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AN EXTENSION OF ORDER PRESERVING OPERATOR INEQUALITY

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Dedicated to Professor Jun Tomiyama
on his 77th birthday
with respect and affection

(*Communicated by F. Hansen*)

Abstract. We discuss an order preserving operator inequality and also we transform it into log majorization.

1. Introduction

A capital letter means a bounded linear operator on a Hilbert space H. An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

THEOREM LH. (Löwner-Heinz inequality, denoted by (LH) briefly).

$$\text{If } A \geqslant B \geqslant 0 \text{ holds, then } A^{\alpha} \geqslant B^{\alpha} \text{ for any } \alpha \in [0,1]. \tag{LH}$$

This was originally proved in [17] and then in [13]. Many nice proofs of (LH) are known. We mention [18] and [2, Theorem 4.2.1]). Although (LH) asserts that $A \geqslant B \geqslant 0$ *ensures* $A^{\alpha} \geqslant B^{\alpha}$ *for any* $\alpha \in [0,1]$, unfortunately $A^{\alpha} \geqslant B^{\alpha}$ *does not always hold for* $\alpha > 1$. The following result has been obtained from this point of view.

THEOREM A. *If* $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.

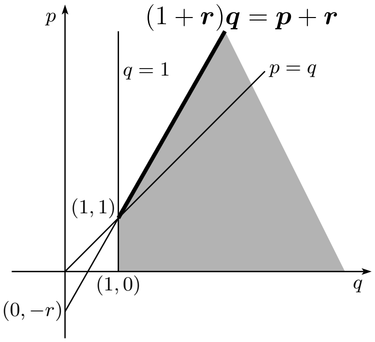

Figure 1

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The original proof of Theorem A is shown in [6], an elementary one-page proof is in [7] and alternative ones are in [3], [14]. It is shown in [19] that the conditions p, q and r in *Figure 1* are best possible.

THEOREM B. *If $A \geqslant B \geqslant 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geqslant 1$,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for $r \geqslant t$ and $s \geqslant 1$.

The original proof of Theorem B is in [8], and an alternative one is in [4], and also an elementary one-page proof is in [9]. Further extensions of Theorem B and related rtesults are in [10], [11], [12] and [15]. It is originally shown in [20] that the exponent value $\dfrac{1-t+r}{(p-t)s+r}$ of the right hand of (1.1) is best possible and alternative ones are in [5], [21]. It is known that the operator inequality (1.1) interpolates Theorem A and an inequality equivalent to the main result of Ando-Hiai log majorization [1] by the parameter $t \in [0,1]$.

In this paper, we show an extension of (1.1) as follows:

If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and p_1, p_2, p_3, $p_4 \geqslant 1$, then

$$A^{1-t+r} \geqslant \left\{A^{\frac{r}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $r \geqslant t$.

We remark that the result stated above yields Theorem B by putting $p_2 = p_3 = 1$.

We discuss an application of our result to log majorization as follows;

(i) *for every $A > 0$, $B \geqslant 0$, $t \in [0,1]$ and p_1, p_2, p_3, $p_4 \geqslant 1$ and $r \geqslant t$,*

$$(A\sharp_{\frac{1}{p_1}}B)^{h} \underset{(\log)}{\succ} A^{1-t+r}\sharp_{\beta}\left\{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\right\}$$

holds, where β and h are as follows;

$$h = \frac{p_1p_2p_4p_4(1-t+r)}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r} \quad \text{and} \quad \beta = \frac{h}{p_1p_2p_3p_4}.$$

This result (i) yields the following known result (ii);

(ii) *for every $A > 0$, $B \geqslant 0$, $0 \leqslant \alpha \leqslant 1$ and each $t \in [0,1]$*

$$(A\sharp_{\alpha}B)^{h} \underset{(\log)}{\succ} A^{1-t+r}\sharp_{\beta}(A^{1-t}\natural_{s}B)$$

holds for $s \geqslant 1$ and $r \geqslant t$, where $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$ and $\beta = \dfrac{h}{s}\alpha$

and also (ii) implies (iii);

(iii) *for every $A, B \geqslant 0$, $0 \leqslant \alpha \leqslant 1$*

$$(A\sharp_{\alpha}B)^{r} \underset{(\log)}{\succ} A^{r}\sharp_{\alpha}B^{r} \quad \text{for } r \geqslant 1.$$

The last result is very useful and fundamental result in log majorization by Ando-Hiai [1].

