Mathematical
Inequalities
& Applications
Volume 12, Number 4 (2009), 805–814

ONE REFINEMENT OF JENSEN'S DISCRETE INEQUALITY AND APPLICATIONS

V. ČULJAK AND B. IVANKOVIĆ

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. Jensen's inequality induces different forms of functionals which enables refinements for many classic inequalities ([5]). Several refinements of Jensen's inequalities were given in [4]. In this paper we refine Jensen's inequality by separating a discrete domain of it. At the end, we give some applications.

1. Introduction

Jensen's inequality plays the crucial role in the Theory of Inequalities. In this paper, C is a convex subset of the linear space X and f is a convex function on C. If $p_1,\dots,p_n \in (0,1)^n$, $\sum_{i=1}^{n} p_i = 1$, and $x_1,\dots,x_n \in C$ is a sequence of vectors, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i). \tag{1}$$

In [5] the authors have investigated the differences $J(f,I,p,x) = \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)$ as a functional depending on a function f, a set of indices I, vectors $x = \{x_i\}_{i\in I}$ and weights $p = \{p_i\}_{i\in I}$ with a constraint $P_I = \sum_{i\in I} p_i \neq 0$. The next refinement of (1) was proven in [5] as a consequence of its Theorem 2.1, part (ii).

COROLLARY 1. *Let $f : C \to \mathbb{R}$ be a convex function on a convex subset C of a real linear space X, $p_i > 0$, and $x_i \in C$. If $\sum_{i=1}^{n} p_i = 1$, then*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant \max_{1\leqslant i<j\leqslant n}\left\{p_i f(x_i) + p_j f(x_j) - (p_i+p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i+p_j}\right)\right\}$$
$$\geqslant 0. \tag{2}$$

Mathematics subject classification (2000): 26D15.
Keywords and phrases: Jensen's inequality; convex functions; linear spaces; AG-inequality.

In [4] the authors obtained a refinement for a convex function in a particular case of its domain.

THEOREM 1. *Let C be a convex subset of a linear space X and $0 \in C$. Suppose $f : C \to \mathbb{R}$ is a convex function on C, $x_j \in C$, $p_i \in (0,1), i \in I_n = \{1,\dots,n\}, n \geqslant 2$ and $\sum_{i=1}^n p_i = 1$.*

If $\sum_{i=1}^n p_i x_i = 0$, then $\frac{p_k}{p_k - 1} x_k \in C$ for all $k \in I_n$ and

$$\sum_{i=1}^n p_i f(x_i) \geqslant \max_{k \in I_n} \left[p_k f(x_k) + (1-p_k) f\left(\frac{p_k}{p_k - 1} \cdot x_k \right) \right]$$
$$\geqslant \min_{k \in I_n} \left[p_k f(x_k) + (1-p_k) f\left(\frac{p_k}{p_k - 1} \cdot x_k \right) \right] \geqslant f(0).$$

In [3] the author refined Jensen's inequality representing the consequence of extracting the single element from $\{x_1,\dots,x_n\}$ together with its weight.

THEOREM 2. *Let $f : C \to \mathbb{R}$ be a convex function defined on the convex subset C of the linear space X, $x_i \in C$, $p_i > 0, i \in I_n = \{1,\dots,n\}, n \geqslant 2$ with $\sum_{i=1}^n p_i = 1$. Then*

$$f\left(\sum_{i=1}^n p_i x_i \right) \leqslant \min_{k \in I_n} \left[(1-p_k) f\left(\frac{\sum_{i=1}^n p_i x_i - p_k x_k}{1-p_k} \right) + p_k f(x_k) \right]$$
$$\leqslant \frac{1}{n} \left[\sum_{k=1}^n (1-p_k) f\left(\frac{\sum_{i=1}^n p_i x_i - p_k x_k}{1-p_k} \right) + \sum_{k=1}^n p_k f(x_k) \right]$$
$$\leqslant \max_{k \in I_n} \left[(1-p_k) f\left(\frac{\sum_{i=1}^n p_i x_i - p_k x_k}{1-p_k} \right) + p_k f(x_k) \right]$$
$$\leqslant \sum_{j=1}^n p_i f(x_i).$$

