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ON ψ– INTERPOLATION SPACES

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In this paper the sequence Banach space $\ell_\psi(\mathbb{Z})$ is defined for a class of convex functions ψ, and properties of the K- and J- interpolation spaces $(E_0,E_1)_{\theta,\psi,K}$ and $(E_0,E_1)_{\theta,\psi,J}$ for a Banach couple $\overline{E}=(E_0,E_1)$ and $\theta\in(0,1)$ are studied.

1. Introduction

Extended a result of Bonsall and Duncan [2], Saito, Kato and Takahashi [11] defined the set Ψ_n of convex and continuous functions on $\Delta_n=\{(s_1,\dots,s_{n-1})\in[0,1]^{n-1}:\sum_{i=1}^{n-1}s_i\leqslant 1\}$ which satisfy some appropriate conditions, and the norm $\|\cdot\|_\psi$ on $\mathbb{C}^n$ for every $\psi\in\Psi_n$. For $\psi\in\Psi_n$ Kato, Saito and Tamura [4] defined the ψ-direct sum of a finite family $X_1,\dots,X_n$ of Banach spaces and proved geometrical properties of this space. Mitani and Saito [8] and Zachariades [13] introduced the sequence spaces ℓ_ψ and $\ell_{\psi,\infty}$ for $\psi\in\Psi_\omega$, where Ψ_ω is a class of convex functions, and studied properties of these spaces. ℓ_p, $1\leqslant p\leqslant\infty$, the Lorentz sequence spaces $d(w,p)$ and the Orlicz sequence spaces ℓ_M are examples of ℓ_ψ spaces. In [13] the ψ-direct sum of a sequence of Banach spaces was defined and geometrical properties of this space were studied.

In this paper we define the Banach space $\ell_\psi(\mathbb{Z})$ and study properties of the K- and J- interpolation spaces $(E_0,E_1)_{X,K}$ and $(E_0,E_1)_{X,J}$ considered for a Banach couple $\overline{E}=(E_0,E_1)$, $\psi\in\Psi_\omega$ and X-weighted sequence space $\ell^\theta_\psi,\theta\in(0,1)$.

2. Preliminaries

Let $\Delta_n=\{(s_1,\dots,s_{n-1})\in[0,1]^{n-1}:\sum_{i=1}^{n-1}s_i\leqslant 1\}$. Saito, Kato and Takahashi [11] denoted by Ψ_n the set of all continuous and convex functions $\psi:\Delta_n\to R$ which satisfy the following conditions:

$$(A_0):\ \psi(0,0,\dots,0)=\psi(1,0,\dots,0)=\psi(0,1,\dots,0)=\dots=\psi(0,0,\dots,1)=1$$

$$(A_1):\ \psi(s_1,\dots,s_{n-1})\geqslant(s_1+\dots+s_{n-1})\psi\left(\frac{s_1}{(s_1+\dots+s_{n-1})},\dots,\frac{s_{n-1}}{(s_1+\dots+s_{n-1})}\right)$$

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$$(A_2): \ \psi(s_1,\ldots,s_{n-1}) \geqslant (1-s_1)\psi\left(0, \frac{s_2}{1-s_1},\ldots,\frac{s_{n-1}}{1-s_1}\right)$$

$$\vdots$$

$$(A_n): \ \psi(s_1,\ldots,s_{n-1}) \geqslant (1-s_{n-1})\psi\left(\frac{s_1}{1-s_{n-1}},\ldots,\frac{s_{n-2}}{1-s_{n-1}},0\right).$$

Let $\Delta^{<\omega} = \{s = (s_n)_{n\in N} \in [0,1]^N : \{n \in N : s_n \neq 0\}$ is finite and $\sum\limits_{n=1}^{\infty} s_n \leqslant 1\}$ and let $\psi : \Delta^{<\omega} \mapsto \mathbb{R}$. In [13] the sequence associated to ψ is the sequence $(\psi_n)_{n=2}^{\infty}$, where $\psi_n : \Delta_n \to \mathbb{R}$ is defined by $\psi_n(s_1,\ldots,s_{n-1}) = \psi(s_1,\ldots,s_{n-1},0,0,\ldots)$ for $n = 2,3,\ldots$, and the set $\Psi_\omega = \{\psi : \Delta^{<\omega} \mapsto R : \psi_n \in \Psi_n$ for every $n = 2,3,\ldots\}$ was defined.

Let be a sequence $(\psi_n)_{n=2}^{\infty}$ such that $\psi_n \in \Psi_n$ and $\psi_n(s_1,\ldots,s_{n-1}) = \psi_{n+1}(s_1,\ldots, s_{n-1},0)$ for every $(s_1,\ldots,s_{n-1}) \in \Delta_n$ and $n = 2,3,\ldots$. For $s = (s_n)_{n\in\mathbb{N}} \in \Delta^{<\omega}$, with $s \neq (0,...,0,...)$, we put $n_s = \max\{n : s_n \neq 0\}$. The function $\psi : \Delta^{<\omega} \mapsto \mathbb{R}$ defined by $\psi(s) = \psi_{n_s+1}(s_1,\ldots,s_{n_s})$ if $s \neq 0$ and $\psi(s) = 1$ if $s = 0$, belongs to Ψ_ω and $(\psi_n)_{n=2}^{\infty}$ is its associated sequence.

Examples of functions in Ψ_ω are

$$\psi_p(s) = \begin{cases} \left[\left(1-\sum\limits_{i=1}^{\infty} s_i\right)^p + \sum\limits_{i=1}^{\infty} s_i^p\right]^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty \\ \sup\left\{1-\sum\limits_{i=1}^{\infty} s_i, s_1, s_2, \ldots, s_n, \ldots\right\} & \text{if } p = \infty \end{cases}$$

for $s = (s_n)_{n\in\mathbb{N}} \in \Delta^{<\omega}$.

Let $c_{00} = \{z = (z_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \{n \in \mathbb{N} : z_n \neq 0\}$ is finite $\}$ and $(e_n)_{n\in\mathbb{N}}$ is the usual basis of c_{00}. Denote $|z| = (|z_n|)_{n\in\mathbb{N}}$. A norm on c_{00} is called absolute if $\|z\| = \||z|\|$ for every $z \in c_{00}$ and normalized if $\|e_n\| = 1$ for every $n = 1,2,\ldots$. For every $\psi \in \Psi_\omega$ the norm $\|.\|_\psi$ is defined for every $z = (z_n)_{n\in\mathbb{N}} \in c_{00}$ as

$$\|z\|_\psi = \begin{cases} \left(\sum\limits_{i=1}^{\infty} |z_i|\right) \psi\left(\frac{|z_2|}{\sum_{i=1}^{\infty}|z_i|},\ldots,\frac{|z_n|}{\sum_{i=1}^{\infty}|z_i|},\ldots\right) & \text{if} z \neq 0, \\ 0 & \text{if} z = 0. \end{cases}$$

Note that if $\psi \in \Psi_\omega$, then $\|.\|_\psi \in AN_\omega$ - the set of all absolute and normalized norms on c_{00}. For any $\|.\| \in AN_\omega$, the function $\psi : \Delta^{<\omega} \to \mathbb{R}$ defined by $\psi(s) = \|(1-\sum_{n=1}^{\infty} s_n, s_1, s_2, \ldots)\|$ belongs to Ψ_ω and $\|(z_1,\ldots,z_n,\ldots)\| = \|(z_1,\ldots,z_n,\ldots)\|_\psi$ for every $(z_1,\ldots,z_n,\ldots) \in c_{00}$. Let us mention the following properties of the functions $\psi \in \Psi_\omega$ and the ψ-norms:

1) $\psi_\infty(s) \leqslant \psi(s) \leqslant 1$ for all $s = (s_n)_{n\in\mathbb{N}} \in \Delta^{<\omega}$.

