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A NOTE ON YOUNG INEQUALITY

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Abstract. In this paper we give an extension of Young inequality establishing lower and upper bound.

1. Introduction

We begin with well-known Young inequality, then we will add some more conditions to get lower and upper estimation of Young inequality.

THEOREM 1.1. (Young) *Let f be a continuous and strictly increasing function on $[0,A]$, for $A>0$. If $f(0)=0$, $a\in[0,A]$ and $b\in[0,f(A)]$, then*

$$\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx \geqslant ab, \tag{1}$$

where f^{-1} is the inverse function of f. Equality in (1) is valid if and only if $b=f(a)$.

Proof. See [4].
The following refinement of Theorem 1.1 is given in [3] :

THEOREM 1.2. *Let f be a continuous, differentiable and strictly increasing function on $[0,A]$, for $A>0$. If $f(0)=0$, $a\in[0,A]$ and $b\in[0,f(A)]$ and $f'(x)$ is strictly monotonous on $[\alpha,\beta]$, then*

$$m\frac{(a-f^{-1}(b))^2}{2} \leqslant \int_0^a f(x)dx+\int_0^b f^{-1}(x)dx-ab \leqslant M\frac{(a-f^{-1}(b))^2}{2} \tag{2}$$

where

$$m=\min\{f'(a),f'(f^{-1}(b))\} \tag{3}$$

and

$$M=\max\{f'(a),f'(f^{-1}(b))\}. \tag{4}$$

Equalities in (2) *are valid if and only if $b=f(a)$.*

In this paper we will give improvement of Theorem 1.2 and several related results. For purposes of the next section we introduce the following notations

$$\alpha=\min\{a,f^{-1}(b)\},\quad \beta=\max\{a,f^{-1}(b)\}.$$

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2. Main results

Below, we will principally work on a measure space $(X,\mathscr{B},\lambda)$, where $X=[\alpha,\beta]$, $\mathscr{B}$ is a $\sigma-$algebra of Borel sets on $[\alpha,\beta]$, and λ is a Lebesgue measure. For a positive measurable a.e. bounded function g on $[\alpha,\beta]$ we define "$p-$norm" $\|g\|_p$ of g :

$$\|g\|_p=\begin{cases}\left(\int_\alpha^\beta g^p(t)dt\right)^{\frac{1}{p}}, & p\neq 0,-\infty,\infty;\\ \inf g, & p=-\infty;\\ \sup g, & p=\infty.\end{cases} \tag{5}$$

Further, we define

$$C_p=\begin{cases}\left(\frac{|a-f^{-1}(b)|^{p+1}}{p+1}\right)^{\frac{1}{p}}, & p\neq 0,-\infty,\infty;\\ |a-f^{-1}(b)|, & p=\infty;\\ 0, & p=-\infty.\end{cases} \tag{6}$$

THEOREM 2.1. *Let f be a differentiable and strictly increasing function on $[0,A]$ for $A>0$. If $f(0)=0$, $a\in[0,A]$ and $b\in[0,f(A)]$, and if f' is a.e. continuous with respect to Lebesgue measure on $[\alpha,\beta]$ then*

$$C_s\|f'\|_t\leqslant\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx-ab\leqslant C_q\|f'\|_p, \tag{7}$$

for all pairs (s,t) and (p,q) such that

$$s,t\in\langle-\infty,0\rangle\cup\langle 0,1\rangle,\ \tfrac{1}{s}+\tfrac{1}{t}=1,\ or\ (s,t)=(1,-\infty)\ or\ (s,t)=(-\infty,1)$$

and

$$1<p,g<\infty,\ \tfrac{1}{p}+\tfrac{1}{q}=1,\ or\ (p,q)=(1,\infty)\ or\ (p,q)=(\infty,1)$$

Inequality for the right-hand side in the (7) becomes equality if and only if $b=f(a)$ or

$$f(x)=\begin{cases}c\left(\frac{a^q}{q}-\frac{(a-x)^q}{q}\right), & 0\leqslant x<a;\\ c\left(\frac{a^q}{q}+\frac{(x-a)^q}{q}\right), & x\geqslant a\end{cases}$$

for some $c>0$ and $1<q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$.