In particular, for $p_k = \frac{1}{n}$, $k = 1,\dots,n$, we have the following inequalities:

$$f\left(\frac{1}{n} \sum_{i=1}^n x_i \right) \leqslant \frac{1}{n} \min_{k \in I_n} \left[(n-1) \cdot f\left(\frac{\sum_{i=1}^n x_i - x_k}{n-1} \right) + f(x_k) \right]$$
$$\leqslant \frac{1}{n^2} \left[(n-1) \sum_{k=1}^n f\left(\frac{\sum_{i=1}^n x_i - x_k}{n-1} \right) + \sum_{j=1}^n f(x_j) \right]$$
$$\leqslant \frac{1}{n} \max_{k \in I_n} \left[(n-1) \cdot f\left(\frac{\sum_{i \in I} x_i}{n-1} \right) + f(x_k) \right]$$
$$\leqslant \frac{1}{n} \sum_{i=1}^n f(x_i).$$

The result given in Theorem 2 can be interpreted as an extension of that from Theorem 1 for the case $\sum_{i=1}^{n} p_i x_i \neq 0$. In this paper we give some generalizations of the results given above and as a consequence, (2) is generalized considering it in the manner

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \max_{1\leqslant i<j\leqslant n}\left\{(p_i+p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i+p_j}\right) + \sum_{k\neq i,j} p_k f(x_k)\right\} \leqslant \sum_{i=1}^{n} p_i f(x_i).$$

2. General Result

The refinement in Theorem 2 is due to the separation $I_n = (I_n \setminus \{x_i\}) \cup \{x_i\}$. This separation can be generalized in many different ways. Next refinements are consequences of such generalizations. The main result is given bellow.

THEOREM 2.1. *Let $f: C \to \mathbb{R}$ be a convex function on the convex subset C of the linear space X, $x_i \in C$, $p_i > 0, i \in I_n = \{1,\ldots,n\}, n \geqslant 3$ with $\sum_{i=1}^{n} p_i = 1$.*

If $\mathscr{I} = \{I \subset I_n, I \neq I_n, |I| \geqslant 2\}$, then

$$\begin{aligned}
f\left(\sum_{i=1}^{n} p_i x_i\right) &\leqslant \min_{\mathscr{I}}\left[P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i)\right] \\
&\leqslant \frac{1}{2^n-n-2}\left[\sum_{I\subset I_n} P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \left(2^{n-1}-n\right)\sum_{i=1}^{n} p_i f(x_i)\right] \\
&\leqslant \max_{\mathscr{I}}\left[P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i)\right] \\
&\leqslant \sum_{i=1}^{n} p_i f(x_i).
\end{aligned}$$

In particular,

$$\begin{aligned}
f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) &\leqslant \frac{1}{n}\min_{\mathscr{I}}\left[|I|\, f\left(\frac{\sum_{i\in I} x_i}{|I|}\right) + \sum_{i\in I_n\setminus I} f(x_i)\right] \\
&\leqslant \frac{1}{n}\cdot\frac{1}{2^n-n-2}\left[\sum_{I\subset I_n} |I|\, f\left(\frac{\sum_{i\in I} x_i}{|I|}\right) + \left(2^{n-1}-n\right)\sum_{i=1}^{n} f(x_i)\right] \\
&\leqslant \frac{1}{n}\max_{\mathscr{I}}\left[|I|\, f\left(\frac{\sum_{i\in I} x_i}{|I|}\right) + \sum_{i\in I_n\setminus I} f(x_i)\right] \\
&\leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i).
\end{aligned}$$