2) $\|.\|_\infty \leqslant \|.\|_\psi \leqslant \|.\|_1$.

3) If $z = (z_i)_{i=1}^{\infty}$ and $w = (w_i)_{i=1}^{\infty}$ with $|z_i| \leqslant |w_i|$ for every $i = 1,\ldots,n,\ldots$, then $\|z\|_\psi \leqslant \|w\|_\psi$.

(The corresponding properties of the functions from Ψ_n are proved in [11].)

In [8] and [13] the spaces ℓ_ψ and $\ell_{\psi,\infty}$ were defined independently. In the following we present the construction of these spaces as it is presented in [13].

For $n = 1,2,\dots$ let the projection $P_n : \mathbb{C}^{\mathbb{N}} \mapsto \mathbb{C}^n$ defined by $P_n(z_1,\dots,z_n,\dots) = (z_1,\dots,z_n)$. If $\psi \in \Psi_\omega$ and $(\psi_n)_{n=2}^\infty$ is its associated sequence, then for every $z \in c_{00}$ the sequence $(\|P_n(z)\|_{\psi_n})_{n=2}^\infty$ is non-decreasing.

For $\psi \in \Psi_\omega$ the space ℓ_ψ is defined to be the completion of the space $(c_{00}, \|\cdot\|_\psi)$ and the space $\ell_{\psi,\infty}$ is defined as the space $\ell_{\psi,\infty} = \{z \in \mathbb{C}^{\mathbb{N}} : \sup_n \|P_n(z)\|_{\psi_n} < +\infty\}$, equipped with the norm $\|z\|_{\psi,\infty} = \sup_n \|P_n(z)\|_{\psi_n}$. For $\psi \in \Psi_\omega$ the spaces ℓ_ψ and $\ell_{\psi,\infty}$ are Banach spaces and ℓ_ψ is a closed subspace of $\ell_{\psi,\infty}$. The following proposition was proved in [13]

PROPOSITION 2.1. *Let* $\psi \in \Psi_\omega$.
i) The sequence $(e_n)_{n\in\mathbb{N}}$ *is a monotone and unconditional basis of* ℓ_ψ.
ii) $(e_n)_{n\in\mathbb{N}}$ *is a boundedly complete basis of* ℓ_ψ *if and only if* $\ell_\psi = \ell_{\psi,\infty}$.
iii) If $(e_n)_{n\in\mathbb{N}}$ *is a shrinking basis of* ℓ_ψ, *then* $\ell_{\psi,\infty}$ *is isometric to the second dual of* ℓ_ψ.
iv) $(e_n)_{n\in\mathbb{N}}$ *is a shrinking and boundedly complete basis o* ℓ_ψ *if and only if* ℓ_ψ *is reflexive.*

Typical examples of ℓ_ψ and $\ell_{\psi,\infty}$ spaces are the spaces $\ell_p = \ell_{\psi_p} = \ell_{\psi_p,\infty}$, $1 \leqslant p < \infty$, the space $c_0 = \ell_{\psi_\infty}$ and the space $\ell_\infty = \ell_{\psi_\infty,\infty}$. Examples of ℓ_ψ spaces are also the Lorentz sequence spaces $d(w,p)$ and the Orlicz sequence spaces ℓ_M [13].

Mitani, Oshiro and Saito [7] defined the dual function ψ^* of $\psi \in \Psi_n$ as

$$\psi^*(s_1,\dots,s_{n-1}) = \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{(1-t_1-\dots-t_{n-1})(1-s_1-\dots-s_{n-1})+t_1s_1+\dots+t_{n-1}s_{n-1}}{\psi(t_1,\dots,t_{n-1})}$$

for $(s_1,\dots,s_{n-1}) \in \Delta_n$. They also proved that $\psi^* \in \Psi_n$, $(\psi^*)^* = \psi$ and $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$. For ψ and ψ^* the following generalized Hölder inequality is proved.

PROPOSITION 2.2. *Let* $\psi \in \Psi_n$. *Then we have* $|<x,y>| \leqslant \|x\|_\psi \|y\|_{\psi^*}$ *for any* $x,y \in \mathbb{C}^n$.

In [13] the dual function of a $\psi \in \Psi_\omega$ was defined and the following properties were proved.

DEFINITION 2.3. Let $\psi \in \Psi_\omega$. If $(\psi_n)_{n\in\mathbb{N}}$ is the associated sequence of $\psi \in \Psi_\omega$ and ψ_n^* is the dual function of ψ_n for $n = 2,3,....$, the dual function $\psi*$ of ψ is defined to be the function with associated sequence $(\psi_n^*)_{n=2}^\infty$.

LEMMA 2.4. *If* $\psi \in \Psi_\omega, z = (z_n)_{n\in\mathbb{N}} \in \ell_\psi$ *and* $w = (w_n)_{n\in\mathbb{N}} \in \ell_{\psi^*}$, *then* $\sum_{n=1}^\infty |w_n z_n| \leqslant \|w\|_{\psi^*} \|z\|_\psi$.

THEOREM 2.5. *Let* $\psi \in \Psi_\omega$. *If* (e_n) *is a shrinking basis of* ℓ_ψ, *then the dual space of* ℓ_ψ *is isometric to* ℓ_{ψ^*}.

For $\psi \in \Psi_\omega$ the ψ-direct sum of a sequence of Banach spaces (X_n) is the space

$$\left(\sum_{n=1}^{\infty} \bigoplus X_n\right)_{\psi} = \{x = (x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} X_n : (\|x_n\|)_{n\in\mathbb{N}} \in \ell_\psi\}$$

equipped with norm $\|x\| = \|(\|x_n\|)_{n\in\mathbb{N}}\|_\psi$ [13].

For every $\psi \in \Psi_\omega$ the space $(\sum_{n=1}^{\infty} \bigoplus X_n)_\psi$ is a Banach space.

