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ON THE GODUNOVA–LEVIN–SCHUR CLASS OF FUNCTIONS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In 1985 Godunova and Levin have considered the following class of functions. A function $f: I \to \mathbb{R}$ is said to belong to the class $Q(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$, satisfies the inequality:

$$f((1-t)x+ty) \leqslant \frac{f(x)}{1-t} + \frac{f(y)}{t}$$

Here I is an interval of $\mathbb{R}$.

It is known that all nonnegative quasiconvex functions belong to this class and this class of functions coincides with the class of Schur functions $\mathscr{S}(I)$, that is, with the class of nonnegative functions that satisfy the inequality

$$\sum f(x)(x-y)(x-z) \geqslant 0 \qquad \text{for every} \qquad x,y,z \in I$$

The aim of this paper is to survey some important properties of functions belonging to these classes of functions and to prove some new results concerning properties of functions from them.

1. Introduction

The following inequality is known as the Schur inequality.

Theorem. Let x, y, z be nonnegative real numbers. Then for every $r > 0$ the following inequality holds:

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geqslant 0 \tag{1.1}$$

Equality holds if and only if $x = y = z$ or if two of x, y, z are equal and the third is zero.

In case the exponent r is an even number then inequality (1.1) holds for every x, y, z real numbers.

One of the reasons for which Schur's inequality is studied is related to its applications to geometric programming.

Geometric programming is a part of nonlinear programming where both the objective function and constraints are polynomials with positive coefficients (posynomials), that is

$$P(x_1, x_2, \ldots, x_n) = \sum_{|\alpha| \leqslant m} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

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where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a n-dimensional vector with components natural numbers, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ and all coefficients a_α are nonnegative numbers. Expanding terms in (1.1) we get

$$\sum x^{r+2} + xyz\left(\sum x^{r-1}\right) \geqslant \sum x^{r+1}y + \sum x^{r+1}z$$

Therefore Schur's inequality is equivalent to an inequality between two posynomials.

DEFINITION 1.1. Let D be a subset of $\mathbb{R}$ containing at least two elements and $f : D \to \mathbb{R}$ be a map. Denote by $S(f,x,y,z)$ the sum

$$f(x)(x-y)(x-z) + f(y)(y-z)(y-x) + f(z)(z-x)(z-y)$$

We shall say that a function $f : D \to \mathbb{R}$ belongs to the class $\mathscr{S}(D)$ of Schur functions if the following inequality holds:

$$S(f,x,y,z) \geqslant 0 \quad \text{for every} \quad x,y,z \in D \tag{1.2}$$

One can easily see that all the functions from $\mathscr{S}(D)$ are nonnegative.

In [7] E. M. Wright had generalized the Schur's inequality, showing that the inequality (1.1) holds if the function $f(x) = x^r$ is replaced with a nonnegative convex function or with a nonnegative monotone function.

Consequently nonnegative convex functions and nonnegative monotone functions belong to the class of Schur functions defined on some interval D.

In [2] Godunova and Levin had introduced the following class of functions:

If D is an interval of $\mathbb{R}$ a function $f : D \to \mathbb{R}$ is said to belong to the class $Q(D)$ if it is nonnegative and for all $x,y \in D$ and $t \in (0,1)$, the following inequality holds:

$$f((1-t)x+ty) \leqslant \frac{f(x)}{1-t} + \frac{f(y)}{t} \tag{1.3}$$

Of course one can extend the definition of the Godunova and Levin class of functions $Q(D)$ in the case D is a subset of $\mathbb{R}$ containing at least two elements. Therefore we shall say that a function $f : D \to \mathbb{R}$ is said to belong to the class $Q(D)$ if it is nonnegative and for all $x,y \in D$ and $t \in (0,1)$, such that $(1-t)x+ty \in D$ the inequality (1.3) holds.

In [2] Godunova and Levin have shown that $\mathscr{S}(D)$, the class of all Schur functions defined on D, coincides with the Godunova-Levin class of functions $Q(D)$.

In the second paragraph we shall survey some important properties of the functions belonging to the Godunova-Levin-Schur class of functions.

In the third paragraph we shall prove some new properties of Godunova-Levin-Schur functions.

