Mathematical
Inequalities
& Applications
Volume 13, Number 1 (2010), 63–81

WEIGHT CHARACTERIZATION OF THE TRACE INEQUALITY FOR THE GENERALIZED RIEMANN–LIOUVILLE TRANSFORM IN $L^{p(x)}$ SPACES

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Dedicated to Professor Josip Pecaric on the occasion of his 60th birthday

(*Communicated by L.-E. Persson*)

Abstract. Necessary and sufficient conditions on a weight governing the trace inequality for the Riemann-Liouville transform with variable parameter $R_{\alpha(x)}$ in $L^{p(x)}$ spaces are established provided that p and q satisfy the log-Hölder continuity condition. Weighted criteria for the compactness of $R_{\alpha(x)}$ from $L^{p(x)}$ to $L_v^{q(x)}$ are also derived.

1. Introduction

Our aim is to find criteria for the Riemann-Liouville operator

$$R_{\alpha(x)}f(x) = \int_0^x f(t)(x-t)^{\alpha(x)-1}dt, \quad x > 0,$$

to be bounded/compact from $L^{p(\cdot)}(I)$ to $L_v^{q(\cdot)}(I)$, where

$$1 < \inf_I p \leqslant p(x) \leqslant q(x) \leqslant \sup_I q < \infty, \quad \inf_I(\alpha - 1/p) > 0,$$

provided that p and q satisfy the weak Lipschitz (log-Hölder continuity) condition. Here I denotes the interval $[0,a)$ ($0 < a \leqslant \infty$). When p and q are general measurable functions, we obtain integral-type sufficient condition (which is also necessary for constant p and q) guaranteeing the trace inequality for $R_{\alpha(x)}$ in variable exponent Lebesgue spaces.

The space $L^{p(\cdot)}$ is the special case of the Musielak-Orlicz space. The basis of the variable exponent Lebesgue spaces were developed by W. Orlicz and J. Musielak (see [37], [33] and [34]).

Criteria for the boundedness/compactness of the operator

$$R_\gamma f(x) = \int_0^x f(t)(x-t)^{\gamma-1}dt, \quad x > 0,$$

Mathematics subject classification (2010): 46E30, 47B34.

Keywords and phrases: Lebesgue spaces with variable exponent, Riemann-Liouville operator, trace inequality, boundedness, compactness.

Paper MIA-13-06

from $L^p(R_+)$ to $L^q_v(R_+)$ (γ, p and q are constants, and $\gamma > 1/p$) were established in [31] and [38]. Solution of the two-weight problem for the operator R_γ when γ is constant and is more than 1, was given in the papers [29], [47] (see also [36]). For two-weight Hardy-type inequalities we refer to [15], [30], [27], [26] and references therein.

The boundedness of classical integral operators in $L^{p(\cdot)}$ spaces has been investigated in the papers [1], [2], [3], [4], [5], [7], [35], [39], [40] (see also [16], [41] and references therein).

Two-weight inequalities for the Hardy operator

$$Hf(x) = \int_0^x f(t)dt, \quad x > 0,$$

in $L^{p(\cdot)}$ spaces were derived in [8], [10] and [24] (in fact in [24] two-weight problem for the Volterra-type operator involving the kernel $(x-y)^{\gamma-1}$ only for $\gamma > 1$ was studied in $L^{p(x)}$ spaces). Further, for weighted estimates for maximal functions, singular integrals and potentials in $L^{p(\cdot)}$ spaces we refer to [6], [11], [12], [18], [19], [20], [21], [22], [23], [42], [43] (see also [16], [41]).

Solution of the the two-weight problem in the non-diagonal case for one-sided potentials in classical Lebesgue spaces was given in [9], [13], [28]. Finally we notice that the criteria established in [28] are of Sawyer type.

Throughout the paper constants (often different constants in the same series of inequalities) will mainly be denoted by c.

2. Preliminaries

Let Ω be a domain in $\mathbf{R}^n$. We denote

$$p_-(E) := \inf_E p, \qquad p_+(E) := \sup_E p$$

for a measurable set $E \subseteq \Omega$. Suppose that p is a measurable function on Ω and $1 < p_-(\Omega) \leqslant p(x) \leqslant p_+(\Omega) < \infty$. Denote by ρ a weight function on Ω, i.e. ρ is an almost everywhere positive measurable function on Ω. We say that a measurable function f on Ω belongs to $L^{p(\cdot)}_\rho(\Omega)$ (or to $L^{p(x)}_\rho(\Omega)$) if

$$S_{p(\cdot),\rho}(f) = \int\limits_\Omega \big|f(x)\rho(x)\big|^{p(x)}dx < \infty.$$

It is is a Banach space with the norm (see e.g. [16], [25], [39], [41], [48])

$$\|f\|_{L^{p(\cdot)}_\rho(\Omega)} = \inf\big\{\lambda > 0 : S_{p(\cdot),\rho}\big(f/\lambda\big) \leqslant 1\big\}.$$

If $\rho \equiv 1$, then we use the symbol $L^{p(\cdot)}(\Omega)$ (resp. $S_{p(\cdot)}$) instead of $L^{p(\cdot)}_\rho(\Omega)$ (resp. $S_{p(\cdot),\rho}$). It is clear that $\|f\|_{L^{p(\cdot)}_\rho(\Omega)} = \|f\rho\|_{L^{p(\cdot)}(\Omega)}$.

Let $\mathscr{P}(\Omega)$ be the class of all measurable functions p, $p:\Omega\to R$, such that the Hardy-Littlewood maximal operator

$$M_\Omega f(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_{Q\cap\Omega}|f(y)|dy,\quad x\in\Omega,$$

where the supremum is taken over cubes Q containing x and satisfying $|Q\cap\Omega|>0$, is bounded in $L^{p(\cdot)}(\Omega)$.

In the sequel we will denote by Z and Z_- the set of all integers and the set of non-positive integers respectively.

To prove the main results we need some statements:

PROPOSITION A. ([25], [39]) *Let E be a measurable subset of Ω. Suppose that $1<p_-(E)\leqslant p_+(E)<\infty$. Then the following inequalities hold:*

$$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)}\leqslant S_p(f\chi_E)\leqslant\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)},\quad \|f\|_{L^{p(\cdot)}(E)}\leqslant 1;$$

$$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)}\leqslant S_p(f\chi_E)\leqslant\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)},\quad \|f\|_{L^{p(\cdot)}(E)}\geqslant 1;$$

$$\Big|\int_E f(x)g(x)dx\Big|\leqslant\Big(\frac{1}{p_-(E)}+\frac{1}{(p_+(E))'}\Big)\|f\|_{L^{p(\cdot)}(E)}\,\|g\|_{L^{p'(\cdot)}(E)},$$

where $p'(x)=\frac{p(x)}{p(x)-1}$ and $1<p_-(E)\leqslant p(x)\leqslant p_+(E)<\infty$.