Examples of ψ-direct sums are the ℓ_p-direct sums for $\psi = \psi_p, 1 \leqslant p < \infty$, Lorentz and Orlicz direct sums.

Let $\overline{E} = (E_0, E_1)$ be a Banach couple, i.e. E_0 and E_1 be two Banach spaces algebraically and topologically imbedded in a separated topological linear space U. The space $\Delta\overline{E}$ consists of the elements common to E_0 and E_1 and its norm is defined by $\|a\|_{\Delta\overline{E}} = \max(\|a\|_{E_0}, \|a\|_{E_1})$. The space $\Sigma\overline{E}$ consists of the elements of the form $a = a_0 + a_1$ with $a_0 \in E_0$ and $a_1 \in E_1$, and its norm is defined by $\|a\|_{\Sigma\overline{E}} = inf\{\|a_0\|_{E_0} + \|a_1\|_{E_1} : a = a_0 + a_1,\ a_0 \in E_0,\ a_1 \in E_1\}$. We mention the definition of Peetre K- and J-functionals.

The K-functional is defined by $K(t,a) = K(t,a,E_0,E_1) = \inf\{\|a_0\|_{E_0} + t\|a_1\|_{E_1} : a = a_0 + a_1,\ a_0 \in E_0,\ a_1 \in E_1\}$ for every $a \in \Sigma\overline{E}$. The J functional is defined by $J(t,a) = J(t,a,E_0,E_1) = \max(\|a\|_{E_0}, t\|a\|_{E_1})$ for every $a \in \Delta\overline{E}$.

Let X be a Banach sequence space on $\mathbb{Z}$. Under some conditions, the K-method space and the J-method space were considered (see [5], [10], [3]). The K-method space $K_X(\overline{E}) = (E_0,E_1)_{X,K}$ is the Banach space of all $a \in \Sigma\overline{E}$ for which $(K(2^n,a,E_0,E_1))_{n\in\mathbb{Z}} \in X$ with the associated norm $\|a\| = \|(K(2^n,a,E_0,E_1))_{n\in\mathbb{Z}}\|_X$. This space is an exact interpolation space. That is, for every two Banach couples $\overline{A} = (A_0,A_1)$ and $\overline{B} = (B_0,B_1)$ if $T : \Sigma\overline{A} \to \Sigma\overline{B}$, with $T|_{A_i} : A_i \to B_i$ for $i = 0,1$, then $\|Ta\|_{K_X(\overline{B})} \leqslant M\|a\|_{K_X(\overline{A})}$ for every $a \in K_X(\overline{A})$, where $M = \max(\|T|_{A_0}\|, \|T|_{A_1}\|)$.

The J-method space $J_X(\overline{E})$ is the Banach space of all $a \in \Sigma\overline{E}$ for which there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\Delta\overline{E}$ such that $a = \sum_{n=-\infty}^{n=\infty} u_n$ (convergence in $\Sigma\overline{E}$) and $(J(2^n,u_n,E_0,E_1))_{n\in\mathbb{Z}} \in X$, equipped with norm $\|a\| = \inf\{\|(J(2^n,u_n,E_0,E_1))_{n\in\mathbb{Z}}\| : a = \sum_{n=-\infty}^{n=\infty} u_n,\ u_n \in \Delta\overline{E},\ (J(2^n,u_n,E_0,E_1))_{n\in\mathbb{Z}} \in X\}$. This space is also an exact interpolation space.

3. The spaces $(E_0,E_1)_{\theta,\psi,K}$ and $(E_0,E_1)_{\theta,\psi,J}$

Let $\tau : \mathbb{N} \to \mathbb{Z}$ defined by

$$\tau(l) = \begin{cases} \dfrac{l-1}{2} & \text{if } l = 1,3,\ldots \\ -\dfrac{l}{2} & \text{if } l = 2,4,\ldots \end{cases}$$

and $T : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{N}}$ defined by $T((z_n))_{n\in\mathbb{Z}} = (z_{\tau(l)})_{l\in\mathbb{N}}$.

For $\psi \in \Psi_\omega$ we define $\ell_\psi(\mathbb{Z}) = \{z = (z_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : T(z) \in \ell_\psi\}$ and $\|z\|_{\ell_\psi(\mathbb{Z})} = \|T(z)\|_\psi$. Where no confusion is possible we write $\|\cdot\|_\psi$ instead $\|\cdot\|_{\ell_\psi(\mathbb{Z})}$.

If $D = \{z = (z_n)_{n\in\mathbb{Z}} \in \ell_\psi(\mathbb{Z}) : \{n \in \mathbb{Z} : z_n \neq 0\}$ is finite $\}$, from the definition of the spaces $\ell_\psi(\mathbb{Z})$ and ℓ_ψ we obtain that D is a dense subset of $\ell_\psi(\mathbb{Z})$ and $\|z\|_\psi = \left(\sum_{n=-\infty}^{+\infty}|z_n|\right)\psi\left(\frac{|z_{-1}|}{\sum_{n=-\infty}^{+\infty}|z_n|}, \frac{|z_1|}{\sum_{n=-\infty}^{+\infty}|z_n|}, \frac{|z_{-2}|}{\sum_{n=-\infty}^{+\infty}|z_n|}, \frac{|z_2|}{\sum_{n=-\infty}^{+\infty}|z_n|}, \ldots\right)$ for every $z = (z_n)_{n\in\mathbb{Z}} \in D$.

In AN_ω many norms are invariant under permutations, that is $\|(x_n)_{n\in\mathbb{N}}\| = \left\|(x_{\pi(n)})_{n\in\mathbb{N}}\right\|$ for every permutation $\pi : \mathbb{N} \to \mathbb{N}$, but there are norms which do not have this property. For example, let $\psi : \Delta_\omega \to \mathbb{R}$ is defined by $\psi(s) = \max\{1-s_1, s_1, s_2, \ldots\}$ for every $s = (s_n)_{n\in\mathbb{N}} \in \Delta_\omega$. It is easy to see that $\psi \in \Psi_\omega$ and $\|(z_n)_{n\in\mathbb{N}}\|_\psi = \max\{|z_1| + \sum_{n=3}^{\infty}|z_n|, |z_2|\}$. Thus, $\|(1,2,3,0,0,\ldots)\|_\psi = 4$ and $\|(2,0,1,3,0,0,\ldots)\|_\psi = 6$. So, $\|\cdot\|_\psi$ is not invariant under permutations.

DEFINITION 3.1. Let $t = (t_n)_{n\in\mathbb{N}}$, $s = (s_n)_{n\in\mathbb{N}}$ in $\Delta^{<\omega}$.

i) t, s are called equivalent if there exists a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $s_n = t_{\pi(n)}$ for every $n \in \mathbb{N}$.

ii) t, s are called complementary if there exists a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and $n_1 \in \mathbb{N}$ such that $s_n = t_{\pi(n)}$ for every $n \neq n_1$ and $s_{n_1} + \sum_{n=1}^{\infty} t_n = t_{\pi(n_1)} + \sum_{n=1}^{\infty} s_n = 1$.