2. A survey of the properties of the Godunova-Levin-Schur functions

In the present paragraph we shall denote by D a subset of $\mathbb{R}$ containing at least two elements. We shall denote with $\mathscr{S}(D)$ the class of Godunova-Levin-Schur functions defined on D. The class of the Godunova-Levin-Schur functions was studied in a series of papers [1], [3], [4, pp. 410–413], [5] and [6].

In [6] Varošanec had introduced a very general class of functions known as the class of h-convex functions. More precisely, let I be an interval of $\mathbb{R}$ and $h:(0,1)\to\mathbb{R}$ be a nonnegative function with the property that there exists $t_0\in(0,1)$ such that $h(t_0)>0$. A function $f:I\to\mathbb{R}$ is called a h-convex function if f is nonnegative and for all $x,y\in I,\alpha\in(0,1)$ we have

$$f(\alpha x+(1-\alpha)y)\leqslant h(\alpha)f(x)+h(1-\alpha)f(y) \tag{2.1}$$

If inequality in (2.1) is reversed, then f is said to be h-concave. Denote by $SX(h,I)$ the class of all h-convex functions. The notion of h-convex function is of course more general than the notion of Godunova-Levin-Schur function. The class of h-convex functions contains in case that special selections are made for the function h some important classess of functions. Obviously, if $h(\alpha)=\alpha,\alpha\in(0,1)$ then all nonnegative convex functions are h-convex functions. If $h(\alpha)=\frac{1}{\alpha},\alpha\in(0,1)$ then $SX(h,I)=\mathscr{S}(I)$. If $h(\alpha)=1,\alpha\in(0,1)$ then $SX(h,I)$ contains the class $P(I)$ of all $P-$functions defined on I. By a $P-$function we understand a nonnegative function $f:I\to\mathbb{R}$ with the property that

$$f(\alpha x+(1-\alpha)y)\leqslant f(x)+f(y)\quad\text{for all } x,y\in I$$

The paper [6] contains many interesting properties of the h-convex functions.

In [3], was proved the following version of the famous Jensen inequality for convex functions.

THEOREM 2.1. *Let D be an interval of $\mathbb{R}$, $n\geqslant 2$, $w_1,w_2,\ldots,w_n$ be real numbers and $f\in\mathscr{S}(D)$. If $v_n=w_1+w_2+\ldots+w_n$ then for every $x_1,x_2,\ldots,x_n\in I$ the following inequality holds*

$$f\left(\frac{1}{v_n}\sum_{i=1}^{n}w_ix_i\right)\leqslant v_n\sum_{i=1}^{n}\frac{f(x_i)}{w_i}$$

Let $I=[a_0,b_o]$ be an interval of the real line, $a,b\in I$, $a<b$ and $f:I\to\mathbb{R}$ be a convex function. The following inequality is known as the Hadamard's inequality:

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}$$

In [1] were proved two sharp integral inequalities of Hadamard type for the Godunova-Levin-Schur functions.

THEOREM 2.2. [1] *Let I be an interval of $\mathbb{R}$, $a,b\in I$, $a<b$ and $f\in\mathscr{S}(I)$ a function integrable on $[a,b]$. Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{4}{b-a}\int_a^b f(x)\,dx \qquad 2.2$$

and

$$\frac{1}{b-a}\int_a^b p(x) f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{2.3}$$

where $p(x) = \frac{(b-x)(x-a)}{(b-a)^2}, x \in I$.

The constant 4 *in (2.2) is the best possible.*

In [5] were proved the following properties of the Godunova-Levin-Schur functions:

PROPOSITION 2.1. [5] *The following assertions hold:*

1^0 *If* $f \in \mathscr{S}(D)$ *then* $f \geqslant 0$.

2^0 *If* $f \in \mathscr{S}(D)$ *and for some* $a,b \in D, a < b$, *we have* $f(a) = f(b) = 0$ *then* $f(x) = 0$ *for every* $x \in [a,b] \cap D$.

3^0 *If there exists* $a,b \in D$, $a < b$, *such that* $\frac{a+b}{2} \in D$ *and*

$$f\left(\frac{a+b}{2}\right) > 2f(a) + 2f(b)$$

then $f \notin \mathscr{S}(D)$.

