lambda(x,y))\varphi(y)\\&=\lambda(x,y)\varphi(a)+(1-\lambda(x,y))\varphi(y),\end{aligned}$$

thus

$$\varphi(a) = \varphi(y),$$

which according to the choice of y is impossible. Now we conclude that $J = I$. This contradiction proves that φ is one-to-one. $\square$

Now as an immediate consequence of Theorem 1 and Lemma 1 we get the following theorem.

THEOREM 2. *Let $\lambda : I^2 \to (0,1)$ be a function, $n,m : I^2 \to I$ be continuous strict means. If there exists a continuous, nonconstant solution of* (II)*, then for every lower semicontinuous function $f : I \to \mathbb{R}$ satisfying inequality* (I)*, the function $f \circ \varphi^{-1}$ is convex on $\varphi(I)$.*

Finally, we give some applications of Theorem 1 and Theorem 2.

EXAMPLE 1. Consider the functional inequality

$$\frac{1}{3}f\left(\frac{3}{4}x + \frac{1}{4}y\right) + \frac{2}{3}f\left(\frac{1}{8}x + \frac{7}{8}y\right) \leqslant \frac{1}{3}f(x) + \frac{2}{3}f(y), \quad x,y \in I. \tag{11}$$

Since $\varphi = \mathrm{id}$ is a solution of the corresponding functional equation, therefore, a lower semicontinuous function $f : I \to \mathbb{R}$ is a solution of inequality (11) if and only if it is convex.

EXAMPLE 2. Let $M : I^2 \to I$ be a continuous strict mean and satisfies the bisymmetry equation:

$$M(M(s,t), M(u,v)) = M(M(s,u), M(t,v)),$$

for all $s,t,u,v \in I$, and $\lambda : I^2 \to (0,1)$ be a weight function of mean M i.e.:

$$M(x,y) = \lambda(x,y)x + (1 - \lambda(x,y))y, \quad x,y \in I.$$

Consider the functional inequality

$$\left(T_f^{\lambda}\right)(M(x, M(x,y)), M(y, M(x,y))) \leqslant \left(T_f^{\lambda}\right)(x,y), \tag{12}$$

for all $x,y \in I$. Function $\varphi = \mathrm{id}$ is a solution of the corresponding functional equation, therefore, a lower semicontinuous function $f : I \to \mathbb{R}$ is a solution of (12) if and only if it is convex.

EXAMPLE 3. Let $M, N : I^2 \to I$ be continuous strict means and $\lambda : I^2 \to (0,1)$ is a weight function of mean $M \otimes N$. Consider the functional inequality

$$\left(T_f^{\lambda}\right)(M(x,y), N(x,y)) \leqslant \left(T_f^{\lambda}\right)(x,y), \tag{13}$$

for all $x,y \in I$. Function $\varphi = \mathrm{id}$ is a solution of the corresponding functional equation, therefore, a lower semicontinuous function $f : I \to \mathbb{R}$ is a solution of (1) if and only if it is convex.

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