DEFINITION 3.2. A function $\psi \in \Psi_\omega$ is called invariant if $\psi(s) = \psi(t)$ for every equivalent or complementary s, $t \in \Delta^{<\omega}$.

LEMMA 3.3. *Let $\psi \in \Psi_\omega$ be invariant and $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation. Then $\|(z_n)_{n\in\mathbb{N}}\|_\psi = \left\|(z_{\pi(n)})_{n\in\mathbb{N}}\right\|_\psi$ for every $(z_n)_{n\in\mathbb{N}} \in c_{00}$.*

Proof. Let $z = (z_n)_{n\in\mathbb{N}} \in c_{00}$ and $w = (z_{\pi(n)})_{n\in\mathbb{N}}$. It is clear that $\sum_{n\in\mathbb{N}}|z_n| = \sum_{n\in\mathbb{N}}|z_{\pi(n)}|$. We set $\sigma_k = \frac{|z_{k+1}|}{\sum_{n\in\mathbb{N}}|z_n|}$, $\tau_k = \frac{|z_{\pi(k+1)}|}{\sum_{n\in\mathbb{N}}|z_n|}$ for $k = 1,2,\ldots.$, $\sigma = (\sigma_k)_{k\in\mathbb{N}}$ and $\tau = (\tau_k)_{k\in\mathbb{N}}$. If $\pi(1) = 1$, we set $\phi : \mathbb{N} \to \mathbb{N}$ defined by $\phi(n) = \pi(n+1) - 1$. Then it is easy to see that ϕ is a permutation and $\sigma_{\phi(n)} = \tau_n$ for every $n \in \mathbb{N}$. So, σ and τ are equivalent. If $\pi(1) \neq 1$, we put $n_1 = \pi(1)$ and $n_2 = \pi^{-1}(1)$. Then the function $\phi : \mathbb{N} \to \mathbb{N}$ defined by $\phi(n) = \pi(n+1) - 1$ if $n \neq n_2 - 1$ and $\phi(n_2 - 1) = n_1 - 1$, is a permutation with property $\tau_n = \sigma_{\phi(n)}$ for $n \neq n_2 - 1$. It is clear that $\sigma_{\phi(n_2-1)} = \sigma_{n_1-1} = \sigma_{\pi(1)-1} = \frac{|z_{\pi(1)}|}{\sum_{n\in\mathbb{N}}|z_n|}$ and $\tau_{n_2-1} = \frac{|z_{\pi(n_2)}|}{\sum_{n\in\mathbb{N}}|z_n|} = \frac{|z_1|}{\sum_{n\in\mathbb{N}}|z_n|}$. So, $\tau_{n_2-1} + \sum_{n\in\mathbb{N}}\sigma_n = \sigma_{\phi(n_2-1)} + \sum_{n\in\mathbb{N}}\tau_n = 1$. Hence, σ and τ are complementary. Therefore, $\|z\|_\psi = \|w\|_\psi$. □

PROPOSITION 3.4. *Let $\psi \in \Psi_\omega$ be invariant and $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation. Then $\|(z_n)_{n\in\mathbb{N}}\|_\psi = \|(z_{\pi(n)})_{n\in\mathbb{N}}\|_\psi$ for every $(z_n)_{n\in\mathbb{N}} \in \ell_\psi$.*

Proof. Let $(z_n)_{n\in\mathbb{N}} \in \ell_\psi$ and $w = (z_{\pi(n)})_{n\in\mathbb{N}}$. We put $\tilde{z}_k = (z_1,\ldots,z_k,0,\ldots)$ and $\tilde{w}_k = (z_{\pi(1)},\ldots,z_{\pi(k)},0,\ldots)$ for $k = 1,2,\ldots$ Let $k \in \mathbb{N}$. For every $i = 1,\ldots,k$ there exists

j_i such that $z_i = z_{\pi(j_i)}$. Let $m = max\{j_i : i = 1,...,k\}$. We put $\tilde{v}_k = (v_n)_{n\in\mathbb{N}}$, where $v_{j_i} = z_{\pi(j_i)} = z_i$ for $i = 1,...,k$ and $v_n = 0$ for $n \neq j_i, i = 1,...,k$. Then, from lemma 3.3 and from properties of ψ-norm, we obtain $\|\tilde{z}_k\| = \|\tilde{v}_k\| \leqslant \|\tilde{w}_m\|$. Therefore $\|z\|_\psi \leqslant \|w\|_\psi$. Similarly, we obtain $\|w\|_\psi \leqslant \|z\|_\psi$. So $\|z\|_\psi = \|w\|_\psi$. □

It is easy to prove the following lemma.

LEMMA 3.5. *Let $\psi \in \Psi_\omega$ be invariant and $\pi : \mathbb{Z} \to \mathbb{Z}$ be a permutation. If $z = (z_n)_{n\in\mathbb{Z}} \in \ell_\psi(\mathbb{Z})$ and $w = (z_{\pi(n)})_{n\in\mathbb{Z}}$, then*
i) $w \in \ell_\psi(\mathbb{Z})$, and
ii) $T(w) = (z_{\pi(\tau(l))})_{l\in\mathbb{N}}$.

From proposition 3.4 and lemma 3.5 we obtain the following corollary.

COROLLARY 3.6. *Let $\psi \in \Psi_\omega$ be invariant and $\pi : \mathbb{Z} \to \mathbb{Z}$ be a permutation. Then $\|(z_n)_{n\in\mathbb{Z}}\|_\psi = \|(z_{\pi(n)})_{n\in\mathbb{Z}}\|_\psi$ for every $(z_n)_{n\in\mathbb{Z}} \in \ell_\psi(\mathbb{Z})$.*

Let $(X_n)_{n\in\mathbb{Z}}$ be a family of Banach spaces. We define

$$\left(\sum_{n=-\infty}^{+\infty} \bigoplus X_n\right)_\psi = \left\{(x_n)_{n\in\mathbb{Z}} : (\|x_n\|)_{n\in\mathbb{Z}} \in \ell_\psi(\mathbb{Z})\right\}$$

and $\|x\| = \|T((\|x_n\|)_{n\in\mathbb{Z}})\|_\psi$ for every $x = (x_n)_{n\in\mathbb{Z}} \in \left(\sum_{n=-\infty}^{+\infty} \bigoplus X_n\right)_\psi$.

