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ASYMPTOTIC ESTIMATES FOR APPROXIMATION NUMBERS OF THE HARDY OPERATOR IN Q–BANACH SPACES

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. The asymptotic estimates of the approximation numbers of the weighted Hardy operator are given.

1. Introduction

Let $\mathscr{B}(X,Y)$ denote the set of all linear bounded operators $T : X \to Y$ acting from a Banach space X into a q-Banach space Y, where $0 < q \leqslant 1$ and the triangle inequality has the form $\|x+y\|_Y^q \leqslant \|x\|_Y^q + \|y\|_Y^q$ for all $x, y \in Y$.

For any positive integer n, the *n-th approximation number* of $T \in \mathscr{B}(X,Y)$ is defined by

$$a_n(T) = \inf\{\|T-L\|_{X\to Y} : L \in \mathscr{B}(X,Y), \operatorname{rank} L \leqslant n-1\}, \quad \operatorname{rank} L = \dim \mathscr{R}(L).$$

The approximation numbers have the following properties: for $S, T \in \mathscr{B}(X,Y)$ and $R \in \mathscr{B}(Y,Z)$

(i) $\|T\| = a_1(T) \geqslant a_2(T) \geqslant \cdots \geqslant 0$;

(ii) $a_{n+m-1}(RS) \leqslant a_n(R)a_m(S), \quad n,m \in \mathbb{N}$,

(iii) If Y is q-Banach space ($0 < q \leqslant 1$), then $a^q_{n+m-1}(T+S) \leqslant a^q_n(T) + a^q_m(S)$, $n,m \in \mathbb{N}$.

In recent years, a significant amount of attention was paid to the study of the approximation numbers of the Hardy operator $T : L_p(0,\infty) \to L_q(0,\infty)$, $1 < p,q < \infty$ in Lebesgue spaces on the semiaxis. This direction of research was started in the papers [1-2] with asymptotic estimates of $a_n(T)$, as $n \to \infty$, for $T : L_p(0,\infty) \to L_p(0,\infty)$, $1 < p < \infty$, continued in [3] for $T : L_p(0,\infty) \to L_q(0,\infty)$, $1 < p,q < \infty$, and received a substantial development in the recent monograph [4]. The main results of [4] was extended for the one weight Riemann-Liouville operator in [5].

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The present paper supplements investigations of [1-5] for the operator $T : L_p(0,\infty) \to L_q(0,\infty)$

$$Tf(x) := v(x)\int_0^x f(y)\,dy \qquad (1)$$

in a new case $0 < q < 1 < p < \infty$, where the weight function $v(x)$ supposed to be non-negative and measurable on $(0,\infty)$.

2. Preliminary Lemmas

First we receive estimates of the approximation numbers on a finite interval.

Throughout the paper we suppose that $0 < q < 1 < p < \infty$ and define $\frac{1}{r} := 1 - \frac{1}{p} + \frac{1}{q}$. Note that $0 < r < 1$.

LEMMA 1. *Let $v \in L_q(I)$. Then the following estimate holds on an interval $I = (a,b) \subseteq (0,\infty)$*

$$\|Tf\|_{L_q(I)} \leqslant |I|^{1/p'}\|v\|_{L_q(I)}\,\|f\|_{L_p(I)} \qquad \textit{for all } f \in L_p(I).$$

Proof. Follows by Hölder's inequality with p and p'. □

Denote $J_v(I) := |I|^{1/p'}\|v\|_{L_q(I)}$. In Lemmas 2-3 we prove the properties of $J_v(I)$, when $I \subset (0,\infty)$ is a finite interval. *A disjoint partition* of an interval $I \subset (0,\infty)$ means a decomposition $I = \bigsqcup_{k=1}^N I_k$ with non-overlapping interiors of the intervals $\{I_k\} \subseteq I$.