PROPOSITION 2.2. [5] *Let* $f : D \to \mathbb{R}$ *be a map. Suppose that there exist two positive constants* m, M *such that:*

$$0 < m \leqslant f(x) \leqslant M \leqslant 4m \quad \text{for every } x \in D$$

Then f *is a Godunova-Levin-Schur map on* D.

PROPOSITION 2.3. [5] *Let* $f : D \to \mathbb{R}$, *be a map. Suppose that there exist two positive constants* m, M *such that:*

$$0 < m \leqslant f(x) \leqslant M \qquad \text{for every } x \in D$$

For every $a \geqslant 0$ *consider the map* $f_a : D \to \mathbb{R}$, $f_a(x) = f(x) + a$, $x \in D$.

Then for every $a \geqslant \max\left(\frac{M-4m}{3}, 0\right)$ *the map* f_a *is a Godunova-Levin-Schur map on* D.

PROPOSITION 2.4. [5] *Let* $f(x) = \left(x^2-1\right)^2$, $x \in \mathbb{R}$. *For every* $a \geqslant 0$ *consider the map* $f_a : \mathbb{R} \to \mathbb{R}$, $f_a(x) = f(x) + a$, $x \in \mathbb{R}$. *Then the following assertions hold:*

1^0 *For every* $a \in [0, \frac{1}{3})$, $f_a \notin \mathscr{S}(\mathbb{R})$.

2^0 *For every* $a \in [1,\infty)$, $f_a \in \mathscr{S}(\mathbb{R})$

3^0 *For every* $a \in \mathbb{R}$, f_a *is not quasiconvex and is not the sum of two positive monotone functions.*

PROPOSITION 2.5. [5] *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a map with the property that* $f \circ \phi \in \mathscr{S}(\mathbb{R})$ *for every* $f \in \mathscr{S}(\mathbb{R})$. *Then* ϕ *is monotone.*

PROPOSITION 2.6. [5] *Let* $f : \left[0, \frac{2}{3}\right] \to \mathbf{R}$, $f(x) = x - x^2$, $x \in \left[0, \frac{2}{3}\right]$. *Then* $f \in \mathscr{S}\left(\left[0, \frac{2}{3}\right]\right)$.

3. Some new properties of the Godunova-Levin-Schur functions

In this paragraph we shall prove some new properties of Schur functions. We shall denote by D a subset of $\mathbb{R}$ containing at least two elements.

PROPOSITION 3.1. *Suppose that* $\inf D = -\infty$. *Let* $a \in D$, $f \in \mathscr{S}(D)$ *be such that* $f(x) = 0$ *for every* $x \in (-\infty, a] \cap D$. *Then* f *is increasing on* D

Proof. Let $x, y, z \in D$ be such that $x \leqslant a < y < z$. Since $f \in \mathscr{S}(D)$ we have

$$0 \leqslant S(f, x, y, z) = f(y)(y - z)(y - x) + f(z)(z - x)(z - y)$$

hence

$$f(z)(z - x) - f(y)(y - x) \geqslant 0 \text{ for every } x \in (-\infty, a] \cap D$$

thus

$$x(f(y) - f(z)) + zf(z) - yf(y) \geqslant 0 \text{ for every } x \in (-\infty, a] \cap D$$

Letting $x \to -\infty$ in the preceding inequality we obtain $f(y) \leqslant f(z)$. Consequently we proved that f is increasing. □

PROPOSITION 3.2. *Suppose that* $\sup D = +\infty$. *Let* $a \in D$, $f \in \mathscr{S}(D)$ *be such that* $f(x) = 0$ *for every* $x \in [a, +\infty) \cap D$. *Then* f *is decreasing on* D.