PROPOSITION A'. ([25], [39], [48]) *Let $1\leqslant r(x)\leqslant p(x)$ and let E be a bounded subset of Ω. Then there exists a positive constant c depending only on E, r and p such that*

$$\|f\|_{L^{r(\cdot)}(E)}\leqslant c\|f\|_{L^{p(\cdot)}(E)}.$$

DEFINITION. We say that p satisfies the weak Lipschitz (log-Hölder continuity) condition on $E\subset\Omega$ $(p\in WL(E))$, if there there is a positive constant A such that for all x and y in E with $0<|x-y|<1/2$ the inequality

$$|p(x)-p(y)|\leqslant A/(-\ln|x-y|)$$

holds.

The next result was obtained in [4].

THEOREM A. *Let Ω be a bounded open set in R^n and let $1<p_-(\Omega)\leqslant p_+(\Omega)<\infty$. Then the operator M_Ω is bounded in $L^{p(\cdot)}(\Omega)$ if $p\in WL(\Omega)$.*

The next lemma was proved in [4].

LEMMA 1. *Let I be an interval in R_+. Then $p\in WL(I)$ if and only if there exists a positive constant C such that*

$$|J|^{p_-(J)-p_+(J)}\leqslant c$$

for all intervals $J\subseteq I$ of I with $|J|>0$.

Let

$$E_k:=[2^k,2^{k+1});\quad I_k:=[2^{k-1},2^{k+1}).$$

We shall need a slight modification of Theorem 2 from [24] (see also [11] for the case $p(x)=q(x)$).

LEMMA 2. *Let* $1 < p_-(R_+) \leqslant p(x) \leqslant q(x) \leqslant q_+(R_+) < \infty$ *and let* $p, q \in WL(R_+)$ *and let* $p(x) \leqslant q(x)$. *Suppose that* $p(x) \equiv p_c = \text{const}$, $q(x) \equiv q_c = \text{const}$ *when* $x > a$ *for some positive number* a. *Then there exists a positive constant* c *such that*

$$\sum_k \|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)} \|g\chi_{I_k}\|_{L^{q'(\cdot)}(R_+)} \leqslant c\|f\|_{L^{p(\cdot)}(R_+)} \|g\|_{L^{q'(\cdot)}(R_+)}$$

for all $f \in L^{p(\cdot)}(R_+)$ *and* $g \in L^{q'(\cdot)}(R_+)$.

Proof. For simplicity assume that $a = 1$. Let us split the sum as follows:

$$\sum_i \|f\chi_{I_i}\|_{L^{p(\cdot)}(R_+)} \|g\chi_{I_i}\|_{L^{q'(\cdot)}(R_+)} = \sum_{i \leqslant 2} + \sum_{i > 2} := J_1 + J_2.$$

Since $p(x) = p_c = \text{const}$, $q(x) \equiv q_c \equiv \text{const}$ on the set $(1, \infty)$, using Hölder's inequality and the fact $p_c \leqslant q_c$, we have

$$J_2 = \sum_{i>2} \|f\chi_{I_i}\|_{L^{p_c}(R_+)} \|g\chi_{I_i}\|_{L^{(q_c)'}(R_+)} \leqslant c\|f\|_{L^{p(\cdot)}(R_+)} \|g\|_{L^{q'(\cdot)}(R_+)}.$$

Now let us estimate J_1. Suppose that $\|f\|_{L^{p(\cdot)}(R_+)} \leqslant 1$ and $\|g\|_{L^{q'(\cdot)}(R_+)} \leqslant 1$. First notice that since $q, q' \in WL(\mathbb{R}^n)$, therefore, by Proposition A and Lemma 1 we have

$$\begin{aligned} |I_k|^{1/q_+(I_k)} &\approx \|\chi_{I_k}\|_{L^{q(\cdot)}(R_+)} \approx |I_k|^{1/q_-(I_k)}; \\ |I_k|^{1/(q')_+(E_k)} &\approx \|\chi_{E_k}\|_{L^{q'(\cdot)}(R_+)} \approx |I_k|^{1/(q')_-(I_k)}, \end{aligned}$$

where $k \leqslant 2$. Hence Hölder's inequality (see Proposition A) yields

$$\begin{aligned} J_1 &\leqslant c\sum_{k\leqslant 2} \int_0^8 \frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)} \|g\chi_{I_k}\|_{L^{q'(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{q(\cdot)}(R_+)} \|\chi_{E_k}\|_{L^{q'(\cdot)}(R_+)}} \chi_{E_k}(x)dx \\ &\leqslant c\int_0^8 \sum_{k\leqslant 2} \frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)} \|g\chi_{I_k}\|_{L^{q'(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{q(\cdot)}(R_+)} \|\chi_{I_k}\|_{L^{q'(\cdot)}(R_+)}} \chi_{E_k}(x)dx \\ &\leqslant c\left\| \sum_{k\leqslant 2} \frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{q(\cdot)}(R_+)}} \chi_{E_k}(\cdot) \right\|_{L^{q(\cdot)}((0,8))} \left\| \sum_{k\leqslant 2} \frac{\|g\chi_{I_k}\|_{L^{p'(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{p'(\cdot)}(R_+)}} \chi_{E_k}(\cdot) \right\|_{L^{q'(\cdot)}((0,8))} \\ &:= cS_1 \cdot S_2. \end{aligned}$$

Now observe that

$$I(q) \leqslant cI(p),$$

where

$$\begin{aligned} I(q) &:= \left\| \sum_{k\leqslant 2} \frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{q(\cdot)}(R_+)}} \chi_{E_k}(\cdot) \right\|_{L^{q(\cdot)}((0,8))}; \\ I(p) &:= \left\| \sum_{k\leqslant 2} \frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}} \chi_{E_k}(\cdot) \right\|_{L^{p(\cdot)}((0,8))}. \end{aligned}$$

Indeed, suppose that $I(p) \leqslant 1$. Taking into account Proposition A and Lemma 1, we have

$$\sum_{k\leqslant 2}\frac{1}{|I_k|}\int\limits_{E_k}\|f\chi_{I_k}\|^{p(x)}_{L^{p(\cdot)}(R_+)}dx \leqslant c\int\limits_0^8\Big(\sum_{k\leqslant 2}\frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}\chi_{E_k}(x)\Big)^{p(x)}dx \leqslant c.$$

Consequently, since $q(x) \geqslant p(x)$, $E_k \subset I_k$ and $\|f\|_{L^{p(\cdot)}(R_+)} \leqslant 1$, we find that

$$\sum_{k\leqslant 2}\frac{1}{|I_k|}\int\limits_{E_k}\|f\chi_{I_k}\|^{q(x)}_{L^{p(\cdot)}(R_+)}dx \leqslant \sum_{k\leqslant 2}\frac{1}{|I_k|}\int\limits_{E_k}\|f\chi_{I_k}\|^{p(x)}_{L^{p(\cdot)}(R_+)}dx \leqslant c.$$

This implies that $I(q) \leqslant c$.

