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AN EXTENSION OF THE FUGLEDE–PUTNAM'S THEOREM TO CLASS A OPERATORS

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Abstract. The familiar Fuglede-Putnam's Theorem is as follows (see [5], [9] and [11]): If A and B are normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$. In this paper, the hypothesis on A and B can be relaxed by using a Hilbert-Schmidt operator X: Let A be a class A operator and let B^* be an invertible class A operator such that $AX = XB$ for a Hilbert-Schmidt operator X. Then $A^*X = XB^*$. As a consequence of this result, we obtain that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel. Some properties of log-hyponormal operators are also given.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H. For any operator A in $B(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*,A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self commutator of A), and consider the following standard definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geqslant 0$, p-hyponormal if $(|A|^{2p} - |A^*|^{2p}) \geqslant 0\,(0 < p \leqslant 1)$. A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$\log(A^*A) \geqslant \log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [18]. However it is very interesting that we may regards log-hyponormal operators as 0-hyponormal operators [18, 19] . The idea of log-hyponormal operator is due to Ando [3] and the first paper in which log-hyponormality appeared is [10]. See ([2, 18, 19, 21] for properties of log-hyponormal operators.

We say that an operator $A \in B(H)$ belongs to the class A if $|A^2| \geqslant |A|^2$. Class A was first introduced by Furuta-Ito-Yamazaki [9] as a subclass of paranormal operators which includes the classes of p-hyponormal and log-hyponormal operators. The following theorem is one of the results associated with class A.

THEOREM 1.1. [9]
(1) Every log-hyponormal operator is a class A operator.
(2) Every class A operator is a paranormal operator.

The familiar Fuglede-Putnam's theorem is as follows (see [5], [9] and [11]):

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THEOREM 1.2. *If A and B are normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

S. K. Berberian [4] relaxes the hypothesis on A and B in Theorem 1.1 as the cost of requiring X to be Hilbert-Schmidt class. H. K. Cha [6] showed that the hyponormality in the result of Berberian [4] can be replaced by the quasihyponormality of A and B^* under some additional conditions. Lee ([14], Theorem 4) showed that the quasihyponormalty in the above result can replaced by the (p,k)-quasihyponormality of A and B^* with the additional condition $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leqslant 1$. In [15] the author showed that Lee's result remains true without the additional condition $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leqslant 1$. In this paper we will show that the (p,k)-quasihyponormality can be replaced by A and B^* class A operators. Let $\delta_{A,B}$ be the generalized derivation defined on $B(H)$ by $\delta_{A,B}(X) = AX - XB$. As a consequence of this result, we obtain that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel.

Let $T \in B(H)$ be compact, and let $s_1(T) \geqslant s_2(T) \geqslant ... \geqslant 0$ denote the singular values of T , i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. We say that the operator T belongs to the Schatten p-class C_p if $\|T\|_p = [\sum_{i=1}^{\infty} s_j(T)^p]^{\frac{1}{p}} = [\mathrm{tr}\,|T|^p]^{\frac{1}{p}} < \infty, 1 \leqslant p < \infty$, where tr denotes the trace functional. Hence $C_1(H)$ is the trace class, $C_2(H)$ is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with $\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$ denoting the usual operator norm. For the general theory of the Schatten p- classes the reader is referred to ([16], [17]).

2. Main results

LEMMA 2.1. *Let $A,B \in B(H)$. If $\tau X = AXB$ on $C_2(H)$ with $A > 0$ and $B > 0$, then* $\log \tau X = \log AX + X \log B$.

Proof. It is well-known that For $X > 0$,

$$\log X = \lim_{p \downarrow 0} \frac{X^p - 1}{p}.$$

It is easy to see that $\tau X > 0$ and $\tau^p X = A^p X B^p$.

$$\begin{aligned} \log \tau X &= \lim_{p \downarrow 0} \frac{\tau^p - 1}{p} X = \lim_{p \downarrow 0} \frac{1}{p}(A^p X B^p - X) \\ &= \lim_{p \downarrow 0} \frac{A^p - 1}{p} X B^p + \lim_{p \downarrow 0} X \frac{B^p - 1}{p} \\ &= \log AX + X \log B \end{aligned}$$

THEOREM 2.1. *Let A and B be operators in $B(H)$. If A and B^* are* log-*hyponormal operators then, the operator τ is also* log*-hyponormal.*

Proof. Since $|\tau|X = |A|X|B^*|$, we have

$$\log|\tau|X = \log|A|X + X\log|B^*|, \log|\tau^*|X = \log|A^*|X + X\log|B|$$

$$(\log|\tau| - \log|\tau^*|)X = (\log|A| - \log|A^*|)X + X(\log|B^*| - \log|B|).$$

Hence,

$$\begin{aligned} &\langle(\log|\tau| - \log|\tau^*|)X, X\rangle \\ &= \operatorname{tr}(X^*(\log|A| - \log|A^*|)X + X^*X(\log|B^*| - \log|B|)) \\ &= \operatorname{tr}(X^*(\log|A| - \log|A^*|)X) + \operatorname{tr}(|X|(\log|B^*| - \log|B|)|X|) \geqslant 0. \end{aligned}$$

This completes the proof.