Proof. Take a separation $I_n = I \cup (I_n \setminus I)$, name $J = I_n \setminus I$ and note that $P_I + P_J = 1$. Do the further estimation using Jensen's inequality (1) twice, by observing $\sum_{i \in J} \frac{p_i}{P_J} = 1$:

$$\begin{aligned} f\left(\sum_{i=1}^n p_i x_i\right) &= f\left(P_I \cdot \frac{\sum_{i\in I} p_i x_i}{P_I} + P_J \cdot \frac{\sum_{i\in J} p_i x_i}{P_J}\right) \\ &\leqslant P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + P_J \cdot f\left(\frac{\sum_{i\in J} p_i x_i}{P_J}\right) \\ &\leqslant P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in J} p_i f(x_i) \\ &\leqslant \sum_{i\in I} p_i f(x_i) + \sum_{i\in J} p_i f(x_i) \\ &= \sum_{i=1}^n p_i f(x_i). \end{aligned}$$

For further estimation we extract the following inequality which holds for every $I \subseteq I_n$:

$$f\left(\sum_{i=1}^n p_i x_i\right) \leqslant P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i) \leqslant \sum_{i=1}^n p_i f(x_i). \tag{3}$$

The statement in the Theorem follows from taking the min and max of the

$$A_I = P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i),$$

over the every $I \subset I_n$ such that $|I| \geqslant 2$. We use the fact

$$\min_{I\subset I_n} A_I \leqslant \frac{1}{N} \sum_{I\subset I_n} A_I \leqslant \max_{I\subset I_n} A_I, \tag{4}$$

where $N = 2^n - n - 2$ represents the number of subsets from I_n except $\emptyset$, the base set I_n and the subsets of the kind $|I| = 1$. Taking any of these subsets, the refinement is shutting down.

Note that $\sum_{I\subset I_n} \sum_{i\in I_n\setminus I} p_i f(x_i) = \left(2^{n-1} - (n-1) - 1\right) \sum_{i=1}^n p_i f(x_i)$, because every $p_i f(x_i)$ appears as many times as there is a subset $I \subset I_n, |I| \geqslant 2,$ that doesn't contain the index i. □

In the next theorem, subsets of equivalent cardinality are observed.

THEOREM 2.2. *Let* $f : C \to \mathbb{R}$ *be a convex function on the convex subset* C *of the linear space* X, $x_i \in C$, $p_i > 0, i \in I_n = \{1, \ldots, n\}, n \geqslant 3$ *with* $\sum_{i=1}^n p_i = 1$. *For every* *subset $I \subset I_n$ such that $|I| = s \geqslant 2$, we can state the following:*

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\leqslant \min_{|I|=s}\left[P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i)\right] \\ &\leqslant \frac{1}{\binom{n}{s}}\left[\sum_{|I|=s} P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \binom{n-1}{s}\sum_{i=1}^{n} p_i f(x_i)\right] \\ &\leqslant \max_{|I|=s}\left[P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i)\right] \\ &\leqslant \sum_{i=1}^{n} p_i f(x_i). \end{aligned}$$

In particular:

$$\begin{aligned} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) &\leqslant \frac{1}{n}\min_{|I|=s}\left[s\cdot f\left(\frac{\sum_{i\in I} x_i}{s}\right) + \sum_{i\in I_n\setminus I} f(x_i)\right] \\ &\leqslant \frac{1}{\binom{n}{s}}\left[\frac{s}{n}\sum_{|I|=s} f\left(\frac{\sum_{i\in I} x_i}{s}\right) + \binom{n-1}{s}\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right] \\ &\leqslant \frac{1}{n}\max_{|I|=s}\left[s\cdot f\left(\frac{\sum_{i\in I} x_i}{s}\right) + \sum_{i\in I_n\setminus I} f(x_i)\right] \\ &\leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i). \end{aligned}$$

Proof. The statement in the Theorem follows from taking the min and max of the middle side of (3), after choosing every subset $I \subset I_n$, such that $|I| = s, 2 \leqslant s < n$. We use the fact mentioned in (4), where $N = \binom{n}{s}$ represents the number of subsets $I \subset I_n, |I| = s$. Note that $\sum_{I\subset I_n,|I|=s}\sum_{i\in I_n\setminus I} p_i f(x_i) = \left[\binom{n}{s} - \binom{n-1}{s-1}\right]\sum_{i=1}^{n} p_i f(x_i)$, because every $p_i f(x_i)$ in the double sum appears as many times as there are subsets $I \subset I_n$, $|I| = s \geqslant 2$ such that $i \notin I$. The subset $I \subset I_n$, with $|I| = s$ and $i \in I$ is constructed by adding $s-1$ elements from the $n-1$ available once. Algebraically, $\left[\binom{n}{s} - \binom{n-1}{s-1}\right]\sum_{i=1}^{n} p_i f(x_i) = \binom{n-1}{s}\sum_{i=1}^{n} p_i f(x_i)$. $\square$