Let us introduce a function

$$\mathbb{P}(t) = \sum_{k\leqslant 2} p_+(I_k)\chi_{E_k}(t).$$

It is clear that $p(t) \leqslant \mathbb{P}(t)$ because $E_k \subset I_k$. Hence, Proposition A' for $\Omega = (0,8)$ yields

$$I(p) \leqslant c\Big\|\sum_{k\leqslant 2}\frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}\chi_{E_k}(\cdot)\Big\|_{L^{\mathbb{P}(\cdot)}((0,8))}.$$

Then, using the definition of $\mathbb{P}$ and the inequality $\|\chi_{I_k}\|^{p_+(I_k)}_{L^{p(\cdot)}(R_+)} \geqslant c2^k$, we have

$$\begin{aligned}\int_0^8\Big(\sum_{k\leqslant 2}\frac{\|f\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}{\|\chi_{I_k}\|_{L^{p(\cdot)}(R_+)}}\chi_{E_k}(x)\Big)^{\mathbb{P}(x)}dx &= \int_0^8\Big(\sum_{k\leqslant 2}\frac{\|f\chi_{I_k}\|^{p_+(I_k)}_{L^{p(\cdot)}(R_+)}}{\|\chi_{I_k}\|^{p_+(I_k)}_{L^{p(\cdot)}(R_+)}}\chi_{E_k}(x)\Big)dx\\ &\leqslant c\int_0^8\Big(\sum_{k\leqslant 2}\frac{\|f\chi_{I_k}\|^{p_+(I_k)}_{L^{p(\cdot)}(R_+)}}{2^k}\chi_{E_k}(x)\Big)dx\\ &\leqslant c\sum_{k\leqslant 2}\|f\chi_{I_k}\|^{p_+(I_k)}_{L^{p(\cdot)}(R_+)}\\ &\leqslant c\sum_{k\leqslant 2}\int_{I_k}|f(x)|^{p(x)}dx\\ &\leqslant c\int_{R_+}|f(x)|^{p(x)}dx \leqslant c.\end{aligned}$$

Consequently, the estimates derived above give us

$$S_1 \leqslant c\|f\|_{L^{p(\cdot)}(R_+)}.$$

Analogously, taking into account the fact that $q' \in WL(R_+)$ and arguing as above, we find that

$$S_2 \leqslant c\|g\|_{L^{q'(\cdot)}(R_+)}.$$

□

LEMMA 3. *Let* $I=[0,a]$ *be a bounded interval and let* $p\in WL(I)$. *Suppose that* $1<p_-(I)\leqslant p_+(I)<\infty$ *and* $\alpha(x)>1/p(x)$ *when* $x\in I$. *Then*

$$I(x):=\|(x-\cdot)^{\alpha(x)-1}\chi_{(x/2,x)}(\cdot)\|_{L^{p'(\cdot)}(R_+)}\leqslant cx^{\alpha(x)-1/p(x)},$$

where the positive constant c *does not depend on* x.

Proof. Since $p\in WL(I)$ implies $p'\in WL(I)$, by Lemma 1 we have the following two-sided estimate:

$$(x-t)^{p'(t)}\leqslant c_1(x-t)^{p'(x)}\leqslant c_2(x-t)^{p'(t)},$$

where $0<t<x<a$ and the positive constants c_1 and c_2 depend only on p and a. Consequently,

$$\begin{aligned}\int_{x/2}^{x}(x-t)^{(\alpha(x)-1)p'(t)}dt &\leqslant c\int_{x/2}^{x}(x-t)^{(\alpha(x)-1)p'(x)}dt\\ &= c\int_{0}^{x/2}u^{(\alpha-1)p'(x)}du=c(x/2)^{(\alpha(x)-1)p'(x)+1}\\ &= cx^{(\alpha(x)-1)p'(x)+1}:=S(x).\end{aligned}$$

Suppose that $I(x)\leqslant 1$. Then, since $p\in WL(I)$ is equivalent to the condition $1/p\in WL(I)$, by Proposition A and Lemma 1 we have

$$\begin{aligned}I(x) &\leqslant (S(x))^{1/(p')_+([x/2,x])}=c\Big((x/2)^{1/(p')_-([x/2,x])}\Big)^{(\alpha(x)-1)p'(x)+1}\\ &\leqslant c\Big((x/2)^{1/(p')_+([x/2,x])}\Big)^{(\alpha(x)-1)p'(x)+1}\leqslant c\Big((x/2)^{1/p'(x)}\Big)^{(\alpha(x)-1)p'(x)+1}\\ &\leqslant cx^{\alpha(x)-1/p(x)}.\end{aligned}$$

For $I(x)>1$ the conclusion is trivial. □

THEOREM B. [24] *Let* $p\in WL(I)$, *where* $I=[0,a]$, *where* $0<a<\infty$. *Suppose that* $1<p_-(I)\leqslant p(x)\leqslant q(x)\leqslant q_+(I)<\infty$ *and* $p,q\in WL(I)$. *Then the Hardy operator* H *is bounded from* $L^{p(\cdot)}_w(I)$ *to* $L^{q(\cdot)}_v(I)$ *if and only if*

$$C:=\sup_{0<t<a}\|v(\cdot)\chi_{(t,a)}(\cdot)\|_{L^{q(\cdot)}(I)}\|w^{-1}(\cdot)\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}(I)}<\infty.$$

Moreover, there exist positive constants c_1 *and* c_2 *such that* $c_1C\leqslant\|H\|\leqslant c_2C$.

To formulate the next results we need the notation:

$$p_0(x):=\inf_{y\in[0,x]}p(y);\quad \widetilde{p_0}(x):=\begin{cases}p_0(x), & 0\leqslant x\leqslant a\\ p_c\equiv\text{const}, & x<a\end{cases},$$

where a is a fixed positive number.

THEOREM C. *Let* $I=[0,a]$ $(0<a<\infty)$ *and let* $1\leqslant p_-(I)\leqslant p_0(x)\leqslant q(x)\leqslant q_+(I)<\infty$ *for a.e.* $x\in I$. *Then the condition*

$$\sup_{0<t<a}\int_t^a (v(x))^{q(x)}t^{q(x)/(p_0)'(x)}dx<\infty$$

implies the boundedness of H *from* $L^{p(\cdot)}(I)$ *to* $L_v^{q(\cdot)}(I)$.

THEOREM C'. *Let* $I=R_+$ *and let* $1\leqslant p_-(I)\leqslant p_0(x)\leqslant q(x)\leqslant q_+(I)<\infty$ *for a.e.* $x\in I$. *Suppose that* $q(x)\equiv q_c=$ *const,* $p(x)\equiv p_c=$ *const when* $x>a$ *for some positive number* a. *Then the condition*

$$\sup_{0<t<\infty}\int_t^\infty (v(x))^{q(x)}t^{q(x)/(\widetilde{p_0})'(x)}dx<\infty$$

guarantees the boundedness of H *from* $L^{p(\cdot)}(I)$ *to* $L_v^{q(\cdot)}(I)$.

Theorems C and C' are special cases of Theorems 3.1 and 3.3 of [10].