Inequality for the left-hand side inequality in (7) becomes equality if and only if $b=f(a)$ or

$$f(x)=\begin{cases}c_1\left(\frac{a^s}{s}-\frac{(a-x)^s}{s}\right), & 0\leqslant x<a;\\ c_1\left(\frac{a^s}{s}+\frac{(x-a)^s}{s}\right), & x\geqslant a\end{cases}$$

for some $c_1>0$ and $1<s<\infty$, $\frac{1}{s}+\frac{1}{t}=1$.

Proof. We begin by changing variable $x = f^{-1}(y)$ in second integral below, using integration by parts and then using Fubini theorem:

$$
\begin{aligned}
\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx &= \int_0^a f(x)dx + \int_0^{f^{-1}(b)} yf'(y)dy \\
&= \int_0^a f(x)dx + bf^{-1}(b) - \int_0^{f^{-1}(b)} f(x)dx \\
&= bf^{-1}(b) + \int_{f^{-1}(b)}^a f(x)dx \\
&= ab + \int_{f^{-1}(b)}^a [f(x) - b]dx \\
&= ab + \int_{f^{-1}(b)}^a \left(\int_{f^{-1}(b)}^x f'(u)du \right) dx \\
(Fubini) &= ab + \int_{f^{-1}(b)}^a (a-u)f'(u)du. \qquad (8)
\end{aligned}
$$

From (8) we have

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab = \int_\alpha^\beta |a-u|f'(u)du, \qquad (9)$$

Now, we will first deduce right-hand inequality in (7) .
Firstly,

$$\int_\alpha^\beta |a-u|f'(u)du \leqslant |f^{-1}(b) - a| \int_\alpha^\beta f'(u)du = C_\infty \|f'\|_1. \qquad (10)$$

Secondly,

$$\int_\alpha^\beta |a-u|f'(u)dt \leqslant C_1 \|f'\|_\infty, \qquad (11)$$

Thirdly, using the Hölder inequality ([2], p.113)we have, for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_\alpha^\beta |a-u|f'(t)du \leqslant \left(\int_\alpha^\beta |a-u|^q \right)^{\frac{1}{q}} \left(\int_\alpha^\beta (f')^p(u)du \right)^{\frac{1}{p}} = C_q \|f'\|_p \qquad (12)$$

The left-hand side of (7) can be proved in a quite similar fashion, just using reverse Hölder inequality.

Equality conditions readily follow combining equality conditions in Hölder and Young inequalities.

Now, using Čebyšev's inequality, we will improve bounds given in Theorem 2.1. Čebyšev's theorem states([2], p.197):

THEOREM 2.2. *Let $f, g : [a,b] \to \mathbb{R}$ and $p : [a,b] \to \mathbb{R}_+$ be integrable functions. If f and g are monotonic in the same direction, then*

$$(b-a) \int_a^b p(x)f(x)g(x)dx \geqslant \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx, \qquad (13)$$

provided that integrals exists. If f and g are monotonous in opposite directions, then the reverse of the inequality in (13) *is valid. In both cases, equality in* (13) *holds iff either g or f is constant almost everywhere.*

THEOREM 2.3. *Let f be a differentiable and strictly increasing function on $[0,A]$ for $A > 0$ $f(0) = 0$, $a \in [0,A]$ and $b \in [0, f(A)]$.*

If f' is increasing on $[\alpha,\beta]$ and $f^{-1}(b) < a$, or if f' is decreasing on $[\alpha,\beta]$ and $f^{-1}(b) > a$, then

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab \leqslant \tfrac{(a-f^{-1}(b))}{2}(f(a) - b). \tag{14}$$

If f' is increasing on $[\alpha,\beta]$ and $f^{-1}(b) > a$, or if f' is decreasing on $[\alpha,\beta]$ and $f^{-1}(b) < a$ inequality in (14) *is reversed.*

Inequality in (14) *become equality if and only if $f(x) = cx$, $c > 0$ or $b = f(a)$.*

Proof. Making similar arguments as in the proof of Theorem 2.1, we have

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab = \int_{f^{-1}(b)}^a (a-x)f'(x)dx. \tag{15}$$

Now, we just apply Theorem 2.2 on integral on the right $\int_\alpha^\beta (a-x)f'(x)dx$. 15 with $f(x) = a - x$, $g(x) = f'(x)$ $p(x) = 1$.