2. An order preserving operator inequality

THEOREM 2.1. *If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and p_1, p_2, p_3, $p_4 \geqslant 1$, then the following inequality holds,*

$$A \geqslant \left\{A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{[\{(p_1-t)p_2+t\}p_3-t]p_4+t}}. \tag{2.1}$$

LEMMA A. [8, Lemma 1]. *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^{\lambda} = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

Proof of Theorem 2.1. By putting $r = t$ in (1.1) of Theorem B, we have;
if $A \geqslant B \geqslant 0$ with $A > 0$, then for $t \in [0,1]$

$$\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{\frac{1}{(p_1-t)p_2+t}} \leqslant A \quad \textit{for any } p_1 \geqslant 1 \textit{ and } p_2 \geqslant 1. \tag{2.2}$$

First step. In case $2 \geqslant p_4 \geqslant 1$.
We recall that (2.2) can be described as

$$C^{\frac{1}{q[2]}} \leqslant A \quad \text{where } C = A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}} \text{ and } q[2] = (p_1-t)p_2+t. \tag{2.2'}$$

$(2.2')$ yields the following (2.3)

$$A^{-t} \leqslant C^{\frac{-t}{q[2]}} \quad \text{for any } t \in [0,1] \tag{2.3}$$

by LH and taking inverses of both sides. Also let $q[4]$ be defined by as follows:

$$q[4] = [\{(p_1-t)p_2+t\}p_3-t]p_4+t = (q[2]p_3-t)p_4+t. \tag{2.4}$$

Then we have

$$\begin{aligned}
&\left\{A^{\frac{t}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{q[4]}} \qquad (2.5)\\
&\quad = \left\{A^{\frac{t}{2}}[A^{\frac{-t}{2}}C^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{q[4]}}\\
&\quad = \left\{C^{\frac{p_3}{2}}\left(C^{\frac{p_3}{2}}A^{-t}C^{\frac{p_3}{2}}\right)^{p_4-1}C^{\frac{p_3}{2}}\right\}^{\frac{1}{q[4]}} \quad \text{by Lemma A}\\
&\quad \leqslant \left\{C^{\frac{p_3}{2}}\left(C^{\frac{p_3}{2}}C^{\frac{-t}{q[2]}}C^{\frac{p_3}{2}}\right)^{p_4-1}C^{\frac{p_3}{2}}\right\}^{\frac{1}{q[4]}}\\
&\quad = \left(C^{p_3+(p_3-\frac{t}{q[2]})(p_4-1)}\right)^{\frac{1}{q[4]}}\\
&\quad = \left(C^{\frac{(q[2]p_3-t)p_4+t}{q[2]}}\right)^{\frac{1}{q[4]}}\\
&\quad = C^{\frac{1}{q[2]}} \quad \text{by } (2.4)\\
&\quad \leqslant A \quad \text{by } (2.2')
\end{aligned}$$

and the first inequality holds by (2.3) and LH since $\frac{1}{q[4]}$, $p_4 - 1 \in [0,1]$ in case p_1, p_2, $p_3 \geqslant 1$ and $2 \geqslant p_4 \geqslant 1$.

Second step. In (2.5), put $A_1 = A$ and

$$B_1 = \left\{A^{\frac{t}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{q[4]}}$$
$$= \left\{A^{\frac{t}{2}}[A^{\frac{-t}{2}}C^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{q[4]}}.$$