Let $\overline{E} = (E_0, E_1)$ be a Banach couple, $\theta \in (0,1)$ and $\psi \in \Psi_\omega$. We are going to define the K- and J- interpolation spaces when the Banach sequence space is the space $\ell_\psi^\theta = \{(z_n)_{n\in\mathbb{Z}} \in \mathbb{C}^\mathbb{Z} : (2^{-\theta n}|z_n|)_{n\in\mathbb{Z}} \in \ell_\psi(\mathbb{Z})\}$ with $\|(z_n)_{n\in\mathbb{Z}}\| = \left\|(2^{-\theta n}|z_n|)_{n\in\mathbb{Z}}\right\|_\psi$. It is not difficult to see that ℓ_ψ^θ is K-nontrivial, i.e. $(\min(1,2^n))_{n\in\mathbb{Z}} \in \ell_\psi^\theta$ and J-nontrivial, i.e. $\sup\left\{\sum_{n=-\infty}^{+\infty} \min(1,2^{-n})|a_n| : (a_n)_{n\in\mathbb{Z}} \in B_{\ell_\psi^\theta}\right\} < \infty$(in terminology of [9]).

DEFINITION 3.7. Let $\overline{E} = (E_0, E_1)$ be a Banach couple, $\theta \in (0,1)$ and $\psi \in \Psi_\omega$.

i) The K-interpolation space $(E_0,E_1)_{\theta,\psi,K}$ is the space of all $a \in \Sigma\overline{E}$ such that $\|a\|_{\theta,\psi,K} < \infty$, where

$$\|a\|_{\theta,\psi,K} = \left\| \left(2^{-\theta n} K(2^n, a, E_0, E_1)\right)_{n\in\mathbb{Z}} \right\|_\psi .$$

ii) The J-interpolation space $(E_0,E_1)_{\theta,\psi,J}$ is the space of all $a \in \Sigma\overline{E}$, for which there exists a sequence $(u_n)_{n\in\mathbb{Z}}$ in $\Delta\overline{E}$ such that $a = \sum_{n=-\infty}^{+\infty} u_n$ (convergence in $\Sigma\overline{E}$) and $(J(2^n,u_n,E_0,E_1))_{n\in\mathbb{Z}} \in \ell_\psi^\theta$ equipped with the norm

$$\|a\| = \inf\left\{ \left\|\left(2^{-\theta n} J(2^n,u_n,E_0,E_1)\right)_{n\in\mathbb{Z}}\right\|_\psi : a = \sum_{n=-\infty}^{+\infty} u_n,\ u_n \in \Delta\overline{E}\ \ \forall n \in \mathbb{Z},\right.$$
$$\left. (J(2^n,u_n,E_0,E_1))_{n\in\mathbb{Z}} \in \ell_\psi^\theta \right\}$$

The classical spaces $(E_0,E_1)_{\theta,p}$, $1 \leqslant p < \infty$, are spaces of this type. It is possible to define K- and J- interpolation spaces using the sequence space $\ell_{\psi,\infty}$. This will cover also the classical case $(E_0,E_1)_{\theta,\infty}$.

It is clear that $\Delta\overline{E} \subset (E_0,E_1)_{\theta,\psi,K}$, $(E_0,E_1)_{\theta,\psi,J} \subset \Sigma\overline{E}$.

A consequence of the fundamental lemma of interpolation is the imbedding $(E_0,E_1)_{\theta,\psi,K} \subset (E_0,E_1)_{\theta,\psi,J}$. If the function ψ is invariant, then the norm of the space $\ell_\psi(\mathbb{Z})$ is translation invariant and it is easy to see, that Calderon operator $\Omega : \ell_\psi(\mathbb{Z}) \to \ell_\psi(\mathbb{Z})$, defined by $\Omega((a_n)_{n\in\mathbb{Z}}) = \left(\sum_{m=-\infty}^{\infty} \min(1,2^{n-m})|a_m|\right)_{n\in\mathbb{Z}}$, is bounded. In this case (see [9], [3]) we have that the K- and J-method spaces coincide (with equivalence of norms) It is easy to see that the spaces $(E_0,E_1)_{\theta,\psi,K}$ and $(E_0,E_1)_{\theta,\psi,J}$ are exact interpolation spaces.

In the following theorem we prove that these spaces are of interpolation type θ when ψ is invariant.

THEOREM 3.8. *If (E_0,E_1) is a couple of Banach spaces, $0<\theta<1$ and ψ is an invariant function in Ψ_ω, then the space $(E_0,E_1)_{\theta,\psi}$ is of interpolation type θ. Moreover, for every two Banach couples $\overline{A} = (A_0,A_1)$, $\overline{B} = (B_0,B_1)$ if $T : \Sigma\overline{A} \to \Sigma\overline{B}$, with $T|_{A_i} : A_i \to B_i$ for $i = 0,1$, then $\|Ta\|_{(B_0,B_1)_{\theta,\psi,K}} \leqslant 2^\theta M_0^{1-\theta} M_1^\theta \|a\|_{(A_0,A_1)_{\theta,\psi,K}}$ for every $a \in (A_0,A_1)_{\theta,\psi,K}$, where $M_0 = \left\|T|_{A_0}\right\|$ and $M_1 = \|T|_{A_1}\|$.*

Proof. Let $\psi \in \Psi_\omega$ is invariant, $\overline{A} = (A_0,A_1)$, $\overline{B} = (B_0,B_1)$ two Banach couples and $T : \Sigma\overline{A} \to \Sigma\overline{B}$, with $T_{A_i} : A_i \to B_i$ for $i = 0,1$. We put $M_i = \|T|_{A_i}\|$ for $i = 0,1$. From corollary 3.6 $\|\cdot\|_\psi$ is invariant under permutation and hence it is translation invariant. Since

$$\begin{aligned} K(t,Ta,B_0,B_1) &\leqslant \inf_{a=a_0+a_1} (\|Ta_0\|_{B_0} + t\|Ta_1\|_{B_1}) \leqslant \inf_{a=a_0+a_1} (M_0\|a_0\|_{A_0} + tM_1\|a_1\|_{A_1}) \\ &\leqslant M_0 K\left(\frac{M_1 t}{M_0}, a, A_0, A_1\right) \end{aligned}$$

we get

$$\|Ta\|_{(B_0,B_1)_{\theta,\psi,K}} = \|(2^{-n\theta}K(2^n,Ta,B_0,B_1))_{n\in\mathbb{Z}}\|_\psi \leqslant M_0 \left\|(2^{-n\theta}K\left(\frac{M_1}{M_0}2^n,a,A_0,A_1\right))_{n\in\mathbb{Z}}\right\|_\psi.$$

Let $m \in Z$ be such, that $2^m \leqslant \frac{M_1}{M_0} < 2^{m+1}$. Then

$$\begin{aligned} \|Ta\|_{(B_0,B_1)_{\theta,\psi,K}} &\leqslant M_0\|(2^{-n\theta}K(2^{n+m+1},a,A_0,A_1)))_{n\in\mathbb{Z}}\|_\psi \\ &= M_0\|(2^{-(n+m+1)\theta}2^{(m+1)\theta}K(2^{n+m+1},a,A_0,A_1)))_{n\in\mathbb{Z}}\|_\psi \\ &= M_0 2^{(m+1)\theta}|a\|_{(A_0,A_1)_{\theta,\psi,K}} \\ &\leqslant M_0\left(\frac{M_1}{M_0}2\right)^\theta \|a\|_{(A_0,A_1)_{\theta,\psi,K}} = 2^\theta M_0^{1-\theta}M_1^\theta|a\|_{(A_0,A_1)_{\theta,\psi,K}}. \end{aligned}$$