REMARK 2.4. It is clear that Theorem 2.3 is generalization of Theorem 1.2, because condition for strict monotonicity of f' is dropped, another bound is added, and because the same upper a lower bounds for $\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab$ from Theorem 2.1 are also valid.

Particulary, Remark 2.4 has it's confirmation in the following example(see [1],[3]).

EXAMPLE 2.5. Prove that

$$9 < \int_0^3 \sqrt[4]{x^4+1}dx + \int_1^3 \sqrt[4]{x^4-1}dx < 9.0001. \tag{16}$$

We will take

$$f(x) = \sqrt[4]{x^4+1} - 1$$

and $a = 3$, $b = 2$. Then using

$$\int_0^3 f(x)dx = \int_0^3 \sqrt[4]{x^4+1}dx - 3$$

$$\int_0^2 f^{-1}(x)dx = \int_0^2 \sqrt[4]{(x+1)^4-1}dx = \int_1^3 \sqrt[4]{x^4-1}dx.$$

Because

$$f'(x) = \frac{x^3}{\sqrt[4]{(x^4+1)^3}},$$

using Theorem 2.3 for upper and Remark 2.4 for lower bound we have

$$(\min f')\frac{(a-f^{-1}(b))^2}{2} < \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab < \frac{(a-f^{-1}(b))}{2}(f(a)-b)$$

i.e.

$$9.000042866 < \int_0^3 \sqrt[4]{x^4+1}dx + \int_1^3 \sqrt[4]{x^4-1}dx < 9.000042868880$$

(upper bound here is better than in [3]).

Changing conditions of the theorems we can give new bounds .

THEOREM 2.6. *Let f be a differentiable and strictly increasing function on $[0,A]$ for $A>0$. If $f(0)=0$, $a\in[0,A]$ and $b\in[0,f(A)]$ and f' is convex on $[\alpha,\beta]$, then*

$$\begin{aligned} \frac{(a-f^{-1}(b))^2}{2} f'\left(\frac{a+2f^{-1}(b)}{3}\right) \leqslant \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab \leqslant \\ \leqslant \left(\frac{f'(a)}{6} + \frac{f'(f^{-1}(b))}{3}\right)(a-f^{-1}(b))^2. \end{aligned} \tag{17}$$

If f' is concave then inequalities in (17) are reversed.

Proof. Again, we first conclude

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab = \int_{f^{-1}(b)}^a (a-x)f'(x)dx. \tag{18}$$

After we change variable in integral on the right in (18):

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx - ab = \int_0^1 (a-f^{-1}(b))^2(1-x)f'(xa+(1-x)f^{-1}(b))dx. \tag{19}$$

After we apply discrete Jensen's inequality and and integration, we get desired conclusions for right-hand side of (17). Left-hand side inequality in (17) follows for integral version of Jensen's inequality([2], p.45):

$$\begin{aligned} \int_{f^{-1}(b)}^a (a-x)f'(x)dx \geqslant \frac{(a-f^{-1}(b))^2}{2} f'\left(\frac{\int_{f^{-1}(b)}^a (a-x)xdx}{\int_{f^{-1}(b)}^a (a-x)dx}\right) = \\ = \frac{(a-f^{-1}(b))^2}{2} f'\left(\frac{a+2f^{-1}(b)}{3}\right). \end{aligned}$$

REMARK 2.7. Using Theorem 2.6 we can get even sharper bounds in Example 2.5 : after we examine concavity of f' on $[\alpha,\beta]=[\sqrt[4]{80},3]$ we get

$$9.000042868058 < \int_0^3 \sqrt[4]{x^4+1}dx + \int_1^3 \sqrt[4]{x^4-1}dx < 9.000042868066.$$

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