LEMMA 2. (i) *For $v \in L_q(I)$ and a disjoint partition $I = \bigsqcup_{k=1}^N I_k$ the following inequality holds:*

$$\left(\sum_{k=1}^N J_v(I_k)^r\right)^{1/r} \leqslant J_v(I). \qquad (2)$$

(ii) *For $v_1,\ v_2 \in L_q(I)$ and a disjoint partition $I = \bigsqcup_{k=1}^N I_k$ we have*

$$\left|\sum_{k=1}^N J_{v_1}(I_k)^r - \sum_{k=1}^N J_{v_2}(I_k)^r\right| \leqslant J_{|v_1-v_2|}(I)^r. \qquad (3)$$

Proof. Applying Hölder's inequality with parameters $\frac{p'}{r}$ and $\frac{q}{r}$, we receive

$$\sum_{k=1}^N J_v(I_k)^r \leqslant \left(\sum_{k=1}^N |I_k|\right)^{r/p'}\left(\sum_{k=1}^N \int_{I_k} v(t)^q\,dt\right)^{r/q} = J_v\left(\bigcup_{k=1}^N I_k\right)^r = J_v^r(I).$$

For the proof of (3) we apply an elementary inequality $\left|\|x\|_r^r - \|y\|_r^r\right| \leqslant \|x-y\|_r^r$ in case of $0<r<1$. Then we have

$$\begin{aligned}\left|\sum_{k=1}^{N} J_{v_1}(I_k)^r-\sum_{k=1}^{N} J_{v_2}(I_k)^r\right| &\leqslant \sum_{k=1}^{N}|I_k|^{r/p'}\left|\|\,|v_1|^r\,\|_{L_{q/r}(I_k)}-\|\,|v_2|^r\,\|_{L_{q/r}(I_k)}\right|\\ &\leqslant \sum_{k=1}^{N}|I_k|^{r/p'}\|\,|\,|v_1|^r-|v_2|^r\,|\,\|_{L_{q/r}(I_k)}\\ &\leqslant \sum_{k=1}^{N}|I_k|^{r/p'}\|\,|v_1-v_2|^r\,|\,\|_{L_{q/r}(I_k)}\\ &=\sum_{k=1}^{N}|I_k|^{r/p'}\|v_1-v_2\|^r_{L_q(I_k)}=\sum_{k=1}^{N} J_{|v_1-v_2|}(I_k)^r\\ &\leqslant J_{|v_1-v_2|}(I)^r. \quad \square\end{aligned}$$

LEMMA 3. *Let $I\subset(0,\infty)$ be a finite interval, $v\in L^q(I)$. Then*

(i) $\|v\|_{L_r(I)}=\inf_{\tau}\left\{\left(\sum_{k=1}^{N} J_v(I_k)^r\right)^{1/r}\right\}$, *where the infimum is taken over all disjoint partitions $\tau=\{I_1,\dots,I_N\}$ of I.*

(ii) *for every $n\in\mathbb{N}$ there exists such a disjoint partition $\tau^*=\{I_1^*,I_2^*,\dots I_n^*\}$ of I that*

$$J_v(I_1^*)=J_v(I_2^*)=\ldots=J_v(I_n^*).$$

Proof. Let $\tau=\{I_1,I_2,\dots I_N\}$ be a disjoint partition of I. By Hölder's inequality with parameters $\frac{p'}{r}$ and $\frac{q}{r}$, we receive

$$\|v\|^r_{L_r(I)}=\sum_{k=1}^{N}\int_{I_k} v(t)^r dt\leqslant \sum_{k=1}^{N}|I_k|^{r/p'}\left(\int_{I_k} v(t)^q\,dt\right)^{r/q}=\sum_{k=1}^{N} J_v(I_k)^r$$

and the inequality $\|v\|_{L_r(I)}\leqslant\inf_{\tau}\left\{\left(\sum_{k=1}^{N} J_v(I_k)^r\right)^{1/r}\right\}$ follows. For the proof of the reverse inequality observe that the equality (i) holds for step functions $\eta(t)=\sum_{k=1}^{N}\alpha_k\chi_{I_k}(t)$, $\alpha_k\geqslant 0$:

$$\sum_{k=1}^{N} J_\eta(I_k)^r=\sum_{k=1}^{N}\alpha_k^r|I_k|^{r/p'}|I_k|^{r/q}=\sum_{k=1}^{N}\alpha_k^r|I_k|=\|\eta\|^r_{L_r(I)}.$$

Given $\varepsilon>0$ and $v\in L_q(I)$ let a function $\eta\geqslant 0$ be chosen such that the inequality $\|v-\eta\|_{L_q(I)}\leqslant\varepsilon|I|^{-1/p'}$ holds which by Hölder's inequality implies $\|v-\eta\|_{L_r(I)}\leqslant\varepsilon$. By virtue of inequality (ii) of Lemma 2 we have

$$\left|\sum_{k=1}^{N} J_v(I_k)^r-\sum_{k=1}^{N} J_\eta(I_k)^r\right|\leqslant J_{|v-\eta|}(I)^r\leqslant\varepsilon^r.$$

Thus,

$$\sum_{k=1}^{N} J_v(I_k)^r \leqslant \sum_{k=1}^{N} J_\eta(I_k)^r + \varepsilon^r = \|\eta\|_{L^r(I)}^r + \varepsilon^r \leqslant \|v\|_{L^r(I)}^r + 2\varepsilon^r$$

and (i) follows by taking the infimum and tending $\varepsilon \to 0$.