Let us recall the two-weight criterion for the Hardy operator in classical Lebesgue spaces (see [15], [30], [27] and [26]):

THEOREM D. *Let* r *and* s *be constants such that* $1<r\leqslant s<\infty$. *Suppose that* $0\leqslant a<b\leqslant\infty$. *Let* v *and* w *be non-negative measurable functions on* $[a,b)$. *Then the inequality*

$$\left(\int_a^b\left(v(x)\int_a^x f(t)dt\right)^s dx\right)^{1/s}\leqslant c\left(\int_a^b (w(t)f(t))^r dt\right)^{1/r},\ f\geqslant 0,$$

holds if and only if

$$\sup_{a\leqslant t\leqslant b}\left(\int_t^b v^s(x)dx\right)^{1/s}\left(\int_a^t w^{-r'}(x)dx\right)^{1/r'}<\infty.$$

The following statements are true (see [31], [32], [38]):

THEOREM E. *Let* r, s *and* γ *be constants on* R_+ *such that* $1<r\leqslant s<\infty$, $\gamma>1/r$. *Then*

(i) R_γ *is bounded from* $L^r(R_+)$ *to* $L_v^s(R_+)$ *if and only if*

$$D:=\sup_{t>0}D(t):=\sup_{t>o}\left(\int_t^\infty (v(x)x^{\gamma-1})^s dx\right)^{1/s}t^{1/r'}<\infty;$$

(ii) R_γ *is bounded from* $L^r(R_+)$ *to* $L_v^s(R_+)$ *if and only if* $D<\infty$ *and* $\lim_{t\to 0}D(t)=\lim_{t\to\infty}D(t)=0$.

THEOREM F. *Let r, s and γ be constants in R_+ such that $1 < r \leqslant s < \infty$, $\gamma > 1/r$. Suppose that $0 \leqslant a < b \leqslant \infty$. Then the inequality*

$$\left(\int_a^b v^s(x)\left|\int_a^x f(y)(x-y)^{\gamma-1}dy\right|^s dx\right)^{1/s} \leqslant c\left(\int_a^b |f(x)|^r dx\right)^{1/r}, \tag{1}$$

where the positive constant c is independent of f, $f \in L^r([a,b])$, holds if and only if

$$E := \sup_{a<t<b}\left(\int_t^b (v(x)x^{\gamma-1})^s dx\right)^{1/s}(t-a)^{1/r'} < \infty.$$

Moreover, if c is the best constant in (1), then $c_2E \leqslant c \leqslant c_1E$, where the positive constants c_1 and c_2 depends only on s, r and γ.

3. Trace Inequality (Boundedness criteria)

In this section we derive boundedness criteria for the operator $R_{\alpha(x)}$ from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}_v(I)$, where I is either a bounded interval $[0,a]$ or R_+.

Theorem 3.1. *Let $I = [0,a]$ be a bounded interval and let $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Suppose that $(\alpha - 1/p)_-(I) > 0$. Further, assume that $p,q \in WL(I)$. Then the inequality*

$$\|R_{\alpha(x)}f\|_{L^{q(x)}_v(I)} \leqslant c\|f\|_{L^{p(x)}(I)}, \quad f \in L^{p(\cdot)}(I) \tag{2}$$

holds if and only if

$$A_a := \sup_{0<t<a} A_a(t) := \sup_{0<t<a}\left\|\chi_{(t,a)}(x)\frac{v(x)}{x^{1-\alpha(x)}}\right\|_{L^{q(x)}(I)} t^{1/p'(0)} < \infty.$$

Moreover, there exist positive constants c_1 and c_2 such that

$$c_1A_a \leqslant \|R_{\alpha(x)}\|_{L^{p(x)}(I)\to L^{q(x)}_v(I)} \leqslant c_2A_a.$$

Proof. For the simplicity assume that $a = 1$.

Sufficiency. Suppose that $f \geqslant 0$. Following the arguments from [31], we represent $R_{\alpha(x)}f$ as follows:

$$\begin{aligned}(R_{\alpha(x)}f)(x) &= \int_0^{x/2} f(t)(x-t)^{\alpha(x)-1}dt + \int_{x/2}^x f(t)(x-t)^{\alpha(x)-1}dt \\ &:= (R^{(1)}_{\alpha(x)}f)(x) + (R^{(2)}_{\alpha(x)}f)(x).\end{aligned}$$

Hence

$$\|(R_{\alpha(x)}f)(x)\|_{L^{q(x)}_v(I)} \leqslant c\|(R^{(1)}_{\alpha(\cdot)}f)(x)\|_{L^{q(x)}_v(I)} + \|(R^{(2)}_{\alpha(x)}f)(x)\|_{L^{q(x)}_v(I)} := S^{(1)} + S^{(2)}.$$

It is easy to see that if $0 < t < x/2$, then $x/2 \leqslant x - t$. Consequently, $(x-t)^{\alpha(x)-1} \leqslant cx^{\alpha(x)-1}$, where the positive constant c does not depend on x. Hence, taking into account Theorem B we have

$$S^{(1)} \leqslant c\left\| \frac{v(x)}{x^{1-\alpha(x)}} Hf(x) \right\|_{L^{q(x)}(I)} \leqslant cA_a \|f\|_{L^{p(\cdot)}(I)}.$$

Suppose now that $\|g\|_{L^{q'(x)}(I)} \leqslant 1$. Using Proposition A and Lemmas 2 and 3 we find that

$$\begin{aligned}
&\int_0^1 v(x)\left(\int_{x/2}^x f(t)(x-t)^{\alpha(x)-1}dt\right)g(x)dx \\
&\quad \leqslant c\sum_{k\in Z_-}\int_{E_{k-1}} v(x)\|\chi_{(x/2,x)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \\
&\qquad \times \|\chi_{(x/2,x)}(\cdot)(x-\cdot)^{\alpha(x)-1}\|_{L^{p'(\cdot)}(I)} g(x)dx \\
&\quad \leqslant c\sum_{k\in Z_-}\|\chi_{I_{k-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\int_{E_{k-1}} v(x)x^{\alpha(x)-1/p(x)}g(x)dx \\
&\quad \leqslant c\sum_{k\in Z_-}\|\chi_{I_{k-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\left\|\chi_{E_{k-1}}(x)v(x)x^{\alpha(x)-1/p(x)}\right\|_{L^{q(x)}(I)} \\
&\qquad \times \|\chi_{E_{k-1}}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \\
&\quad \leqslant c2^{k/p'(0)}\sum_{k\in Z_-}\|v(x)x^{\alpha(x)-1}\chi_{E_{k-1}}(x)\|_{L^{q(x)}(I)}\|\chi_{I_{k-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \\
&\qquad \times \|\chi_{E_{k-1}}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \leqslant cA_a\|f\|_{L^{p(\cdot)}(I)}\|g\|_{L^{q'(\cdot)}(I)} \leqslant cA_a\|f\|_{L^{p(\cdot)}(I)}.
\end{aligned}$$

Taking the supremum with respect to g we have the desired result.