Proof. Let $x, y, z \in D$ be such that $x \geqslant a > y > z$. Since $f \in \mathscr{S}(D)$ we have

$$0 \leqslant S(f, x, y, z) = f(y)(y - z)(y - x) + f(z)(z - x)(z - y)$$

hence

$$f(z)(z - x) - f(y)(y - x) \leqslant 0 \text{ for every } x \in [a, +\infty) \cap D$$

thus

$$x(f(y) - f(z)) + zf(z) - yf(y) \leqslant 0 \text{ for every } x \in [a, +\infty) \cap D$$

Letting $x \to +\infty$ in the preceding inequality we obtain $f(y) \leqslant f(z)$. Consequently we proved that f is decreasing. □

PROPOSITION 3.3. *Suppose that* $\inf D = -\infty$ *and* $\sup D = +\infty$. *Let* $\varphi : D \to \mathbb{R}$ *be a map with the property that* $\varphi \cdot f \in \mathscr{S}(D)$ *for every* $f \in \mathscr{S}(D)$. *Then* φ *is a nonnegative constant map.*

Proof. For every $a \in D$ consider the maps $f_a : D \to \mathbb{R}$ and $g_a : D \to \mathbb{R}$

$$f_a(x) = \begin{cases} 1 & \text{if } x \in [a, +\infty) \cap D \\ 0 & \text{if } x \in (-\infty, a) \cap D \end{cases}$$

$g_a(x) = 1 - f_a(x)$ for every $x \in D$. Note that f_a is increasing and g_a is decreasing. Consequently $f_a, g_a \in \mathscr{S}(D)$. Hence $\varphi \cdot f_a \in \mathscr{S}(D)$ and $\varphi \cdot g_a \in \mathscr{S}(D)$. By proposition 3.1. it follows that the map $\varphi \cdot f_a$ is monotone increasing for every $a \in D$, hence the map φ is increasing. By proposition 3.2. it follows that the map $\varphi \cdot g_a$ is monotone decreasing for every $a \in D$, hence the map φ is decreasing. Hence φ is a constant map. □

PROPOSITION 3.4. *Let $f \in \mathscr{S}(\mathbb{R})$ be a map with the property*

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = 0 \tag{3.1}$$

Then $f = 0$.

Proof. Let $z \in \mathbb{R}$, $a = b = \frac{1}{2}$. By (1.3) we have

$$f(z) = f(a(z+x) + b(z-x)) \leqslant \frac{f(z+x)}{a} + \frac{f(z-x)}{b} = 2f(z+x) + 2f(z-x)$$

hence

$$f(z) = \lim_{x \to +\infty} [2f(z+x) + 2f(z-x)] = 2 \lim_{x \to +\infty} f(x) + 2 \lim_{x \to -\infty} f(x) = 0 \qquad \square$$

PROPOSITION 3.5. *Let $f \in \mathscr{S}(D)$ and $x_0 \in D$. Suppose that there exists a sequence $(x_n)_{n \geqslant 1}$ in D such that $\lim_{n \to \infty} x_n = +\infty$ and $\lim_{n \to \infty} x_n f(x_n) = 0$. Then $f = 0$ on $[x_0, +\infty) \cap D$.*

Proof. Let $x_0 < x$. Then there exists a natural number n_x such that $x < x_n$ for every $n \geqslant n_x$. Denote

$$a_n = \frac{x_n - x}{x_n - x_0}, \quad b_n = \frac{x - x_0}{x_n - x_0}, \text{ for every } n \geqslant n_x.$$

Note that a_n, $b_n \in (0,1)$, $a_n +\ b_n = 1$ and $x = a_n x_0 + b x_n$ for every $n \geqslant n_x$. Since $f \in \mathscr{S}(D)$ we have

$$f(x) = f(a_n x_0 + b x_n) \leqslant \frac{f(x_0)}{a_n} + \frac{f(x_n)}{b_n} = \frac{f(x_n)}{b_n} = f(x_n) \frac{x_n - x_0}{x - x_0}$$

whence

$$f(x) \leqslant x_n f(x_n) \frac{1 - (x_0/x_n)}{x - x_0}$$

Letting $n \to \infty$ in the preceding inequality we obtain $f(x) = 0$. □

PROPOSITION 3.6. *Let $f \in \mathscr{S}(D)$. Then the following assertions are equivalent:*

1^0 *f is a quasiconvex map.*

2^0 *There exists a sequence $(\alpha_k)_{k\geqslant 1}$ of positive numbers such that $\lim_{k\to\infty} \alpha_k = +\infty$ and $f^{\alpha_k} \in \mathscr{S}(D)$ for every $k \geqslant 1$.*

Proof. To prove that 1^0 implies 2^0 suppose that f is a quasiconvex map. This implies that f^{α} is quasiconvex for every $\alpha > 0$. Hence for every sequence $(\alpha_k)_{k\geqslant 1}$ of positive numbers such that $\lim_{k\to\infty} \alpha_k = +\infty$ we have that f^{α_k} is quasiconvex. Consequently $f^{\alpha_k} \in \mathscr{S}(D)$ for every $k \geqslant 1$.