The proof is complete. $\square$

PROPOSITION 3.9. *Let $\overline{E} = (E_0, E_1)$ be a Banach couple, $\theta \in (0,1)$ and $\psi \in \Psi_\omega$. Then $2^{-m\theta}K(2^m,a) \leqslant \|a\|_{\theta,\psi,K} \leqslant C_\theta 2^{-m\theta}J(2^m,a)$ for every $a \in \Delta\overline{E}$ and $m \in \mathbb{Z}$, where $C_\theta = \frac{1}{(2^{1-\theta}-1)(2^\theta-1)}$.*

Proof. Let $m,n \in \mathbb{Z}$. From the inequality, $\min\left(1,\frac{t}{s}\right)K(s,a) \leqslant K(t,a)$ [1], by replacing t by 2^n and s by 2^m, we get $2^{-n\theta}\min(1,2^{n-m})K(2^m,a) \leqslant 2^{-n\theta}K(2^n,a)$ and so, we obtain

$$K(2^m,a)\left\|\left(2^{-\theta n}\min(1,2^{n-m})\right)_{n\in\mathbb{Z}}\right\|_\psi \leqslant \|a\|_{\theta,\psi,K}.$$

Therefore

$$K(2^m,a) \leqslant \frac{2^{m\theta}}{A_m}\|a\|_{\theta,\psi,K},$$

where $A_m = \|(2^{(m-n)\theta}\min(1,2^{n-m}))_{n\in\mathbb{Z}}\|_\psi$.

Since $1 \leqslant A_m$ we obtain $2^{-m\theta}K(2^m,a) \leqslant \|a\|_{\theta,\psi,K}$.

Consider now the inequality $K(t,a) \leqslant \min\left(1,\frac{t}{s}\right)J(s,a)$. By replacing t by 2^n and s by 2^m we get

$$2^{-n\theta}K(2^n,a) \leqslant 2^{-m\theta}2^{(m-n)\theta}\min(1,2^{n-m})J(2^m,a).$$

Therefore, we obtain $\|a\|_{\theta,\psi,K} \leqslant A_m 2^{-m\theta}J(2^m,a)$. Since $\|\cdot\|_\psi \leqslant \|\cdot\|_1$ we have

$$A_m \leqslant \sum_{n=-\infty}^{\infty} 2^{(m-n)\theta}\min(1,2^{n-m}) = \sum_{n=-\infty}^{\infty} 2^{-n\theta}\min(1,2^n) = C_\theta.$$

So, $\|a\|_{\theta,\psi,K} \leqslant C_\theta 2^{-m\theta}J(2^m,a)$. □

Using Proposition 3.9 a generalization of so called Lions-Peetre lemma about compactness can be proved but for brevity we will omit here the proof.

THEOREM 3.10. *Let B be a Banach space, $\overline{A} = (A_0,A_1)$ a Banach couple and T be a linear operator. Let $0 < \theta < 1$.*

i) $T : \Sigma\overline{A} \to B$, with $T|_{A_0} : A_0 \to B$ compact and $T|_{A_1} : A_1 \to B$ bounded, then $T : (A_0,A_1)_{\theta,\psi,K} \mapsto B$ is compact.

ii) $T : B \mapsto \Delta\overline{A}$, $T : B \mapsto A_0$ compact and $T : B \mapsto A_1$ bounded, then $T : B \mapsto (A_0,A_1)_{\theta,\psi,J}$ is compact.

Let A be a Banach space, $\psi \in \Psi_\omega$ and $0 < \theta < 1$. The space $\ell_\psi^\theta(A)$ is defined as the space of the sequences $(\alpha_n)_{n\in\mathbb{Z}}$ of A such that $(2^{-\theta n}\|\alpha_n\|)_{n\in\mathbb{Z}} \in \ell_\psi(\mathbb{Z})$, with $\|(\alpha_n)_{n\in\mathbb{Z}}\| = \left\|(2^{-\theta n}\|\alpha_n\|)_{n\in\mathbb{Z}}\right\|$.

THEOREM 3.11. *Let A be a Banach space, $\psi,\psi_0,\psi_1 \in \Psi_\omega$ and $0 < \theta < 1$. If ψ is invariant then $(\ell_{\psi_0}(A), \ell_{\psi_1}^1(A))_{\theta,\psi} = \ell_\psi^\theta(A)$ up to equivalence of norms.*

Proof. Using properties of $\|\cdot\|_\psi$ as well as $K(t,a,\ell_\infty(A),\ell^1_\infty(A)) \sim \sup_j \min(1,t2^j)$ $\|a_j\|_A$ and $K(t,a,\ell_1(A),\ell^1_1(A)) \sim \sum_{n=-\infty}^{n=+\infty} \min(1,t2^n)\|a_n\|_A$, which are proved like in Theorem 1.18.2 [12], we can prove the following chain of embeddings

$$\ell^\theta_\psi(A) \subset (\ell_1(A),\ell^1_1(A))_{\theta,\psi} \subset (\ell_{\psi_0}(A),\ell^1_{\psi_1}(A))_{\theta,\psi} \subset (\ell_\infty(A),\ell^1_\infty(A))_{\theta,\psi} \subset \ell^\theta_\psi(A).$$

So, $(\ell_{\psi_0}(A),\ell^1_{\psi_1}(A))_{\theta,\psi} = \ell^\theta_\psi(A)$ up to equivalence of norms. □

REMARK. The particular case $(\ell_{\psi_0},\ell^1_{\psi_1})_{\theta,\psi} = \ell^\theta_\psi$ can be obtained from results of [9]. Indeed, when ψ is invariant, the space ℓ^θ_ψ is K-nontrivial and J- nontrivial and so $(E_0,E_1)_{\theta,\psi,K} = (E_0,E_1)_{\theta,\psi,J} = (E_0,E_1)_{\theta,\psi}$ for every Banach couple $(E_0,\ E_1)$ and $0<\theta<1$. Therefore, the conditions of Corollary 2.6 [9] are fulfilled and we have $\ell^\theta_\psi = (\ell_1,\ell^1_1)_{\theta,\psi}$. Under the same conditions Corollary 2.9 [9] gives $\ell^\theta_\psi = (\ell_\infty,\ell^1_\infty)_{\theta,\psi}$. Therefore, we get following chain of embeddings

$$\ell^\theta_\psi = (\ell_1,\ell^1_1)_{\theta,\psi} \subset (\ell_{\psi_0},\ell^1_{\psi_1})_{\theta,\psi} \subset (\ell_\infty,\ell^1_\infty)_{\theta,\psi} = \ell^\theta_\psi.$$

Thus $(\ell_{\psi_0},\ell^1_{\psi_1})_{\theta,\psi} = \ell^\theta_\psi$ up to equivalence of norms.