Necessity. Let us take $f_k(x) = \chi_{[0,2^{k-2}]}(x)$, where $k \in Z_-$. Then by Proposition A and Lemma 1 we have

$$\|f_k\|_{L^{p(\cdot)}(I)} \leqslant c2^{k/(p')_+([0,2^{k+2}])} \leqslant c2^{k/p'(0)}.$$

On the other hand,

$$\|R_{\alpha(x)}f\|_{L_v^{q(x)}(I)} \geqslant C\|\chi_{E_{k-1}}(x)v(x)x^{\alpha(x)-1}\|_{L^{q(x)}(I)}.$$

Here we used the estimate $(x-t)^{\alpha(x)-1} \geqslant cx^{\alpha(x)-1}$ when $x \in [2^{k-1}, 2^k]$ and $t < 2^{k-2}$.

Hence

$$\bar{A} := \sup_{k\in Z_-}\bar{A}_k := \sup_{k\in Z_-}\|\chi_{E_{k-1}}(x)v(x)x^{\alpha(x)-1}\|_{L^{q(x)}(I)}2^{k/p'(0)} \leqslant c\|R_{\alpha(x)}\|.$$

Let us now take $t \in I$. Then $t \in [2^{m-1}, 2^m)$ for some $m \in Z_-$. Consequently,

$$A(t) \leqslant \sum_{k=m}^{0}\|\chi_{E_{k-1}}(x)v(x)x^{\alpha(x)-1}\|_{L^{q(x)}(I)}2^{m/p'(0)} \leqslant \bar{A}2^m\sum_{k=m}^{0}2^{-k/p'(0)} \leqslant c\bar{A}.$$

Hence $A < c\|R_{\alpha(x)}\|$. □

Theorem 3.2. *Let* $I = R_+$ *and let* $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. *Suppose that* $(\alpha - 1/p)_-(I) > 0$. *Further, assume that* $p, q \in WL(I)$ *and that* $q(x) \equiv q_c = \text{const}$, $p(x) \equiv p_c = \text{const}$ *outside an interval* $[0,a]$ *for some positive number* a. *Then inequality* (2) *holds if and only if*

$$A_\infty := \sup_{t>0} A_\infty(t) := \sup_{t>0} \left\| \chi_{(t,\infty)}(x) \frac{v(x)}{x^{1-\alpha(x)}} \right\|_{L^{q(x)}(I)} t^{1/P'(t)} < \infty,$$

where

$$P(t) = \begin{cases} p(0), & 0 \leqslant t \leqslant a, \\ p_c, & t > a. \end{cases}$$

Moreover, there are positive constants c_1 *and* c_2 *such that*

$$c_1 A_\infty \leqslant \|R_{\alpha(x)}\|_{L^{p(x)}(I) \to L^{q(x)}_v(I)} \leqslant c_2 A_\infty.$$

Proof. For the simplicity we assume that $a = 1$.

First we prove *sufficiency.* Suppose that $f \geqslant 0$. We have

$$\|R_{\alpha(x)} f\|_{L^{q(x)}_v(R_+)} \leqslant \|R_{\alpha(x)} f\|_{L^{q(x)}_v([0,2])} + \|R_{\alpha(x)} f\|_{L^{q(x)}_v((2,\infty))} := I_1 + I_2.$$

Taking into account Theorem 3.1 we find that the condition $A_\infty < \infty$ implies

$$I_1 \leqslant cA_\infty \|f\|_{L^{p(x)}([0,2])} \leqslant cA_\infty \|f\|_{L^{p(x)}(I)}.$$

For I_2, we have

$$\begin{aligned} I_2 &\leqslant \left\| v(x) \int_0^1 (x-t)^{\alpha(x)-1} f(t)dt \right\|_{L^{q(x)}((2,\infty))} \\ &\quad + \left\| v(x) \int_1^{x/2} (x-t)^{\alpha(x)-1} f(t)dt \right\|_{L^{q(x)}((2,\infty))} \\ &\quad + \left\| v(x) \int_{x/2}^{x} (x-t)^{\alpha(x)-1} f(t)dt \right\|_{L^{q(x)}((2,\infty))} \\ &:= I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned}$$

Notice that when $t \leqslant 1$ and $x \geqslant 2$, then $(x-t)^{\alpha(x)-1} \leqslant cx^{\alpha(x)-1}$. Consequently, using Hölder's inequality we find that

$$\begin{aligned} I_{2,1} &\leqslant c \left\| v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}((2,\infty))} \|f \chi_{[0,1]}\|_{L^{p(\cdot)}(I)} \|\chi_{[0,1]}\|_{L^{p(\cdot)}(I)} \\ &\leqslant c \left\| v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}([1,\infty))} \|f\|_{L^{p(\cdot)}(R_+)} \leqslant cA_\infty \|f\|_{L^{p(\cdot)}(R_+)}. \end{aligned}$$

It is easy to see that the estimate $(x-t)^{\alpha(x)-1} \leqslant cx^{\alpha(x)-1}$ and Theorem D implies

$$\begin{aligned} I_{2,2} &\leqslant c \left\| v(x) x^{\alpha(x)-1} \int_1^x f(t)dt \right\|_{L^{q(x)}([1,\infty))} \\ &\leqslant cA_\infty \|f\|_{L^{q(x)}([1,\infty))} \leqslant cA_\infty \|f\|_{L^{q(x)}(I)}, \end{aligned}$$

while Hölder's inequality for classical Lebesgues spaces yields

$$\begin{aligned}(I_{2,3})^{q_c} &\leqslant c\left(\sum_{k=1}^{+\infty}\int_{E_k} v(x)^{q_c}x^{(\alpha(x)-1)q_c}dx\right)\left(\int_{I_k} f^{p_c}(t)dt\right)^{q_c/p_c} 2^{k/(p_c)'}\\ &\leqslant cA_\infty^{q_c}\|f\|_{L^{p(\cdot)}(R_+)}^{q_c}.\end{aligned}$$

Necessity. Necessity follows in the same way as in the case of Theorem 3.1. In this case we take the test functions $f_t(x) = \chi_{(t/2,t)}(x)$, $t > 0$. The details are omitted.

□

COROLLARY. *Let r and s be constants such that $1 < r \leqslant s < \infty$. Suppose that $\alpha_-(R_+) > 1/r$. Then $R_{\alpha(x)}$ is bounded from $L^r(R_+)$ to $L^s(R_+)$ if and only if*

$$\sup_{t>0}\left(\int_t^\infty \left(\frac{v(x)}{x^{1-\alpha(x)}}\right)^s dx\right)^{1/s} t^{1/r'} < \infty, \quad r' = r/(r-1).$$

Theorem 3.3. *Let $I := [0,a]$, where $a < \infty$. Suppose that p and q are measurable functions on I and $1 < p_-(I) \leqslant p_0(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Suppose also that $\alpha_-(I) > 1/p_-(I)$. If*

$$B_a := \sup_{0<t<a} B_a(t) := \sup_{0<t<a}\int_t^a (v(x)x^{\alpha(x)-1})^{q(x)} t^{q(x)/(p_0)'(x)}dx < \infty,$$

then inequality (2) *holds.*

REMARK 1. The condition $B_a < \infty$ is also necessary for the boundedness of $R_{\alpha(x)}$ from $L^p([0,a])$ to $L^q_v([0,a])$ for constant p and q. This follows immediately from Theorem 3.1 taking $p = \text{const}$, $q = \text{const}$ there.