Then $A_1 \geqslant B_1$ holds *for any* $2 \geqslant p_4 \geqslant 1$ by (2.5). Repeating (2.5) for $A_1 \geqslant B_1$ with $A_1 > 0$, then we have

$$\left\{A_1^{\frac{t'}{2}}[A_1^{\frac{-t'}{2}}\{A_1^{\frac{t'}{2}}(A_1^{\frac{-t'}{2}}B_1^{p'_1}A_1^{\frac{-t'}{2}})^{p'_2}A_1^{\frac{t'}{2}}\}^{p'_3}A_1^{\frac{-t'}{2}}]^{p'_4}A_1^{\frac{t'}{2}}\right\}^{\frac{1}{q'[4]}} \leqslant A_1 \quad \text{for any} \quad 2 \geqslant p'_4 \geqslant 1 \tag{2.6}$$

and $p'_1, p'_2, p'_3 \geqslant 1$ and $t' \in [0,1]$, where $q'[4] = [\{(p'_1 - t')p'_2 + t'\}p'_3 - t']p'_4 + t'$. In (2.6) take p'_1, p'_2, p'_3 and t' as follows;

$$p'_1 = q[4] = [\{(p_1 - t)p_2 + t\}p_3 - t]p_4 + t, \qquad p'_2 = p'_3 = 1 \quad \text{and} \quad t' = t. \tag{2.7}$$

Then we have

$$B_1^{p'_1} = A^{\frac{t}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}} = A^{\frac{t}{2}}[A^{\frac{-t}{2}}C^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}} \tag{2.8}$$

$$q'[4] = [\{(p_1 - t)p_2 + t\}p_3 - t]p_4p'_4 + t \tag{2.9}$$

and (2.6),(2.8) and (2.9) ensure the following (2.10)

$$\begin{aligned}
&\left\{A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}A^{\frac{t}{2}}(A^{\frac{-t}{2}}C^{p_3}A^{\frac{-t}{2}})^{p_4}A^{\frac{t}{2}}A^{\frac{-t}{2}}\right]^{p'_4}A^{\frac{t}{2}}\right\}^{\frac{1}{[\{(p_1-t)p_2+t\}p_3-t]p_4p'_4+t}} \\
&=\left\{A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4p'_4}A^{\frac{t}{2}}\right\}^{\frac{1}{[\{(p_1-t)p_2+t\}p_3-t]p_4p'_4+t}} \\
&\leqslant A \quad \text{holds for any } 4 \geqslant p_4p'_4 \geqslant 1
\end{aligned} \tag{2.10}$$

and repeating this process from (2.5) to (2.10), (2.1) holds for *any* $p_4 \geqslant 1$. □

3. An extension of Theorem B

THEOREM 3.1. *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$ and p_1, p_2, p_3, $p_4 \geqslant 1$,*

$$A^{1-t+r} \geqslant \left\{A^{\frac{r}{2}}\left[A^{\frac{-t}{2}}\left\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\right\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}} \tag{3.1}$$

holds for $r \geqslant t$.

REMARK 3.1. Theorem 3.1 yields Theorem B by putting $p_2 = p_3 = 1$.

Proof of Theorem 3.1. In (2.1) of theorem 2.1, put $A_1 = A$ and

$$B_1 = \left\{A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}\left\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\right\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{[\{(p_1-t)p_2+t\}p_3-t]p_4+t}}$$

Then $A_1 \geqslant B_1$ by (2.1) holds for $t \in [0,1]$ and p_1, p_2, p_3, $p_4 \geqslant 1$, by applying Theorem A,

$$A_1^{1+r_1} \geqslant (A_1^{\frac{r_1}{2}}B_1^{s_1}A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{s_1+r_1}} \quad \text{holds for } s_1 \geqslant 1 \text{ and } r_1 \geqslant 0. \tag{3.2}$$

In (3.2) we have only to put $r_1 = r - t \geqslant 0$ and $s_1 = q[4] \geqslant 1$ to obtain (3.1).