Let (E_0,E_1) be a Banach couple, ψ is an invariant function in Ψ_ω and $0<\theta<1$. We are going to determine the dual space of $(E_0,E_1)_{\theta,\psi}$. If $a\in(E_0,E_1)_{\theta,\psi} = (E_0,E_1)_{\theta,\psi,J}$ then a can be represented in the form $\sum_{n=-\infty}^{n=\infty} u_n$ with $u=(u_n)\in\Delta\overline{E}$ and $\|a\|_{(E_0,E_1)_{\theta,\psi,J}} = \inf\{\left\|\left(2^{-\theta n}J(2^n,u_n,E_0,E_1)\right)_{n\in\mathbb{Z}}\right\|_{\ell_\psi} : a=\sum_{n=-\infty}^{n=\infty} u_n,\ u_n\in\Delta\overline{E}\}$. Let $u=(u_n)\in\Delta\overline{E}$ such that $a=\sum_{n=-\infty}^{n=\infty} u_n$ and $\left(2^{-\theta n}J(2^n,u_n,E_0,E_1)\right)_{n\in\mathbb{Z}} \in \ell_\psi(\mathbb{Z})$. For every $\varepsilon>0$, we can find $c=(c_n)_{n\in\mathbb{Z}}\in D : \|b-c\|_\psi<\varepsilon$ where $b=(b_n)_{n\in\mathbb{Z}} = \left(2^{-\theta n}J(2^n,u_n,E_0,E_1)\right)_{n\in\mathbb{Z}}$. Then

$$\begin{aligned}
&\left\| a - \sum_{n=-N}^{n=N} u_n \right\|_{(E_0,E_1)_{\theta,\psi,J}} \\
&\quad \leqslant \|(0,\ldots,0,b_{N+1},b_{-(N+1)},\ldots)\|_\psi \\
&\quad \leqslant \|(|b_0-c_0|,|b_1-c_1|,|b_{-1}-c_{-1}|,\ldots,|b_N-c_N|,|b_{-N}-c_{-N}|,b_{N+1},\ldots)\|_\psi \\
&\quad = \|b-c\|_\psi < \varepsilon.
\end{aligned}$$

So, we get that $\Delta\overline{E}$ is dense in $(E_0,E_1)_{\theta,\psi}$.

Now, from the embedding (dense) $\Delta\overline{E}\subset(E_0,E_1)_{\theta,\psi}\subset\Sigma\overline{E}$ we get $\Delta\overline{E'}\subset(E_0,E_1)'_{\theta,\psi}$ $\subset\Sigma\overline{E'}$.

THEOREM 3.12. *If $\Delta\overline{E}$ is dense in E_0 and E_1, $0<\theta<1$ and ψ is an invariant function in Ψ_ω such that ℓ_ψ has a shrinking basis, then $(E_0,E_1)'_{\theta,\psi} = (E'_0,E'_1)_{\theta,\psi*}$ (with equivalent norms).*

Sketch of the proof. We mention, that if $\Delta\overline{E}$ is dense in E_0 and E_1, then the following equalities hold $K(t,a',E'_0,E'_1) = \sup_{a\in\Delta\overline{E}} \frac{|<a',a>|}{J(t^{-1},a,E_0,E_1)}$ and $J(t,a',E'_0,E'_1) = \sup_{a\in\Delta\overline{E}} \frac{|<a',a>|}{K(t^{-1},a,E_0,E_1)}$ [1].

As in the proof of the corresponding Theorem 3.7.1 [1], we can show that $(A_0,A_1)'_{\mu,\psi,J} \subset (A'_1,A'_0)_{1-\mu,\psi*,K}$ and $(A'_1,A'_0)_{1-\mu,\psi*,J} \subset (A_0,A_1)'_{\mu,\psi,K}$. The proof of these embeddings goes in the same way, replacing the space $\lambda^{\mu,q}$ by ℓ^{μ}_{ψ}. Note that the spaces ℓ^{μ}_{ψ} and $\ell^{1-\mu}_{\psi*}$ are in duality, generated by the bilinear form $\sum 2^{-n}a_nb_n$ when ℓ_ψ has shrinking basis. As a consequence of fundamental lemma of interpolation theory the K interpolation space is embedded in the J interpolation space for any function from Ψ_ω. Moreover, invariantness of ψ and equality $K(2^{-n},a,E_0,E_1) = 2^{-n}K(2^n,a,E_1,E_0)$ give that $(E_0,E_1)_{\theta,\psi} = (E_1,E_0)_{1-\theta,\psi}$. Since for ψ invariant J and K spaces coincide up to equivalence of norms, these embeddings with $\mu = 1-\theta$, $A_0 = E_1, A_1 = E_0$ give

$$(E_0,E_1)'_{\theta,\psi} = (E_1,E_0)'_{1-\theta,\psi} = (E_1,E_0)'_{1-\theta,\psi,J} \subset (E'_0,E'_1)_{\theta,\psi*,K} \subset (E'_0,E'_1)_{\theta,\psi*,J}$$
$$\subset (E_1,E_0)'_{1-\theta,\psi,K} = (E_1,E_0)'_{1-\theta,\psi} = (E_0,E_1)'_{\theta,\psi}. \quad \square$$

If we want to avoid the condition the space ℓ_ψ to have shrinking basis, then we will arrive not to the space $(E'_0,E'_1)_{\theta,\psi*}$ but to a space, generated by the space $\ell_{\psi*,\infty}$. We are not going to consider here this situation in details.

Let (E_0,E_1) be a Banach couple. Sometimes is more proper to consider the space $(E_0,E_1)_{\theta,\psi,q}$, $0<\theta<1$, $1<q<\infty$ which is defined as follow.

For $n = 1,2,\ldots$ let the space $X_n = E_0+E_1$ with $\|a\| = 2^{-n\theta}K_q(2^n,a,E_0,E_1)$ for $a \in X_n$, where $K_q(t,a,E_0,E_1) = \inf_{a=a_0+a_1}(\|a\|^q_{E_0} + t^q(\|a\|^q_{E_1})^{1/q}$. Then $(E_0,E_1)_{\theta,\psi,q} = \{a \in E_0+E_1 : \|a\|_{\theta,\psi,q} < \infty\}$, where $\|a\|_{\theta,\psi,q} = \left\|\left(2^{n\theta}K_q(2^n,a,E_0,E_1)\right)_{n\in\mathbb{N}}\right\|_\psi$.

Actually we should write $(E_0,E_1)_{\theta,\psi,q,K}$ and $\|a\|_{\theta,\psi,q,K}$ but if there is no misunderstanding, we will omit K.