Proof of Theorem 3.3. For the simplicity assume that $a = 1$. Suppose that $S_{p(\cdot)}(f) \leqslant 1$, where $f \geqslant 0$. We have

$$\begin{aligned}S_{q(\cdot),v}(R_{\alpha(x)}) &\leqslant 2^{q_-(I)-1}\left(\int_0^1\left(v(x)\int_0^{x/2}(x-y)^{\alpha(x)-1}f(y)dy\right)^{q(x)}dx\right.\\ &\qquad \left. + \int_0^1\left(v(x)\int_{x/2}^x (x-y)^{\alpha(x)-1}f(y)dy\right)^{q(x)}dx\right)\\ &:= 2^{q_-(I)-1}(I_1+I_2).\end{aligned}$$

If $0 < y < x/2$, then $(x-y)^{\alpha(x)-1} \leqslant cx^{\alpha(x)-1}$, where the positive constant c does not depend on x. Consequently, using Theorem C we find that

$$I_1 \leqslant \int_0^1\left(v(x)(x/2)^{\alpha(x)-1}\int_0^x f(y)dy\right)^{q(x)}dx \leqslant C.$$

By Hölder's inequality with respect to the exponent $p_0(x)$ we have

$$\begin{aligned} I_2 &\leqslant \int_0^1 (v(x))^{q(x)} \left(\int_{x/2}^x (f(y))^{p_0(x)} dy \right)^{q(x)/p_0(x)} \\ &\quad \times \left(\int_{x/2}^x (x-y)^{(\alpha(x)-1)(p_0)'(x)} dy \right)^{q(x)/(p_0)'(x)} dx. \end{aligned}$$

Now observe that

$$\begin{aligned} \int_{x/2}^x f(y)^{p_0(x)} dy &\leqslant \int_{[x/2,x]\cap\{f\leqslant 1\}} (f(y))^{p_0(x)} dy + \int_{[x/2,x]\cap\{f>1\}} (f(y))^{p_0(x)} dy \\ &\leqslant cx + \int_{x/2}^x (f(y))^{p(y)} dy; \end{aligned}$$

$$\int_{x/2}^x (x-y)^{(\alpha(x)-1)(p_0)'(x)} dy = c_{p,\alpha} x^{(\alpha(x)-1)(p_0)'(x)+1}.$$

The latter equality holds because the condition $\alpha_-(I) > 1/p_-(I)$ guarantees $\alpha(x) > 1/p_0(x)$. Therefore

$$\begin{aligned} I_2 &\leqslant c_{p,q} \left(\int_0^1 (v(x))^{q(x)} x^{q(x)\alpha(x)} dx \right. \\ &\quad \left. + \int_0^1 v(x)^{q(x)} \left(\int_{x/2}^x f(y)^{p(y)} dy \right)^{q(x)/p_0(x)} x^{(\alpha(x)-1)q(x)+q(x)/(p_0)'(x)} dx \right) \\ &:= c_{p,q}(I_{2,1} + I_{2,2}). \end{aligned}$$

For $I_{2,1}$, we find that

$$\begin{aligned} I_{2,1} &= \sum_{k\in Z_-} \int_{E_{k-1}} v(x)^{q(x)} x^{(\alpha(x)-1)q(x)} x^{q(x)} dx \\ &\leqslant \sum_{k\in Z_-} 2^{kq_-(I)/p_+(I)} \int_{E_{k-1}} v(x)^{q(x)} x^{(\alpha(x)-1)q(x)} 2^{(k-1)q(x)/(p_0)'(x)} dx \\ &\leqslant cB \sum_{k\in Z_-} 2^{kq_-(I)/p_+(I)} \leqslant cB < \infty, \end{aligned}$$

while taking into account the fact $q(x)/p_0(x) \geqslant 1$ we have

$$\begin{aligned} I_{2,2} &\leqslant c \sum_{k\in Z_-} \int_{E_{k-1}} (v(x)x^{\alpha(x)-1})^{q(x)} \left(\int_{x/2}^x (f(y))^{p(y)} dy \right) x^{q(x)/(p_0)'(x)} dx \\ &\leqslant c \sum_{k\in Z_-} \left(\int_{E_{k-1}} \left(v(x)x^{\alpha(x)-1}\right)^{q(x)} 2^{(k-1)q(x)/(p_0)'(x)} dx \right) \left(\int_{I_{k-1}} (f(y))^{p(y)} dy \right) \\ &\leqslant cBS_{p(\cdot)}(f) \leqslant cB < \infty. \end{aligned}$$

By combining the above estimates we find that also $I_2 < \infty$ and the proof follows. $\square$

Theorem 3.4. *Let $I := R_+$. Suppose that $p(x)$ and $q(x)$ are measurable functions on I and $1 \leqslant p_-(I) \leqslant p_0(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Suppose also that $\alpha_-(I) > 1/p_+(I) > 0$ and there exists a positive number a such that $q(x) \equiv q_c = const$, $p(x) \equiv p_c = const$ outside $[0,a]$. If*

$$B_\infty := \sup_{0<t<\infty} B_\infty(t) := \sup_{0<t<\infty} \int_t^\infty (v(x)x^{\alpha(x)-1})^{q(x)} t^{q(x)/(\widetilde{p_0})'(x)} dx < \infty, \tag{3}$$

then $R_{\alpha(x)}$ is bounded from $L^{p(x)}(I)$ to $L_v^{q(x)}(I)$.

REMARK 2. Notice that (3) is also necessary for the boundedness of $R_{\alpha(x)}$ from $L^p(R_+)$ to $L_v^q(R_+)$, where p and q are constants (see Corollary).

Proof of Theorem 3.4. Suppose that $f \geqslant 0$ and $S_{p(\cdot)}(f) \leqslant 1$. For the simplicity assume that $a = 1$. We have

$$\begin{aligned} S_{q(\cdot),v}(R_{\alpha(x)}f) &= \int_0^2 \big(v(x)\big)^{q(x)} (R_\alpha f)^{q(x)}(x)dx + \int_2^\infty \big(v(x)\big)^{q(x)} (R_{\alpha(x)}f)^{q_c}(x)dx \\ &:= I_1 + I_2. \end{aligned}$$

Since the condition $B_\infty < \infty$ implies $B_a < \infty$, by Theorem 3.3 we conclude that $I_1 \leqslant c < \infty$, while for I_2, we have

$$\begin{aligned} I_2 &\leqslant c\bigg(\int_2^\infty (v(x))^{q_c} \bigg(\int_0^1 (x-y)^{\alpha(x)-1} f(y)dy \bigg)^{q_c} dx \\ &\quad + \int_2^\infty (v(x))^{q_c} \bigg(\int_1^{x/2} (x-y)^{\alpha(x)-1} f(y)dy \bigg)^{q_c} dx \\ &\quad + \int_2^\infty (v(x))^{q_c} \bigg(\int_{x/2}^x (x-y)^{\alpha(x)-1} f(y)dy \bigg)^{q_c} dx \bigg) \\ &:= c(I_{2,1} + I_{2,2} + I_{2,3}). \end{aligned}$$