To show (ii) we fix $n \in \mathbb{N}$ and define the function Φ on a set of n-partitions of interval I by the formula

$$\Phi(\tau) := \max_{1\leqslant k\leqslant n} J_v(I_k) - \min_{1\leqslant k\leqslant n} J_v(I_k), \quad \tau = \{I_1, \dots, I_n\}.$$

The function $\Phi(\tau)$ continuously depends on the end-points of the intervals defining a partition τ of the interval I. Hence, there is a partition τ^*, $|I_k^*| > 0$, on which the function Φ reaches its minimal value. We show that $\Phi_{\min} = \Phi(\tau^*) = 0$.

Suppose, that there exists a partition $\tau' = \{I_1', \dots, I_n'\} \neq \tau^*$ on which the function $\Phi(\tau)$ reaches the minimal value, but $\Phi'_{\min} = \Phi(\tau') > 0$. Then such a number $k_0 \in \{1, \dots n\}$ can be found, that $J_v(I_{k_0}') = \max_{1\leqslant k\leqslant n} J_v(I_k')$, admitting $J_v(I_{k_0}') > J_v(I_{k_0+1}')$.

If we pertubate $|I_{k_0}'|$ by a small enough $\delta > 0$, for example $|I_{k_0}'| = |x_{k_0} - x_{k_0-1}| > |(x_{k_0} - \delta) - x_{k_0-1}|$, then for the new partition $\tau'' = \{I_1'', \dots, I_n''\} \neq \tau'$, there exist such a number $k_1 \in \{1, \dots n\}$ that $J_v(I_{k_1}'') = \max_{1\leqslant k\leqslant n} J_v(I_k'')$ and $J_v(I_{k_0}') > J_v(I_{k_1}'')$. Then the new partition τ'' is such that $\Phi(\tau') > \Phi(\tau'')$. Having repeated this procedure, we receive a new partition $\tau''' : \Phi(\tau') > \Phi(\tau'') > \Phi(\tau''')$. As a result, it is possible to build new partitions on which function Φ accept smaller value, hence, any partition τ' for which $\Phi(\tau') > 0$, cannot be minimal. Hence, $\Phi_{\min} = \Phi(\tau^*) = 0$ with $J_v(I_1^*) = J_v(I_2^*) = \ldots = J_v(I_n^*)$ and the proof of lemma 3 is completed. $\square$

Let $I = \bigsqcup_k I_k$ be a representation of the interval $I \subseteq (0, \infty)$ as the union of pairwise disjoint finite intervals I_k. It is supposed also, that $v \in L_q(I_k)$.

We define projectors $P_I^c : L_p(I) \to L_p^c(I)$ and $P_I^\circ : L_p(I) \to L_p^\circ(I)$ according to the formulae

$$P^c f(t) = \sum_k \frac{1}{|I_k|} \left(\int_{I_k} f \right) \chi_{I_k}(t) \quad \textit{and} \quad P_I^\circ = id - P_I^c,$$

where $L_p^c(I) = P^c(L_p(I)) = \{ f \in L_p(I) : f(t) = \sum_{k\in\mathscr{K}} \alpha_k \chi_{I_k}(t),\ \alpha_k \in \mathbb{R} \}$ and $L_p^\circ(I) := \left\{ f \in L_p(I) : \int_{I_k} f(y)dy = 0, \quad k \in \mathscr{K} \right\}$,

Thus, the operator T is decomposed into the sum $T = TP^\circ + TP^c$. Observe that $\|P^\circ\| \leqslant 2$, $\|P^c\| \leqslant 1$. If supp$f \subseteq I_k$ and $f \in L_p^\circ(I)$, then $Tf(x) = 0$, if $x \notin I_k = (a_k, b_k)$ since for $x \geqslant b_k$, we have $Tf(x) = v(x) \int_{I_k} f(y)dy = 0$. This implies, that in the subspace $L_p^\circ(I)$ the disjunct property

$$\|Tf\|_{L^q(I)}^q = \sum_{k\in\mathscr{K}} \|Tf\|_{L^q(I_k)}^q, \quad f \in L_p^\circ(I) \tag{4}$$

holds. The following lemma specifies the norm value of operator T when it is narrowed to the subspace $L_p^\circ(I)$ and plays a key role in obtaining of further estimations.