4. Transformation of Theorem 3.1 into Log Majorization

Following after [1], let us define the *log majorization* for positive semidefinte matrices $A, B \geqslant 0$, denoted by $A \underset{(\log)}{\succ} B$ if

$$\prod_{i=1}^{k}\lambda_i(A) \geqslant \prod_{i=1}^{k}\lambda_i(B) \quad \text{for} \quad k = 1,2,...,n-1$$

and

$$\prod_{i=1}^{n}\lambda_i(A) = \prod_{i=1}^{n}\lambda_i(B) \quad \text{i.e., } \det A = \det B$$

where $\lambda_1(A) \geqslant \lambda_2(A) \geqslant ... \geqslant \lambda_n(A)$ and $\lambda_1(B) \geqslant \lambda_2(B) \geqslant ... \geqslant \lambda_n(B)$ are the eigenvalues of A and B, respectively, arranged in decreasing order. When $0 \leqslant \alpha \leqslant 1$, α-power mean of positive invertible matrices $A, B > 0$ is defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ in [16].

Further, $A\sharp_\alpha B$ for $A, B \geqslant 0$ is defined by $A\sharp_\alpha B = \lim_{\epsilon\downarrow 0}(A+\epsilon I)\sharp_\alpha(B+\epsilon I)$.

For the sake of convenience for symbolic expression, we defined $A\natural_s B$ in [8], for any real number $s \geqslant 0$ and for $A > 0$ and $B \geqslant 0$, by the following

$$A\natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^s A^{\frac{1}{2}}$$

$A\natural_s B$ in case $0 \leqslant \alpha \leqslant 1$ just coincides with the usual α-power mean $A\sharp_\alpha B$.

THEOREM 4.1. *For every $A > 0$, $B \geqslant 0$, $t \in [0,1]$ and p_1, p_2, p_3, $p_4 \geqslant 1$ and $r \geqslant t$,*

$$(A\sharp_{\frac{1}{p_1}}B)^h \underset{(\log)}{\succ} A^{1-t+r}\sharp_\beta\left\{A^{\frac{1-t}{2}}\left[A^{\frac{t}{2}}\{A^{\frac{-t}{2}}(A^{\frac{t-1}{2}}BA^{\frac{t-1}{2}})^{p_2}A^{\frac{-t}{2}}\}^{p_3}A^{\frac{t}{2}}\right]^{p_4}A^{\frac{1-t}{2}}\right\} \tag{4.1}$$

holds, that is,

$$(A\sharp_{\frac{1}{p_1}}B)^h \underset{(\log)}{\succ} A^{1-t+r}\sharp_\beta\left\{A^{1-t}\natural_{p_4}\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\right\} \tag{4.2}$$

holds, where β and h are as follows:

$$h = \frac{p_1p_2p_3p_4(1-t+r)}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r} \quad \text{and} \quad \beta = \frac{h}{p_1p_2p_3p_4}.$$

Proof. In the same way in the proof of [1, Theorem 2.1], by arranging the order of homogeneity in (4.1), to prove (4.1) we have only to show that $I \geqslant A\sharp_{\frac{1}{p_1}}B$, equivalently, $A^{-1} \geqslant (A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{p_1}}$ ensures the following inequality

$$I \geqslant A^{1-t+r}\sharp_\beta\Big\{A^{\frac{1-t}{2}}\big[A^{\frac{t}{2}}\{A^{\frac{-t}{2}}(A^{\frac{t-1}{2}}BA^{\frac{t-1}{2}})^{p_2}A^{\frac{-t}{2}}\}^{p_3}A^{\frac{t}{2}}\big]^{p_4}A^{\frac{1-t}{2}}\Big\},$$

for $t \in [0,1]$ and p_1, p_2, p_3, $p_4 \geqslant 1$ and $r \geqslant t$, equivalently,

$$A^{-1+t-r} \geqslant \Big\{A^{\frac{-r}{2}}\big[A^{\frac{t}{2}}\{A^{\frac{-t}{2}}(A^{\frac{t-1}{2}}BA^{\frac{t-1}{2}})^{p_2}A^{\frac{-t}{2}}\}^{p_3}A^{\frac{t}{2}}\big]^{p_4}A^{\frac{-r}{2}}\Big\}^\beta. \tag{4.3}$$