COROLLARY 2.1. *Let τ be as above. Then*

$$\begin{aligned} ||\log|\tau| - \log|\tau^*||| &= r(\log|\tau| - \log|\tau^*|) \\ &= ||\log|A| - \log|A^*||| + ||\log|B^*| - \log|B||| \\ &\leqslant \int\int_{re^{i\theta}\in\sigma(A)} r^{-1}drd\theta + \leqslant \int\int_{re^{i\theta}\in\sigma(B^*)} r^{-1}drd\theta. \end{aligned}$$

Proof. Since $\log|\tau| - \log|\tau^*| \geqslant 0$,

$$||\log|\tau| - \log|\tau^*||| = r(\log|\tau| - \log|\tau^*|).$$

It is well-known that the spectrum of the generalized derivation

$$\delta_{C,D} = CX + XD$$

is $\{z + w : z \in \sigma(C), w \in \sigma(D)\}$. Therefore

$$\sigma(\log|\tau| - \log|\tau^*|) = \{z + w : z \in \sigma(\log|A| - \log|A^*|), w \in \sigma(\log|B^*| - \log|B|)\},$$

and

$$r(\log|\tau| - \log|\tau^*|) = ||\log|A| - \log|A^*||| + ||\log|B^*| - \log|B|||.$$

The last inequality follows from Tanahashi's result [18].

We say that the operator $A \in B(H)$ belongs to the class A if $|A^2| \geqslant |A|^2$. Class A was first introduced by Furuta-Ito-Yamazaki [9] as a subclass of paranormal operators which includes the class of p-hyponormal and $\log$-hyponormal operators.

LEMMA 2.2. [12, Theorem 3.2] *Let A and B be operators in $B(H)$. Then A and B^* are class A operators if and only if the operator τ is a class A operator.*

LEMMA 2.3. [23, Proposition 3] *If $A \in B(H)$ is an invertible paranormal operator, then A^{-1} is also paranormal. In particular this holds for a class A operator.*

Now we are ready to extend Putnam-Fuglede's theorem to class A operators.

THEOREM 2.2. *Let A be a class A operator and B^* be an invertible class A operator. If $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.*

Proof. Recall that if A is a class A operator, then the nonzero eigenvalues of A are normal eigenvalues (i.e., if

$$\lambda \in \sigma_p(A) \setminus \{0\}, \text{ then } \lambda \in \sigma_p(A^*)). \tag{$*$}$$

Let τ be defined on $C_2(H)$ by $\tau Y = AYB^{-1}$ for all $Y \in C_2(H)$. Since B^* is an invertible class A operator, Lemma 2.3 implies that $(B^*)^{-1}$ is also a class A operator. Then it follows from Lemma 2.2 that τ is a class A operator, furthermore, $\tau X = AXB^{-1} = XBB^{-1} = X$ and so, X is an eigenvector of τ. Now by applying (*) we get $\tau^* X = A^*X(B^*)^{-1} = X$, that is, $A^*X = XB^*$ and the proof is achieved.

THEOREM 2.3. *Let A, B be operators in $B(H)$ and $S \in C_2(H)$. Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 \tag{2.1}$$

and

$$\left\|\delta^*_{A,B}(X) + S\right\|_2^2 = \left\|\delta^*_{A,B}(X)\right\|_2^2 + \|S\|_2^2 \tag{2.2}$$

if and only if $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$, for all $X \in C_2(H)$.

Proof. It is well known that the Hilbert-Schmidt class $C_2(H)$ is a Hilbert space under the inner product

$$\langle Y, Z\rangle = \mathrm{tr}(Z^*Y) = \mathrm{tr}(YZ^*).$$

Note that

$$\begin{aligned}\|\delta_{A,B}(X) + S\|^2 &= \|\delta_{A,B}(X)\|^2 + \|S\|^2 + 2\mathrm{Re}(\delta_{A,B}(X), S) \\ &= \|\delta_{A,B}(X)\|^2 + \|S\|^2 + 2\mathrm{Re}(X, \delta^*_{A,B}(S)\end{aligned}$$

and

$$\|\delta^*_{A,B}(X) + S\|^2 = \|\delta^*_{A,B}(X)\|^2 + \|S\|^2 + 2\mathrm{Re}(X, \delta_{A,B}(S).$$

Hence by the equality $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$ we obtain (2.1) and (2.2).

COROLLARY 2.2. *Let A, B be operators in $B(H)$ and $S \in C_2(H)$. Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2$$

and

$$\left\|\delta^*_{A,B}(X) + S\right\|_2^2 = \left\|\delta^*_{A,B}(X)\right\|_2^2 + \|S\|_2^2$$

if and only if A is class A operator and B^* is an invertible class A operator.

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