LEMMA 4. *Let $I \subseteq (0,\infty)$, $\tau = \{I_k\}_{k\in\mathscr{K}}$, $I = \bigsqcup_{k\in\mathscr{K}} I_k$ and $v \in L_q(I_k)$. Then for any function $f \in L_p^\circ(I;\tau)$*

$$\|Tf\|_{L_q(I)} \leqslant \left(\sum_{k\in\mathscr{K}} J_v(I_k)^{r/(1-r)}\right)^{1/r-1} \|f\|_{L_p(I)}. \tag{5}$$

Proof. Applying Hölder's inequality with parameters $\frac{p}{q}$, $\frac{r}{(1-r)q}$ and lemma 1, we have

$$\begin{aligned}\|Tf\|^q_{L_q(I)} = \sum_{k\in\mathscr{K}} \|T(\chi_{I_k} f)\|^q_{L_q(I_k)} &\leqslant \sum_{k\in\mathscr{K}} J_v(I_k)^q \|\chi_{I_k} f\|_p^q \\ &\leqslant \left(\sum_{k\in\mathscr{K}} J_v(I_k)^{r/(1-r)}\right)^{q(1-r)/r} \left(\sum_{k\in\mathscr{K}} \|\chi_{I_k} f\|_p^p\right)^{q/p} \\ &= \left(\sum_{k\in\mathscr{K}} J_v(I_k)^{r/(1-r)}\right)^{q(1-r)/r} \|f\|^q_{L_p(I)}. \qquad \square\end{aligned}$$

LEMMA 5. *Let $I \subseteq (0,\infty)$, $\tau = \{I_k\}_{k\in\mathscr{K}}$, $I = \bigsqcup_{k\in\mathscr{K}} I_k$, $v \in L_q(I_k)$ and $P^\circ : L_p(I) \to L_p^\circ(I,\tau)$. Let a sequence of natural numbers $\{n_k\}_{k\in\mathscr{K}}$ be such that $n = \sum\limits_{k\in\mathscr{K}} (n_k - 1) + 1 < \infty$. Then*

$$a_n(TP^\circ) \leqslant 2\left(\sum_{k\in\mathscr{K}} n_k^{-r/(1-r)} J_v(I_k)^{r/(1-r)}\right)^{1/r-1}. \tag{6}$$

Proof. Since $n = \sum_{k\in\mathscr{K}} (n_k - 1) + 1 < \infty$ only a finite number of n_k can be distinct from 1. For such $n_k \neq 1$, according to lemma 3 we divide corresponding intervals I_k into n_k intervals $I^*_{k,1}, \ldots, I^*_{k,n_k}$ such that $J_v(I^*_{k,1}) = \ldots = J_v(I^*_{k,n_k})$. If $n_k = 1$, we assume $I^*_{k,1} = I_k$. As a result, a new partition $\tau^* = \{I^*_{k,j} : 1 \leqslant j \leqslant n_k, \quad k \in \mathscr{K}\}$ of the interval is constructed. By virtue of the formula (2) we have

$$\left(\sum_{j=1}^{n_k} J_v^r(I^*_{k,j})\right)^{1/r} \leqslant J_v(I_k),$$

and $\left(n_k J_v^r(I^*_{k,j})\right)^{1/r} \leqslant J_v(I_k)$ which implies the inequality

$$J_v(I^*_{k,j}) \leqslant n_k^{-1/r} J_v(I_k). \tag{7}$$

Now we have $P^\circ_*(L_p(I)) = L_p^\circ(I,\tau^*)$ and $P^c_*(L_p(I)) = L_p^c(I,\tau^*)$. Hence,

$$P^\circ - P^c_* P^\circ = P^\circ_* P^\circ = P^\circ_*, \tag{8}$$

$$\operatorname{rank}(P_*^c P^\circ) \leqslant \sum_{k\in\mathscr{K}} (n_k - 1) = n - 1. \tag{9}$$