Put $A_1 = A^{-1}$ and $B_1 = (A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{p_1}}$. By applying Theorem 3.1, we have

$$A_1^{1-t+r} \geqslant \left\{A_1^{\frac{r}{2}}\left[A_1^{\frac{-t}{2}}\{A_1^{\frac{t}{2}}(A_1^{\frac{-t}{2}}B_1^{p_1}A_1^{\frac{-t}{2}})^{p_2}A_1^{\frac{t}{2}}\}^{p_3}A_1^{\frac{-t}{2}}\right]^{p_4}A_1^{\frac{r}{2}}\right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}} \tag{4.4}$$

that is,

$$A^{-1+t-r} \geqslant \Big\{A^{\frac{-r}{2}}\big[A^{\frac{t}{2}}\{A^{\frac{-t}{2}}(A^{\frac{t-1}{2}}BA^{\frac{t-1}{2}})^{p_2}A^{\frac{-t}{2}}\}^{p_3}A^{\frac{t}{2}}\big]^{p_4}A^{\frac{-r}{2}}\Big\}^\beta$$

holds, that is, we have (4.3) as desired, where h and are as follows:

$$h = \frac{p_1p_2p_3p_4(1-t+r)}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r} \quad \text{and} \quad \beta = \frac{h}{p_1p_2p_3p_4}.$$

□

COROLLARY 4.2. *For every $A > 0$, $B \geqslant 0$, and p_1, p_2, p_3, $p_4 \geqslant 1$ and $r \geqslant 1$,*

$$(A\sharp_{\frac{1}{p_1}}B)^h \underset{(\log)}{\succ} A^r\sharp_\beta[A^{\frac{1}{2}}(A^{\frac{-1}{2}}B^{p_2}A^{\frac{-1}{2}})^{p_3}A^{\frac{1}{2}}]^{p_4}, \tag{4.5}$$

holds, that is,

$$(A\sharp_{\frac{1}{p_1}}B)^h \underset{(\log)}{\succ} A^r\sharp_\beta(A\natural_{p_3}B^{p_2})^{p_4} \tag{4.6}$$

holds, where β and h are as follows:

$$h = \frac{p_1p_2p_3p_4r}{[\{(p_1-1)p_2+1\}p_3-1]p_4+r} \quad \textit{and} \quad \beta = \frac{h}{p_1p_2p_3p_4}.$$

Proof. We have only to put $t = 1$ in Theorem 4.1. □

Theorem 4.1 yields the following Theorem C by replacing $\frac{1}{p_1}$ by $\alpha \in [0,1]$, $p_3 = p_4 = 1$ and $p_2 = s \geqslant 1$.

THEOREM C. [8]. *For every* $A > 0$, $B \geqslant 0$, $0 \leqslant \alpha \leqslant 1$ *and each* $t \in [0,1]$

$$(A \sharp_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$$

holds for $s \geqslant 1$ *and* $r \geqslant t$, *where*

$$h = \frac{(1-t+r)s}{(1-\alpha t)s + \alpha r} \quad \textit{and} \quad \beta = \frac{h}{s}\alpha.$$

We state the following two known results.

THEOREM D. [8]. *For every* $A, B \geqslant 0$, $0 \leqslant \alpha \leqslant 1$

$$(A \sharp_\alpha B)^h \underset{(\log)}{\succ} A^r \sharp_{\frac{h}{s}\alpha} B^s \quad \textit{for } r \geqslant 1 \textit{ and } s \geqslant 1.$$

where $h = [\alpha s^{-1} + (1-\alpha) r^{-1}]^{-1}$ *(* h *is the harmonic mean of* s *and* r *).*

THEOREM E. [1]. *For every* $A, B \geqslant 0$, $0 \leqslant \alpha \leqslant 1$

$$(A \sharp_\alpha B)^r \underset{(\log)}{\succ} A^r \sharp_\alpha B^r \quad \textit{for } r \geqslant 1.$$

We remark that Theorem E is very useful and fundamental result in log majorization and Theorem D yields Theorem E putting $r = s$. Theorem C yields Theorem D putting $t = 1$ and also Corollary 4.2 implies Theorem D putting $p_3 = 1$ and replacing $p_2 p_4 \geqslant 1$ by $s \geqslant 1$ and replacing $\frac{1}{p_1}$ by $\alpha \in [0,1]$.

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