The result from [3] is here obtained for $s = n-1$. Actually,

$$\min_{I\subset I_n,|I|=n-1} A_I = \min_{k\in I_n} A_k, \qquad \max_{I\subset I_n,|I|=n-1} A_I = \max_{k\in I_n} A_k$$

and $\sum_{I\subset I_n,|I|=n-1} A_I = \sum_{k=1}^{n} A_k$, where $A_k = (1-p_k) f\left(\frac{\sum_{i=1}^{n} p_i x_i - p_k x_k}{(1-p_k)}\right) + p_k f(x_k)$.

The inequalities from Theorem 2.2 can be rewritten as those in Theorem 2.

Every partition of $I_n = \{1,\ldots,n\}$ gives the statement obtained in the next Theorem.

THEOREM 2.3. *Let $f : C \to \mathbb{R}$ be a convex function on the convex subset C of the linear space X, $x_i \in C$, $p_i \in (0,1), i \in I_n = \{1,\ldots,n\}$, $n \geqslant 4$ and $\sum_{i=1}^{n} p_i = 1$. For every integer k, $4 \leqslant 2k \leqslant n$ there is a partition $I_1 \cup \ldots \cup I_k = I_n$ with $2 \leqslant |I_j| < n$ for $j = 1,\ldots,k$.*
Then

$$\begin{aligned}
f\left(\sum_{i=1}^{n} p_i x_i\right) &\leqslant \min_{j=1,\ldots,k}\left[P_{I_j} f\left(\frac{\sum_{i\in I_j} p_i x_i}{P_{I_j}}\right) + \sum_{i\in I_n\setminus I_j} p_i f(x_i)\right] \\
&\leqslant \frac{1}{k}\left[\sum_{j=1}^{k} P_{I_j} f\left(\frac{\sum_{i\in I_j} p_i x_i}{P_{I_j}}\right) + (k-1)\sum_{i=1}^{n} p_i f(x_i)\right] \\
&\leqslant \max_{j=1,\ldots,k}\left[P_{I_j} f\left(\frac{\sum_{i\in I_j} p_i x_i}{P_{I_j}}\right) + \sum_{i\in I_n\setminus I_j} p_i f(x_i)\right] \\
&\leqslant \sum_{i=1}^{n} p_i f(x_i)
\end{aligned}$$

holds.

Proof. Every subset $I_j \subset I_n$ induced its complement $I_n \setminus I_j$ and (3) is valid with the substitutions: $I \to I_j$. For $A_{I_j} = P_{I_j} f\left(\frac{\sum_{i\in I_j} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I_j} p_i f(x_i)$ we take the min and max over $j = 1,...,k$ and use the fact that

$$\min_{j=1,\ldots,k} A_{I_j} \leqslant \frac{1}{k}\sum_{j=1}^{k} A_{I_j} \leqslant \max_{j=1,\ldots,k} A_{I_j}.$$

Note: $\sum_{j=1}^{k}\sum_{i\in I_n\setminus I_j} p_i f(x_i) = (k-1)\sum_{i=1}^{n} p_i f(x_i).$ □

Theorem 2.1 ensures the next refinement of (2).