Using (9), lemma 4 and (7), we find

$$\begin{aligned}\|(TP^\circ - TP_*^c P^\circ)f\|_{L_q(I)} = \|T(P_*^\circ f)\|_{L_q(I)} &\leqslant \left(\sum_{k\in\mathscr{K}}\sum_{j\leqslant n_k} J_v(I_{k,j}^*)^{\frac{r}{1-r}}\right)^{\frac{1}{r}-1} \|P_*^\circ f\|_{L_p(I)} \\ &\leqslant 2\left(\sum_{k\in\mathscr{K}} n_k^{1-\frac{1}{1-r}} J_v(I_k)^{\frac{r}{1-r}}\right)^{1/r-1} \|f\|_{L_p(I)} \\ &= 2\left(\sum_{k\in\mathscr{K}} n_k^{\frac{-r}{1-r}} J_v(I_k)^{\frac{r}{1-r}}\right)^{1/r-1} \|f\|_{L_p(I)}\end{aligned}$$

for all functions $f \in L_p(I)$. It follows from formula (9) that

$$\operatorname{rank}(TP_*^c P^\circ) \leqslant \operatorname{rank}(P_*^c P^\circ) \leqslant n - 1.$$

Then

$$a_n(TP^\circ) = \inf_{\operatorname{rank}(TP_*^c P^\circ)\leqslant n-1} \|TP^\circ - TP_*^c P^\circ\|$$

and

$$a_n(TP^\circ) \leqslant 2\left(\sum_{k\in\mathscr{K}} n_k^{-r/(1-r)} J_v(I_k)^{r/(1-r)}\right)^{1/r-1}. \qquad \square$$

3. Main results

THEOREM 1. *Let* $I \subseteq (0,\infty)$, $\tau = \{I_k\}_{k\in\mathscr{K}}$, $I = \bigsqcup_{k\in\mathscr{K}} I_k$, $v \in L_q(I_k)$ *for any* $k \in \mathscr{K}$ *and* $P^\circ : L_p(I) \to L_p^\circ(I,\tau)$. *Then*

$$a_n(TP^\circ) \leqslant 2n^{-1}\left(\sum_{k\in\mathscr{K}} J_v(I_k)^r\right)^{1/r}. \tag{10}$$

Proof. Let $\sum_{k\in\mathscr{K}} J_v(I_k)^r < \infty$. For $k \in \mathscr{K}$ we choose such $n_k \in \mathbb{N}$, that the inequality

$$n_k - 1 < n \cdot \frac{J_v(I_k)^r}{\sum_{j\in\mathscr{K}} J_v(I_j)^r} \leqslant n_k$$

holds. Then

$$n_k^{-r/(1-r)} J_v(I_k)^{r/(1-r)} \leqslant n^{-r/(1-r)} \left(\sum_{j\in\mathscr{K}} J_v(I_j)^r\right)^{r/(1-r)} J_v(I_k)^r$$

and by lemma 5 the estimate (10) follows. $\square$

THEOREM 2. *Let $I \subset (0,\infty)$ be a finite interval and let $v \in L_q(I)$. Then*

$$\limsup_{n\to\infty} n a_n(T) \leqslant 2\|v\|_{L_r(I)}. \tag{11}$$

Proof. Given $\varepsilon > 0$ by lemma 3 there exist a partition $\tau = \{I_1, I_2, \dots I_N\}$ of interval I such that

$$\left(\sum_{k=1}^{N} J_v(I_k)^r\right)^{1/r} \leqslant (1+\varepsilon)\|v\|_{L_r(I)}. \tag{12}$$

Let $T = TP^\circ + TP^c$, where $P^\circ : L_p(I) \to L_p^\circ(I,\tau)$, $\quad P^c : L_p(I) \to L_p^c(I,\tau)$. Note, that rank$\,(TP^c) \leqslant N$ and $a_{N+1}(TP^c) = 0$. Using the property of the approximation numbers, (12) and theorem 1, we receive

$$\limsup_{n\to\infty} n a_n(T) \leqslant \limsup_{n\to\infty} n a_n(TP^\circ) \leqslant 2\left(\sum_{k=1}^{N} J_v(I_k)^r\right)^{1/r} \leqslant 2(1+\varepsilon)\|v\|_{L_r(I)}.$$

Then (11) follows, when $\varepsilon \to 0$. $\square$

To obtain the upper estimaes for the approximation numbers of the Hardy operator on the semiaxis we need the following extension of the result from [6] about the a-numbers of a diagonal operator.

Let $0 < q < 1 < p < \infty$, $\quad \frac{1}{r} = \frac{1}{q} + \frac{1}{p'}$, $\quad \frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.