To prove that 2^0 implies 1^0 suppose that there exists a sequence $(\alpha_k)_{k\geqslant 1}$ of positive numbers such that $\lim_{k\to\infty} \alpha_k = +\infty$ and $f^{\alpha_k} \in \mathscr{S}(D)$ for every $k \geqslant 1$. Let $x, y \in D$ be such that $0 \leqslant f(x) \leqslant f(y)$. We shall prove that f is a quasiconvex map on $[x,y] \cap D$. We shall study two cases: the case $f(y) = 0$ and the case $f(y) > 0$.

If $f(y) = 0$ then $f(t) = 0$ for every $t \in [x,y] \cap D$.

If $f(y) > 0$ and $a, b \in (0,1)$, $a + b = 1$ and $ax + by \in D$ then

$$f^{\alpha_k}(ax+by) \leqslant \frac{f^{\alpha_k}(x)}{a} + \frac{f^{\alpha_k}(y)}{b}$$

If we divide the preceding inequality by $f^{\alpha_k}(y)$ we obtain

$$\left[\frac{f(ax+by)}{f(y)}\right]^{\alpha_k} \leqslant \frac{1}{a}\left[\frac{f(x)}{f(y)}\right]^{\alpha_k} + \frac{1}{b} \leqslant \frac{1}{a} + \frac{1}{b}$$

hence

$$\frac{f(ax+by)}{f(y)} \leqslant \lim_{k\to\infty} \left(\frac{1}{a} + \frac{1}{b}\right)^{\frac{1}{\alpha_k}} = 1$$

Thus $f(ax+by) \leqslant f(y) = \max(f(x), f(y))$. $\square$

PROPOSITION 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic map. Then the following assertions are equivalent:*

1^0 $f \in \mathscr{S}(\mathbb{R})$

2^0 $0 \leqslant \sup f \leqslant 4 \inf f$

Proof. To prove that 1^0 implies 2^0 suppose that $f \in \mathscr{S}(\mathbb{R})$ and $x, y \in \mathbb{R}$. Denote by T the period of f, $n_0 = \left[\frac{|x-y|}{T}\right] + 1$. By $[c]$ we denoted the greatest integer less or equal to c. Consider the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$, $a_n = \frac{x-y+nT}{2nT}$, $b_n = 1 - a_n$, $n \geqslant 1$. Note that a_n, $b_n \in (0,1)$ for every $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{1}{2}$. Letting $n \to \infty$ in the following inequality

$$\begin{aligned} f(y) &= f(a_n(x-nT)+b_n(x+nT)) \\ &\leqslant \frac{f(x-nT)}{a_n}+\frac{f(x+nT)}{b_n}=\frac{f(x)}{a_n}+\frac{f(x)}{b_n}=\frac{f(x)}{a_nb_n} \end{aligned}$$

we obtain $f(y) \leqslant 4f(x)$. Since x and y were chosen arbitrarily it follows that $\sup f \leqslant 4\inf f$

To prove that 2^0 implies 1^0 suppose that $0 \leqslant \sup f \leqslant 4\inf f$.

Let $x,y \in \mathbb{R}$, $a,b \in (0,1)$, $a+b=1$. From the inequalities

$$f(ax+by) \leqslant \sup f \leqslant 4\inf f \leqslant \left(\frac{1}{a}+\frac{1}{b}\right)\inf f \leqslant \frac{f(x)}{a}+\frac{f(x)}{b}$$

it follows that $f \in \mathscr{S}(\mathbb{R})$. □

THEOREM 3.8. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function of class C^1 with the following properties:*

$$\lim_{x\to-\infty} f'(x) = -\infty \text{ and } \lim_{x\to+\infty} f'(x)=+\infty \tag{3.2}$$

For every $a\in\mathbb{R}$ consider the map $f_a:\mathbb{R}\to\mathbb{R}$, $f_a(x)=f(x)+a$, $x\in\mathbb{R}$. Then there exists $a\in\mathbb{R}$ such that $f_a\in\mathscr{S}(\mathbb{R})$.