Using Hölder's inequality for Lebesgue spaces with variable exponent (see Proposition A) we have

$$\begin{aligned} I_{2,1} &\leqslant c \int_2^\infty (v(x))^{q_c} \bigg(\int_0^1 (x-y)^{\alpha(x)-1} f(y)dy \bigg)^{q_c} dx \\ &\leqslant c\bigg(\int_2^\infty (v(x)(x/2)^{\alpha(x)-1})^{q_c} dx \bigg) \|f\|_{L^{p(\cdot)}(I)} \|\chi_{[0,1]}\|_{L^{p'(\cdot)}(I)} \leqslant cB_1, \end{aligned}$$

while Theorem C' yields

$$I_{2,2} \leqslant c \int_1^\infty \big(v(x)x^{\alpha(x)-1}\big)^{q_c} \bigg(\int_1^x f(y)dy \bigg)^{q_c} dx \leqslant cB_\infty.$$

Now applying the condition $\alpha(x) > 1/(p_0)'(x)$ we find that

$$\begin{aligned} I_{2,3} &= \sum_{k=1}^{\infty} \int_{E_k} (v(x))^{q_c} \left(\int_{x/2}^{x} f(y)(x-y)^{\alpha(x)-1} dy \right)^{q_c} dx \\ &\leqslant \sum_{k=1}^{\infty} \int_{E_k} (v(x))^{q_c} \left(\int_{x/2}^{x} (f(y))^{p_c} dy \right)^{q_c/p_c} \left(\int_{x/2}^{x} (x-y)^{(\alpha(x)-1)(p_c)'} dy \right)^{q_c/(p_c)'} \\ &\leqslant c \sum_{k=1}^{\infty} \left(\int_{E_k} (v(x))^{q_c} (x/2)^{(\alpha(x)-1)q_c + q_c/(p_c)'} dx \right) \left(\int_{E_k} (f(y))^{p(y)} dy \right) \\ &\leqslant cB_\infty \sum_{k=1}^{\infty} \int_{E_k} (f(y))^{p(y)} dy \leqslant cB_\infty < \infty. \end{aligned}$$

Summarizing the estimates for I_1 and I_2 we have the desired result. $\square$

4. Compactness

In this section we give the criteria for which the operator $R_{\alpha(x)}$ is compact from $L^{p(\cdot)}(I)$ to $L_v^{q(\cdot)}(I)$.

Integral-type necessary conditions and sufficient conditions governing the compactness of the Hardy operator H from $L^{p(\cdot)}(I)$ to $L_v^{q(\cdot)}(I)$ were established in [10]. We refer also to [12] for the compactness of the potential-type operators in weighted $L^{p(\cdot)}$ spaces with special weights.

To prove the main results we shall need the following statement which can be found, e.g., in [12].

THEOREM G. *Let $p(x)$ and $q(x)$ be measurable functions on an interval $I \subseteq R_+$. Suppose that $1 < p_-(I) \leqslant p_+(I) < \infty$ and $1 < q_-(I) \leqslant q_+(I) < \infty$. If*

$$\Big\| \|k(x,y)\|_{L^{p'(y)}(I)} \Big\|_{L^{q(x)}(I)} < \infty,$$

where k is a non-negative kernel, then the operator

$$Kf(x) = \int_I k(x,y) f(y) dy$$

is compact from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

Theorem 4.1. *Let $I = [0,a]$, $0 < a < \infty$, and let $1 < p_-(I) \leqslant p_+(I) \leqslant q_-(I) \leqslant q_+(I) < \infty$. Suppose that $(\alpha - 1/p)_-(I) > 0$. Further, assume that $p, q \in WL(I)$. Then $I_{\alpha(x)}$ is compact from $L^{p(\cdot)}(I)$ to $L_v^{q(\cdot)}(I)$ if and only if*

(i) $$A_a < \infty;$$

(ii) $$\lim_{t\to 0} A_a(t) = 0,$$

where A_a and $A_a(t)$ are defined in Theorem 3.1.

Proof. Sufficiency. For the simplicity assume that $a = 1$. We represent $R_{\alpha(x)}$ as follows:

$$R_{\alpha(x)}f(x) = R^{(1)}_{\alpha(x)}f(x) + R^{(2)}_{\alpha(x)}f(x),$$

where

$$R^{(1)}_{\alpha(x)}f(x) = \chi_{[0,\beta]}(x)R_{\alpha(x)}f(x), \quad R^{(2)}_{\alpha(x)}f(x) = \chi_{(\beta,1]}(x)R_{\alpha(x)}f(x)$$

and $0 < \beta < 1$. Observe that by Hölder's inequality, Proposition A and Lemma 3 we have the following estimates:

$$\begin{aligned}
&\Big\|\chi_{(\beta,1]}(x)v(x)\big\|\chi_{[0,x]}(y)(x-y)^{\alpha(x)-1}\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\quad\leqslant \Big\|\chi_{(\beta,1]}(x)v(x)\big\|\chi_{[0,x/2]}(y)(x-y)^{\alpha(x)-1}\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\qquad+\Big\|\chi_{(\beta,1]}(x)v(x)\big\|\chi_{(x/2,x]}(y)(x-y)^{\alpha(x)-1}\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\quad\leqslant c\big\|\chi_{(\beta,1]}(x)v(x)x^{\alpha(x)-1/p(x)}\big\|_{L^{q(x)}(I)} + c\big\|\chi_{(\beta,1]}(x)v(x)x^{\alpha(x)-1/p(x)}\big\|_{L^{q(x)}(I)} < \infty,
\end{aligned}$$

because $A_1 < \infty$. Consequently, by Theorem G, $R^{(1)}_{\alpha(x)}$ is compact.

Further, according to Theorem 3.1 we have

$$\|R_{\alpha(x)} - R^{(1)}_{\alpha(x)}\|_{L^{q(\cdot)}(I)\to L^{q(\cdot)}_v(I)} \leqslant \|R^{(2)}_{\alpha(x)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}_v(I)} \leqslant c \sup_{0<t<\beta} A_1(t),$$

where the positive constant c depends only on p, q and α. Passing β to 0 we have that $R_{\alpha(x)}$ is compact as a limit of compact operators.