We consider the diagonal operator $D : \ell_p \to \ell_q$, which is given by

$$D\{x_k\} = \{\sigma_k x_k\}, \qquad x = \{x_k\} \in \ell_p, \qquad \sigma_1 \geqslant \sigma_2 \geqslant \dots \geqslant 0.$$

Applying Hölder's inequality with parameters $\frac{p}{q}$, $\frac{p}{p-q}$ we obtain

$$\|D\|_{\ell_p\to\ell_q} = \sup_{x\neq 0} \frac{\|Dx\|_{\ell_q}}{\|x\|_{\ell_p}} = \sup_{x\neq 0} \frac{\left(\sum_{k=1}^{\infty} |\sigma_k x_k|^q\right)^{1/q}}{\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}}$$

and

$$\|D\|_{\ell_p\to\ell_q} \leqslant \left(\sum_{k=1}^{\infty} \sigma_k^s\right)^{1/s} < \infty. \tag{13}$$

We define the operator $P_n : \ell_p \to \ell_p$ by

$$P_n(x_1, x_2, \dots, x_n, x_{n+1}, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

and put $L = DP_{n-1}$. In this case $rank L < n$, and we find

$$a_n(D : \ell_p \to \ell_q) \leqslant \|D - L\|_{\ell_p\to\ell_q} \leqslant \left(\sum_{k\geqslant n} \sigma_k^s\right)^{1/s} < \infty. \tag{14}$$

THEOREM 3. *Let* $0<q<1<p<\infty$, $\frac{1}{r}=\frac{1}{q}+\frac{1}{p'}$, *and the diagonal operator* $D:\ell_p\to\ell_q$, $D\{x_k\}=\{\sigma_k x_k\}$, $x=\{x_k\}\in\ell_p$ *is defined by a sequence of real numbers* $\{\sigma_k\}$ *such that* $\sigma_1\geqslant\sigma_2\geqslant\ldots\geqslant 0$. *Then*

$$\sup_n n^q a_n^q(D)=\|\{a_n^q(D)\}\|_{\ell_{1/q,\infty}}\leqslant C\|\{\sigma_k\}\|^q_{\ell_{r,\infty}}. \tag{15}$$

Proof. In the special case where $\sigma_k=k^{-1/r}$ we shall denote the diagonal operator D_r, which is given by $D_r x_k=k^{-1/r}x_k$. Applying the inequality (14) we get

$$a_n^q(D_r)\leqslant\left(\sum_{k\geqslant n}\frac{1}{k^{s/r}}\right)^{q/s}\leqslant C\left(\frac{1}{n^{1/r-1/s}}\right)^q=Cn^{-q}.$$

Hence,

$$a_n^q(D)\leqslant\sup_k|\sigma_k k^{1/r}|^q a_n^q(D_r)\ll\|\{\sigma_k\}\|^q_{\ell_{r,\infty}}a_n^q(D_r),$$

and we conclude

$$\sup_n n^q a_n^q(D)=\|\{a_n^q(D)\}\|_{\ell_{1/q,\infty}}\leqslant C\|\{\sigma_k\}\|^q_{\ell_{r,\infty}}. \qquad\Box$$

Set $\Delta_k=[2^k,2^{k+1})$, $|\Delta_k|^{1/p'}=2^{k/p'}$, and introduce the following notations

$$\|v\|_r:=\left(\int_0^\infty|v(x)|^r dx\right)^{1/r},\quad |v|_r:=\left(\sum_{k\in\mathbb{Z}}J(\Delta_k)^r\right)^{1/r},$$

$$|v|_{r,\infty}:=\|\{\delta_k\}\|_{\ell_{r,\infty}}=\sup_{k\geqslant 0}(k+1)^{1/r}\delta_k^*+\sup_{k<0}|k|^{1/r}\delta_k^+, \tag{16}$$

where $\{\delta_k^*\}_{k\geqslant 0}$ and $\{\delta_k^+\}_{k<0}$ are decreasing and increasing rearrangements of $\delta_k:=J(\Delta_k)$. Using Hölder's inequality and properties of Lorentz spaces we have