Proof. If $t\in\mathbb{R}$ denote $t_+=\frac{t+|t|}{2}$ and $t_-=\frac{|t|-t}{2}$. Then $f'=(f')_+-(f')_-$. Let $g,h:\mathbb{R}\to\mathbb{R}$ be functions of class C^1 such that $g'=(f')_+$ and $h'=-(f')_-$, $g(0)=h(0)$.

From (3.2) it follows that there exists $c>0$ such that $(f')_+=0$ on $(-\infty,-c]$ and $(f')_-=0$ on $(c,+\infty)$. Consequently $g=$constant on $(-\infty,-c]$ and $h=$constant on $(c,+\infty)$. Since g is monotone increasing and h is monotone decreasing it follows that g and h are bounded from below. Consequently there exists $a\in\mathbb{R}$ such that $g+\frac{a}{2}$ and $h+\frac{a}{2}$ are nonnegative, hence they belong to $\mathscr{S}(\mathbb{R})$. Since

$$f_a=\left(g+\frac{a}{2}\right)+\left(h+\frac{a}{2}\right)$$

it follows that $f_a\in\mathscr{S}(\mathbb{R})$. □

THEOREM 3.9. *Let $f:\mathbb{R}\to\mathbb{R}$ be a rational function of the following type*

$$f(x)=\frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions generated by monic polynomials and $Q(x)>0$ for every $x\in\mathbb{R}$. Denote $m=\deg P$, $n=\deg Q$. For every $a\in\mathbb{R}$ consider the map $f_a:\mathbb{R}\to\mathbb{R}$, $f_a(x)=f(x)+a$, $x\in\mathbb{R}$.

Then the following assertions hold:

1^0 *If $0\leqslant m<n$, there exists $x_0\in\mathbb{R}$ such that $f(x_0)>0$ and f is nonnegative then $f\notin\mathscr{S}(\mathbb{R})$.*

2^0 *If* $m = n$ *then there exists* $a \in \mathbb{R}$ *such that* $f_a \in \mathscr{S}(\mathbb{R})$.
3^0 *If* $m > n$ *and* $m - n$ *is even then there exists* $a \in \mathbb{R}$ *such that* $f_a \in \mathscr{S}(\mathbb{R})$.
4^0 *If* $m > n$ *and* $m - n$ *is odd then for every* $a \in \mathbb{R}$ *we have that* $f_a \notin \mathscr{S}(\mathbb{R})$.

Proof. To prove 1^0 note that $0 \leqslant m < n$ implies that (3.1) holds. By proposition 3.4. it results that $f \notin \mathscr{S}(\mathbb{R})$. Suppose now that $m = n$. Since

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = 1$$

it follows that f is bounded on $\mathbb{R}$. Let M_1 and M_2 be such that $M_1 \leqslant f(x) \leqslant M_2$ for every $x \in \mathbb{R}$. We shall prove that

$$a > \max\left[|M_1|, \frac{1}{3}(M_2 - 4M_1)\right] \tag{3.3}$$

implies that $f_a \in \mathscr{S}(\mathbb{R})$. Note that (3.3) implies $0 < M_1 + a \leqslant f(x) + a \leqslant M_2 + a \leqslant 4(M_1 + a)$. From proposition 2.2. f_a is a Godunova-Levin-Schur map on $\mathbb{R}$. Thus assertion 2^0 is proved.

To prove 3^0 note that $m > n$ and $m - n$ is even implies that

$$\lim_{x \to -\infty} f'(x) = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{P(x)}{xQ(x)} = -\infty$$

$$\lim_{x \to +\infty} f'(x) = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{P(x)}{xQ(x)} = +\infty$$

By theorem 3.8. there exists $a \in \mathbb{R}$ such that $f_a \in \mathscr{S}(\mathbb{R})$.

If $m > n$ and $m - n$ is odd then for every $a \in \mathbb{R}$ we have $\lim_{x \to -\infty} f_a(x) = -\infty$. This implies that f_a fails to be nonnegative, hence $f_a \notin \mathscr{S}(\mathbb{R})$. $\square$

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