The space $(E_0,E_1)_{\theta,\psi,q}$ is the subspace of the constant sequences in $(\sum_{n=1}^{\infty} \bigoplus X_{n,q})_\psi$. If $\delta(\varepsilon) = \min(\delta_{E_0}(\varepsilon),\delta_{E_1}(\varepsilon))$, where $\delta_{E_0}(\varepsilon)$ and $\delta_{E_1}(\varepsilon)$ are the modulus of convexity of E_0 and E_1, from [6] we have that $\inf_{n\in\mathbb{N}}\delta_{X_{n,q}}(\varepsilon) \geqslant \delta_{\ell^2_q}\left(\frac{\varepsilon}{2}\delta\left(\frac{\varepsilon}{2}\right)\right)$. Therefore, from [13] we obtain the following corollary.

COROLLARY 3.13. *Let (E_0,E_1) be a Banach couple, $1<q<+\infty$ and $\psi \in \Psi_\omega$.*

i) If ℓ_ψ is uniformly convex and the spaces E_0 and E_1 are uniformly convex (resp. uniformly non-square), then the space $(E_0,E_1)_{\theta,\psi,q}$ is uniformly convex (resp. uniformly non-square).

ii) If ℓ_ψ is uniformly smooth and the spaces E_0 and E_1 are uniformly smooth, then the space $(E_0,E_1)_{\theta,\psi,q}$ is uniformly smooth.

Moreover, for $\varphi \in \Psi_2$ we can define generalized K_φ functional as $K_\varphi(t,a,E_0,E_1) = \inf_{a=a_0+a_1}\|(\|a_0\|_{E_0} + t\|a_1\|_{E_1})\|_\varphi = \inf_{a=a_0+a_1}(\|a_0\|_{E_0}+t\|a_1\|_{E_1})\varphi\left(\frac{t\|a\|_{E_1}}{\|a_0\|_{E_0}+t\|a_1\|_{E_1}}\right)$. When $\varphi(t) = \varphi_1(t) \equiv 1$ we get Peetre K-functional. When $\varphi(t) = \varphi_q(t) = ((1-t)^q + t^q)^{\frac{1}{q}}$ we get Holmstedt-Peetre K-functional $K_q(t,a,E_0,E_1)$. The function $\varphi(t)$ is convex, $\varphi(0) = 1, \varphi(1) = 1, \max(t,1-t) \leqslant \varphi(t) \leqslant 1$ and $\inf\varphi(t) \geqslant \min(t,1-t) = \frac{1}{2}$, i.e.

$\frac{1}{2} \leqslant \varphi(t) \leqslant 1$. Hence for $K_\varphi(t,a,E_0,E_1) = K_\varphi(t,a)$ we have

$$\frac{1}{2}K(t,a) \leqslant K_\varphi(t,a) \leqslant K(t,a) = \inf_{a=a_0+a_1}(\|a_0\|_{E_0} + t\|a\|_{E_1}).$$

But $K(t,a) \leqslant \min(1,t)\|a\|_{\Delta(E)}$ and $K(t,a) \geqslant \min(1,t)\|a\|_{\Sigma(E)}$. Hence

$$\frac{1}{2}\min(1,t)\|a\|_{\Sigma(E)} \leqslant K_\psi(t,a) \leqslant \min(1,t)\|a\|_{\Delta(E)},$$

i.e. we get an intermediate space E $(\Delta(E) \subset E \subset \Sigma(E))$, where $E = (E_0,E_1)_{\theta,p,\phi}$ has norm

$$\|a\|_{\theta,p,\varphi} = \left\| \left(\left\{ 2^{-\theta n} K_\varphi(2^n,a,E_0,E_1) \right\} \right)_{n\in\mathbb{Z}} \right\|_{l_p}.$$

Because of the inequalities $\frac{1}{2}\|a\|_{\theta,p} \leqslant \|a\|_{\theta,p,\varphi} \leqslant \|a\|_{\theta,p}$ where $\|a\|_{\theta,p}$ is the norm of the space $(E_0,E_1)_{\theta,p}$ generated by the usual functional $K(t,a)$,we get again the space $(E_0,E_1)_{\theta,p}$ with equivalent norm $\|a\|_{\theta,p,\varphi}$. We can of course consider another generalization, namely the space $(E_0,E_1)_{\theta,\psi,\varphi}$, with norm

$$\|a\|_{\theta,\psi,\varphi} = \left\| \left(2^{-\theta n} K_\varphi(2^n,a,E_0,E_1) \right)_{n\in\mathbb{Z}} \right\|_{\psi}.$$

As above we see that this norm is equivalent to the norm of the space $(E_0,E_1)_{\theta,\psi,}$. So, we see that different second functions, namely φ's, give equivalent norms.

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REFERENCES

[1] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces: An introduction*, Springer-Verlag, Berlin, 1976.

[2] F. F. BONSALL AND J. DUNCAN, *Numerical Ranges II*, Lecture Note Series, **10**, London Math. Soc., London, 1973.

[3] Y. BRUDNYI AND N. KRUGLJAK, *Interpolation functors and interpolation spaces*, Vol. **1**, North-Holland, Amsterdam/New York/Oxford, Tokyo, 1991.

[4] M. KATO, K.-S. SAITO AND T. TAMURA, *On ψ-direct sums of Banach spaces and convexity*, J. Aust. Math. Soc., **75** (2003), 413–422.

[5] S. G. KREIN, Y. PETUNIN AND E. SEMENOV, *Interpolation of operators*, Transl. of Math. Monographs, Vol. **54**, AMS, Providence, Rhode Island, 1982.

[6] D. N. KUTZAROVA, L. Y. NIKOLOVA AND T. ZACHARIADES, *Real interpolation for families of Banach spaces and convexity*, Math. Nachr., **171** (1995), 259–268.

[7] K.-I. MITANI, S. OSHIRO AND K.-S. SAITO, *Smoothness of ψ direct sums of Banach spaces*, Math. Inequal. and Appl., **1** (2005), 147–157.

[8] K.-I. MITANI, K.-S. SAITO, *On generalized ℓ_p-spaces*, Hiroshima Math. J., **37** (2007), 1–12.

[9] P. NILSSON, *Reiteration theorem for real interpolation and approximation spaces*, Ann. Mat. Pura Appl., **132** (1982), 291–330.

[10] V. OVCHINNIKOV, *Method of orbits in interpolation theory*, Math. Rep., **1** (1984), 349–516.

[11] K.-S. SAITO, M. KATO AND Y. TAKAHASHI, *Absolute norms on $\mathbb{C}^n$*, J. Math. Anal. Appl., **252** (2000), 879–905.

[12] H. TRIEBEL, *Interpolation Theory. Function Spaces. Diferential Operators*, North-Holland, Amsterdam, 1978.

[13] T. ZACHARIADES, *On ψ-direct sums of Banach sequence spaces and convexity*, to appear in Rocky Mountain Journal of Math.

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