$$\|v\|_r\leqslant|v|_r,\qquad |v|_{r,\infty}\leqslant c|v|_r.$$

Let subspaces $L_p^\circ((0,\infty),\Delta)$ and $L_p^c((0,\infty),\Delta)$ be constructed with respect to the partition $(0,\infty)=\bigsqcup_{k\in\mathbb{Z}}\Delta_k$ and $P_I^\circ:L_p(0,\infty)\to L_p^\circ(0,\infty)$, $P_I^c:L_p(0,\infty)\to L_p^c(0,\infty)$ be the projectors corresponding to this partition. Then we have representation $T=TP^\circ+TP^c$ and by property of the approximation numbers we have the following estimate

$$a_{2n-1}^q(T)\leqslant a_n^q(TP^\circ)+a_n^q(TP^c). \tag{17}$$

LEMMA 6. *If* $0<q<1<p<\infty$, *then the inequality*

$$\sup_n n^q a_n^q(TP^\circ)\leqslant 2^q|v|_r^q$$

holds.

Proof. The proof follows by application of theorem 1 with $I=(0,\infty)$ and with partition $\tau=\{\Delta_k\}_{k\in\mathbb{Z}}$. $\Box$

LEMMA 7. *Let* $0 < q < 1 < p < \infty$, *then*

$$\sup_n n^q a_n^q(TP^c) \leqslant c|v|_r^q.$$

Proof. Let the map $\varphi_p : l_p \to L_p^c(0,\infty)$ is defined by

$$\varphi_p : \{x_k\}_{k\in\mathbb{Z}} \to \sum_{k\in\mathbb{Z}} x_k|\Delta_k|^{-1/p}\chi_{\Delta_k}(t).$$

Then

$$TP^c\varphi_p : l_p \to L_q(0,\infty).$$

If $t \in \Delta_k$, we have representation

$$(TP^c\varphi_p\, x)(t) = v(t)x_k|\Delta_k|^{-1/p}(t-2^k) + v(t)\sum_{j<k} x_j|\Delta_j|^{-1/p'} = R_1x(t) + R_2x(t).$$

To estimate the first summand we define diagonal operator $D : \ell_p \to \ell_q$ by the formula $D\{x_k\} = \{\delta_k x_k\}$, where $\delta_k = J_v(\Delta_k)$, $k \in \mathbb{Z}$. Now we define the operator $Y : \ell_q \to L_q(0,\infty)$ by the formula

$$Y(x) = \sum_{k\in\mathbb{Z}} x_k g_k,$$

where $x = \{x_k\}_{k\in\mathbb{Z}} \in \ell_q$, and the functions $g_k(t) := \dfrac{v(t)(t-2^k)}{\|v\|_{L_q(\Delta_k)}|\Delta_k|}\chi_{\Delta_k}(t), \quad t > 0$. Note, that $\|g_k\|_{L_q} \leqslant 1$ and $\|Y\|_{\ell_q\to L_q} \leqslant 1$. For $x = \{x_k\}_{k\in\mathbb{Z}} \in \ell_q$ and $t > 0$ we obtain

$$(YDx)(t) = \sum_{k\in\mathbb{Z}} x_k|\Delta_k|^{1/p'}\|v\|_{L_q(\Delta_k)}g_k(t) = \sum_{k\in\mathbb{Z}} x_k|\Delta_k|^{-1/p}(t-2^k)v(t)\chi_{\Delta_k}(t) = R_1x(t).$$

Thus, $R_1 = YD$ and

$$a_n(R_1) \leqslant \|Y\|a_n(D) \leqslant a_n(D).$$

Now we define the operator $V : \ell_p \to \ell_p$ by the formula $Vx := \left\{\sum_{j<k} x_j \frac{|\Delta_j|^{1/p'}}{|\Delta_k|^{1/p'}}\right\}_{k\in\mathbb{Z}}$. By the Young inequality we obtain

$$\|Vx\|_p = \left\|\sum_{k-j>0} x_j 2^{-(k-j)/p'}\right\|_p = \left\|\sum_{l=1}^{\infty} x_{k-l}\, 2^{-l/p'}\right\|_p \leqslant \sum_{l=1}^{\infty} \frac{1}{2^{l/p'}}\|x\|_p \leqslant C\|x\|_p.$$