Necessity. Suppose that $f_t(x) = t^{-1/p(0)}\chi_{[0,t/2)}(x)$. Hence for all $\varphi \in L^{p'(x)}(I)$ we have

$$\begin{aligned}
\left|\int_0^1 f_t(x)\varphi(x)dx\right| &\leqslant k(p)\|f_t(\cdot)\|_{L^{p(\cdot)}(I)}\|\varphi(\cdot)\chi_{[0,t/2)}(\cdot)\|_{L^{p'(\cdot)}(I)}\\
&\leqslant ct^{-1/p(0)}t^{1/p_+([0,t/2])}\|\varphi(\cdot)\chi_{[0,t/2)}(\cdot)\|_{L^{p'(\cdot)}(I)}\\
&\leqslant c\|\varphi(\cdot)\chi_{[0,t/2)}(\cdot)\|_{L^{p'(\cdot)}(I)} \to 0
\end{aligned}$$

as $t \to 0$. Hence, f_t converges weakly to 0 as $t \to 0$. Further, it is obvious that

$$\begin{aligned}
\|R_{\alpha(x)}f_t\|_{L^{q(\cdot)}_v(I)} &\geqslant \Big\|\chi_{[t,1)}(x)v(x)\Big(\int_0^{t/2}(x-t)^{\alpha(x)-1}dt\Big)\Big\|_{L^{q(\cdot)}(I)}t^{-1/p(0)}\\
&\geqslant ct^{-1/p'(0)}\Big\|\chi_{[t,1)}(x)v(x)x^{\alpha(x)-1}\Big\|_{L^{q(x)}(I)}.
\end{aligned}$$

Finally we conclude that $\lim_{t\to 0}A_1(t) = 0$ because the compact operator maps weakly convergent sequence into strongly convergent one. □

Theorem 4.2. *Let* $I = R_+$ *and let* $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. *Suppose that* $p(x) \equiv p_c = \text{const}$ *and* $q(x) \equiv q_c = \text{const}$ *when* $x > a$ *for some positive* *constant* a. *Let* $(\alpha - 1/p)_-(I) > 0$. *Further, assume that* $p, q \in WL(I)$. *Then* $R_{\alpha(x)}$ *is compact from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}_v(I)$ *if and only if*

(i) $$A_\infty < \infty;$$

(ii) $$\lim_{t\to 0} A_\infty(t) = \lim_{t\to\infty} A_\infty(t) = 0,$$

where A_∞ *and* $A_\infty(t)$ *are defined in Theorem* 3.2.

Proof. For the simplicity assume that $a = 1$. To prove *sufficiency* we use the representation $R_{\alpha(x)}f = \sum_{n=1}^{5} R^{(n)}_{\alpha(x)}f$, where

$$\begin{aligned}
R^{(1)}_{\alpha(x)}f(x) &= \chi_{[0,\beta)}(x)(R_{\alpha(x)}f)(x),\\
R^{(2)}_{\alpha(x)}f(x) &= \chi_{[\beta,\gamma)}(x)R_{\alpha(x)}(\chi_{[0,\beta/2]}f)(x),\\
R^{(3)}_{\alpha(x)}f(x) &= \chi_{[\beta,\gamma)}(x)R_{\alpha(x)}(\chi_{[\beta/2,\infty)}f)(x),\\
R^{(4)}_{\alpha(x)}f(x) &= \chi_{[\gamma,\infty)}R_{\alpha(x)}(\chi_{[0,\gamma/2)}f)(x),\\
R^{(5)}_{\alpha(x)}f(x) &= \chi_{[\gamma,\infty)}(x)R_{\alpha(x)}(\chi_{[\gamma/2,\infty)}f)(x),
\end{aligned}$$

where $0 < \beta < 1/2 < 2 < \gamma < \infty$. Now observe that

$$\begin{aligned}
&\Big\|\chi_{[\beta,\gamma)}(x)v(x)\Big\|\chi_{[0,\beta/2)}(y)(x-y)^{\alpha(x)-1}\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\qquad\leqslant c\Big\|\chi_{[\beta,\gamma)}(x)v(x)x^{\alpha(x)-1}\Big\|_{L^{p'(x)}(I)}\Big\|\chi_{[0,\beta/2)}\Big\|_{L^{p'(\cdot)}(I)} < \infty
\end{aligned}$$

because $A_\infty < \infty$. Further,

$$\begin{aligned}
&\Big\|\chi_{[\beta,\gamma)}(x)v(x)\Big\|\chi_{[\beta/2,\infty)}(y)(x-y)^{\alpha(x)-1}\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\qquad\leqslant \Big\|\chi_{[\beta,\gamma)}(x)v(x)\Big\|\chi_{[\beta/2,x/2)}(y)(x-y)^{\alpha(x)-1}\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\qquad\quad+\Big\|\chi_{[\beta,\gamma)}(x)v(x)\Big\|\chi_{[x/2,x)}(y)(x-y)^{\alpha(x)-1}\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\qquad:= I_1 + I_2.
\end{aligned}$$

It is easy to see that

$$I_1 \leqslant c\Big\|\chi_{[\beta,\gamma)}(x)v(x)x^{\alpha(x)-1}\Big\|_{L^{q(x)}(I)}\Big\|\chi_{[\beta/2,\gamma/2)}(\cdot)\Big\|_{L^{p'(\cdot)}(I)} < \infty.$$

Using the condition $p \in WL(I)$ and Lemma 3 we shall see that $I_2 < \infty$. Besides

$$\begin{aligned}
&\Big\|\chi_{[\gamma,\infty)}(x)v(x)\Big\|\chi_{[0,\gamma/2)}(y)(x-y)^{\alpha(x)-1}\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
&\qquad\leqslant \Big\|\chi_{[\gamma,\infty)}(x)v(x)x^{\alpha(x)-1}\Big\|_{L^{q(x)}(I)}\Big\|\chi_{[0,\gamma/2)}(\cdot)\Big\|_{L^{p'(\cdot)}(I)} < \infty
\end{aligned}$$

since $A_\infty < \infty$. Applying Theorem 3.1 we see that

$$\|R^{(1)}_{\alpha(x)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} = \|R_{\alpha(x)}\|_{L^{p(\cdot)}([0,\beta))\to L^{q(\cdot)}([0,\beta))} \leqslant c \sup_{0<t<\beta} A_\infty(t) \to 0$$

as $\beta \to 0$.

Arguing as in the proof of sufficiency of Theorem 3.2 (see also Theorem F for constant α), we see that the inequality

$$\left\|R^{(5)}_{\alpha(x)} f(x)\right\|_{L^{p(x)}([\gamma,\infty))\to L^{q(x)}([\gamma,\infty))} \leqslant c \sup_{t>\gamma/2} \left(\int_t^\infty (v(x))^{q_c} x^{\alpha(x)-1} dx\right)^{1/q_c} (t-\gamma)^{1/(p_c)'}$$

holds. The latter term tends to 0 when $\gamma \to \infty$ because $\lim_{t\to\infty} A_\infty(t) = 0$. Finally we conclude that $R_{\alpha(x)}$ is compact.

Necessity. The condition $A_\infty < \infty$ is a consequence of Theorem 3.2. The fact $\lim_{t\to 0} A_\infty(t)$ follows in the same manner as in the proof of necessity of Theorem 4.1. To show that $\lim_{t\to\infty} A_\infty(t) = 0$ we argue as above using the facts that p and q are constants outside $[0,a]$ and $R_{\alpha(x)}$ is compact from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}_v(I)$ if and only if the operator

$$W_{\alpha,v} f(x) = \int_x^\infty v(y) f(y)(y-x)^{\alpha(y)-1} dy$$

is compact from $L^{q'(x)}(I)$ to $L^{p'(x)}(I)$. □

Acknowledgement. The authors express their gratitude to the referee for valuable remarks and suggestions.

The second and third authors were partially supported by the INTAS grant “Variable Exponent Analysis”, Nr. 06-1000017-8792 and the Georgian National Science Foundation Grant No. GNSF/ST07/3-169.

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(Received April 2, 2008)

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