COROLLARY 2.1. *Under the conditions of Theorem 2.1, we obtain*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant \max_{I}\left[\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)\right] \geqslant 0.$$

In particular, for $p_i = \frac{1}{n}$, $i \in I_n$*:*

$$\sum_{i=1}^{n} f(x_i) - n \cdot f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geqslant \max_{I}\left[\sum_{i\in I} f(x_i) - |I| \cdot f\left(\frac{\sum_{i\in I} x_i}{|I|}\right)\right].$$

Proof. Subtracting $\sum_{i=1}^{n} p_i f(x_i)$ from every side of (3),we obtain that for every choice of $I \subseteq I_n = \{1,\ldots,n\}$ there is a statement:

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) \geqslant 0. \tag{5}$$

Taking the max of the right side in (5) for $I \subset I_n, |I| \geqslant 2$, the proof is making through. Similarly, taking $p_i = \frac{1}{n}$, we obtain the second inequality. □

The main result from [4] given in Theorem 1 is generalized and refined as a consequence of Theorem 2.1.

COROLLARY 2.2. *Conditions are taken from Theorem* 2.1. *Suppose* $0 \in C$. *For* $J \subset I_n$ *note* $P_J = \sum_{i\in J} p_i$. *If*

$$\sum_{i=1}^{n} p_i x_i = 0, \tag{6}$$

then

$$\sum_{i=1}^{n} p_i f(x_i) \geqslant \max_{0<|J|<n-1}\left[\sum_{i\in J} p_i f(x_i) + (1-P_J) f\left(\frac{\sum_{i\in J} p_i x_i}{P_J - 1}\right)\right] \geqslant f(0).$$

In particular, for $p_i = \frac{1}{n}$, *there are*

$$\sum_{i=1}^{n} f(x_i) \geqslant \max_{0<|J|<n-1}\left[\sum_{i\in J} f(x_i) + (n-|J|) f\left(\frac{\sum_{i\in J} x_i}{|J| - n}\right)\right] \geqslant n f(0),$$

Proof. We substitute condition (6) in (3), observing it for $J = I_n \backslash I$.

$$f(0) \leqslant (1 - P_{I_n\backslash I}) f\left(\frac{-\sum_{i\in I_n\backslash I} p_i x_i}{1 - P_{I_n\backslash I}}\right) + \sum_{i\in I_n\backslash I} p_i f(x_i) \leqslant \sum_{i=1}^{n} p_i f(x_i),$$

and take the max for $1 \leqslant |I_n \setminus I| \leqslant n-2$. □

3. Applications

Some convex functions admit consequences of the results that have been obtained above.

APPLICATION 1. *If $(X,\|\ \|)$ is a normed linear space, then the function $f:X\to\mathbb{R}$ defined by $f(x)=\|x\|^p$ is a convex function for $p\geqslant 1$.*

Taking $x_i\in X,\ p_i>0,\ i\in I_n=\{1,\dots,n\}, n>2$ with $\sum_{i=1}^n p_i=1$, we get the next chain of inequalities for $I\subset I_n$ such that $|I|\geqslant 2$.

$$\begin{aligned}\left\|\sum_{i=1}^n p_i x_i\right\|^p &\leqslant \min_I\left[P_I^{1-p}\left\|\sum_{i\in I}p_i x_i\right\|^p+\sum_{i\in I_n\setminus I}p_i\|x_i\|^p\right]\\ &\leqslant \frac{1}{2^n-n-2}\left[\sum_I P_I^{1-p}\left\|\sum_{i\in I}p_i x_i\right\|^p+(2^{n-1}-n)\sum_{i=1}^n p_i\|x_i\|^p\right]\\ &\leqslant \max_I\left[P_I^{1-p}\left\|\sum_{i\in I}p_i x_i\right\|^p+\sum_{i\in I_n\setminus I}p_i\|x_i\|^p\right]\\ &\leqslant \sum_{i=1}^n p_i\|x_i\|^p.\end{aligned}$$

In particular, the un-weighted refinement is given as

$$\begin{aligned}n^{1-p}\left\|\sum_{i=1}^n x_i\right\|^p &\leqslant \min_I\left[|I|^{1-p}\left\|\sum_{i\in I}x_i\right\|^p+\sum_{i\in I_n\setminus I}\|x_i\|^p\right]\\ &\leqslant \frac{1}{2^n-n-2}\cdot\left[\sum_I |I|^{1-p}\left\|\sum_{i\in I}x_i\right\|^p+(2^{n-1}-n)\sum_{i=1}^n\|x_i\|^p\right]\\ &\leqslant \max_I\left[|I|^{1-p}\left\|\sum_{i\in I}x_i\right\|^p+\sum_{i\in I_n\setminus I}\|x_i\|^p\right]\\ &\leqslant \sum_{i=1}^n\|x_i\|^p.\end{aligned}$$