Also, we introduce the operator $W_q : \ell_q \to L_q(0,\infty)$ by

$$W_q(x)(t) := \frac{v(t)\chi_{\Delta_k}(t)}{\|v\chi_{\Delta_k}\|_q}.$$

As a result we have $R_2 = W_qDV$, hence,

$$a_n(R_2) \leqslant \|W_q\|a_n(D)\|V\| \leqslant Ca_n(D)$$

and

$$a_{2n-1}(TP^c) \leqslant Ca_n(D). \tag{18}$$

Since $D = D^+ + D^-$ and $a^q_{2n-1}(D) \leqslant a^q_n(D^+) + a^q_n(D^-)$, than applying theorem 3 to diagonal operators D^+ and D^-, formed by $\{\delta_k\}_{k\geqslant 0}$ and $\{\delta_k\}_{k<0}$, respectively, we deduce

$$\begin{aligned}(2n-1)^q a^q_{2n-1}(D) &\leqslant c_1 \sup_n n^q a^q_n(D^+) + c_2 \sup_n n^q a^q_n(D^-)\\ &\leqslant C_1 \|\{\delta_k\}_{k\geqslant 0}\|^q_{\ell_{r,\infty}} + C_2 \|\{\delta_k\}_{k<0}\|^q_{\ell_{r,\infty}} \leqslant C|v|^q_{r,\infty},\end{aligned}$$

from which

$$\sup_n n^q a^q_n(D) \leqslant C|v|^q_{r,\infty} \leqslant C|v|^q_r.$$

With similar arguments in view of inequality (18) we prove the estimation

$$\sup_n n^q a^q_n(T^c_v) \leqslant c|v|^q_r. \qquad \square$$

From the proof of lemmas 6-7 and inequality (18) we get

$$(2n-1)^q a^q_{2n-1}(T) \leqslant 4^q \left(\sup_n n^q a^q_n(TP^\circ) + \sup_n n^q a^q_n(TP^c) \right) \leqslant c^q\,|v|^q_r. \tag{19}$$

Our main result is the following.

THEOREM 4. *Let $0 < q < 1 < p < \infty$, and let $T : L_p(0,\infty) \to L_q(0,\infty)$. Then, for some constants c_1, c_2 that are either absolute or dependent only on p and q, the following estimates hold:*

$$\sup_n n a_n(T) \leqslant c_1\,|v|_r \tag{20}$$

and if $|v|_r < \infty$, *then*

$$\limsup_{n\to\infty} n\, a_n(T) \leqslant c_2 \left(\int_0^\infty |v(x)|^r dx \right)^{1/r}. \tag{21}$$

Proof. The proof of inequality (20) follows from lemmas 6-7 and inequality (19). Let's prove the second part of the theorem. If $|v|_r < \infty$, then for a given $\varepsilon > 0$ we choose a natural number K so that

$$\left[\sum_{|k|\geqslant K} 2^{kr/p'} \left(\int_{\Delta_k} v(s)^q ds \right)^{r/q} \right]^{1/r} \leqslant \varepsilon.$$

On finite interval $I = [2^{-K}, 2^K]$ we define the function $v_1 := v \cdot \chi_I$ and put $v_2 := v - v_1$, then the operator $T = T_{v_1} + T_{v_2}$. Since $|v_2|_r \leqslant \varepsilon$, then (17) implies

$$\sup_n n^q a^q_n(T_{v_2}) \leqslant c^q\,|v_2|^q_r \leqslant c^q\,\varepsilon^q.$$

On the other hand, $v_1 \in L_q(I)$, and it is possible to apply theorem 2 to operator T_{v_1}. Thus, according to the property (iii) of the approximation numbers we obtain

$$(2n-1)^q a_{2n-1}^q(T) \leqslant (2n-1)^q \left(a_n^q(T_{v_1}) + a_n^q(T_{v_2})\right)$$
$$\leqslant 2^q \left(n^q a_n^q(T_{v_1}) + \sup_n n^q a_n^q(T_{v_2})\right) \leqslant 2^q \left(n^q a_n^q(T_{v_1}) + c^q \varepsilon^q\right).$$

Applying elementary inequality $(a+b)^{\frac{1}{q}} \leqslant 2^{\frac{1}{q}-1}\left(a^{\frac{1}{q}} + b^{\frac{1}{q}}\right)$, $0 < q < 1$, we get

$$(2n-1)a_{2n-1}(T) \leqslant 2^{1/q}\left(na_n(T_{v_1}) + c\varepsilon\right)$$
$$\limsup_{n\to\infty} n\, a_n(T) \leqslant 2^{1+\frac{1}{q}} \|v_1\|_{L_r(I)} + c\varepsilon \leqslant 2^{1+\frac{1}{q}} \|v\|_{L_r} + c\varepsilon.$$

Limiting process with $\varepsilon \to 0$ completes the proof of theorem 4. □

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