The arithmetic mean-geometric mean inequality is well known in the literature. If $x_i,p_i>0,\ i\in I_n=\{1,\dots,n\}$ and $\sum_{i=1}^n p_i=1$, then

$$\sum_{j=1}^n p_j x_j\geqslant\prod_{j=1}^n x_j^{p_j}.$$

The equality holds in the case $x_1=x_2=\ldots=x_n$.

APPLICATION 2. *Let* $x_i, p_i > 0,\ i \in I_n = \{1,\ldots,n\}, n \geqslant 3,$ *with* $\sum_{i=1}^{n} p_i = 1$. *Applying the concave function* $f : (0,\infty) \to \mathbb{R}$, *defined as* $f(x) = \ln x$, *we are obtaining the next chain of inequalities. If* $I \subset I_n$ *and* $|I| \geqslant 2$, *then*

$$\begin{aligned}\sum_{i=1}^{n} p_i x_i &\geqslant \max_I \left[\left(\sum_{i\in I}\frac{p_i x_i}{P_I}\right)^{P_I} \cdot \prod_{i\in I_n\setminus I} x_i^{p_i}\right] \\ &\geqslant \left[\prod_I \left(\sum_{i\in I}\frac{p_i x_i}{P_I}\right)^{P_I} \cdot \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{2^{n-1}-n}\right]^{\frac{1}{2^n-n-2}} \\ &\geqslant \min_I \left[\left(\sum_{i\in I}\frac{p_i x_i}{P_I}\right)^{P_I} \cdot \prod_{i\in I_n\setminus I} x_i^{p_i}\right] \\ &\geqslant \prod_{i=1}^{n} x_i^{p_i}\end{aligned}$$

In particular, for the un-weighted case:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} x_i &\geqslant \max_I \left[\left(\frac{1}{|I|}\sum_{i\in I} x_i\right)^{|I|} \cdot \prod_{i\in I_n\setminus I} x_i\right]^{\frac{1}{n}} \\ &\geqslant \left[\prod_I \left(\frac{1}{|I|}\sum_{i\in I} x_i\right)^{|I|} \cdot \left(\prod_{i=1}^{n} x_i\right)^{2^{n-1}-n}\right]^{\frac{1}{n(2^n-n-2)}} \\ &\geqslant \min_I \left[\left(\frac{1}{|I|}\sum_{i\in I} x_i\right)^{|I|} \cdot \prod_{i\in I_n\setminus I} x_i\right]^{\frac{1}{n}} \\ &\geqslant \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\end{aligned}$$

REFERENCES

[1] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex functions, partial orderings, and statistical applications,* Mathematics in science and engineering, Georgia Institute of Tehnology, 1992.
[2] G.H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge, 1934.
[3] S.S. DRAGOMIR, *A refinement of Jensen's inequality with applications for f-divergence measures*, JIPAM, J. Inequal. Pure Appl. Math., **10** (2007), article 15. (electronic)
[4] N.S. BARNET AND S.S. DRAGOMIR, *On some inequalities for convex functions with applications in normed spaces*, MIA, Mathematical Inequalities and Applications, preprint, 2008.
[5] S.S. DRAGOMIR, J. PEČARIĆ AND L.E. PERSSON, *Properties of some functionals related to Jensen's inequality*, Acta Math. Hungarica, **70**, 1–2 (1996), 129–143.

(Received October 29, 2008)

V. Čuljak
Department of Mathematics
Faculty of Civil Engineering
University of Zagreb
Kaciceva 26
10 000 Zagreb
Croatia
e-mail: vera@master.grad.hr

B. Ivanković
Faculty of Transport and Traffic Engineering
University of Zagreb
Vukelićeva 4
10 000 Zagreb
Croatia
e-mail: ivankovb@fpz.hr