CONTINUITY PROPERTIES FOR RIESZ POTENTIALS OF
FUNCTIONS IN MORREY SPACES OF VARIABLE EXPONENT

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Abstract. Our aim in this paper is to deal with continuity properties for Riesz potentials of functions in Morrey spaces of variable exponent. The modulus of continuity is determined by the structure of Morrey space.

1. Introduction

For $0 < \alpha < n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $U_\alpha f$ of order α by

$$U_\alpha f(x) = \int_{\mathbf{R}^n} |x - y|^{\alpha - n} f(y) dy.$$

Here it is natural to assume that $U_\alpha |f| \not\equiv \infty$, which is equivalent to

$$\int_{\mathbf{R}^n} (1 + |y|)^{\alpha - n} |f(y)| dy < \infty; \tag{1.1}$$

for this fact, see [8, Theorem 1.1, Chapter 2]. If f is a locally integrable function on $\mathbf{R}^n$ satisfying (1.1) and the Morrey condition

$$\sup_{x \in \mathbf{R}^n, r > 0} r^{-\nu} \int_{B(x,r)} |f(y)|^p dy < \infty,$$

then it is known (see e.g. Sawano-Tanaka [17]) that $U_\alpha f$ is Hölder continuous on $\mathbf{R}^n$ whenever $\nu > n - \alpha p$, where $1 < p < \infty$ and $B(x, r)$ denotes the open ball centered at x of radius r; in case $\nu \leqslant n - \alpha p$, we have Sobolev's type inequality for $U_\alpha f$ as was proved in the paper by Adams [1].

In the present paper, we are concerned with the continuity properties of $U_\alpha f$ for locally integrable functions f on $\mathbf{R}^n$ satisfying (1.1) and

$$\int_0^\infty t^{\alpha p - n} \varphi_1(t) \left(\int_{B(x,2t)} |f(y)|^p \varphi_2(|f(y)|) dy \right) \frac{dt}{t} < \infty, \tag{1.2}$$

where φ_i $(i = 1, 2)$ are positive monotone functions on the interval $[0, \infty)$ satisfying

$$C^{-1}\varphi_i(r) \leqslant \varphi_i(r^2) \leqslant C\varphi_i(r) \qquad \text{whenever } r > 0$$

with a positive constant C. When $\varphi_1(r) = (\log(2 + r^{-1}))^{\delta_1}$ and $\varphi_2(r) = (\log(2 + r))^{\delta_2}$, our Theorem 2.1 shows that $U_\alpha f$ is continuous if and only if $\delta_1 + \delta_2 > p$. In case $\delta_1 + \delta_2 \leqslant p$, we show by an example that $U_\alpha f$ might not be continuous; in this case, Nakai [13, Theorem 2.2] obtained Trudinger's inequality for $U_\alpha f$.

Next, following Kováčik and Rákosník [7], we consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$, which is called a variable exponent. In connection with (1.2), for $0 \leqslant v \leqslant n$ and a real number β, we define the $L^{p(\cdot), v, \beta}$ norm by

$$\|f\|_{p(\cdot), v, \beta} = \inf\left\{\lambda > 0 : \sup_{x \in \mathbf{R}^n} \int_0^\infty r^{-v}(\log(2 + r^{-1}))^\beta \right.$$
$$\left. \times \left(\int_{B(x, r)} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy\right) \frac{dr}{r} \leqslant 1\right\}$$

and denote by $L^{p(\cdot), v, \beta}$ the space of all measurable functions f on $\mathbf{R}^n$ with $\|f\|_{p(\cdot), v, \beta} < \infty$. This space $L^{p(\cdot), v, \beta}$ is referred to as a generalized Morrey space of variable exponent. In this paper we deal with $p(\cdot)$ satisfying the following $\log$-Hölder condition

$$p(y) \geqslant p_0 + \frac{a \log(\log(1/|x_0 - y|))}{\log(1/|x_0 - y|)} - \frac{b}{\log(1/|x - y|)}$$

for $y \in B(x_0, r_0)$, where $p_0 = (n - v)/\alpha$ and a, b are positive constants. Our second aim in this paper is to study the continuity for Riesz potentials of functions in the generalized Morrey space $L^{p(\cdot), v, \beta}$ of variable exponent, as an extension of Futamura and the authors [2], [4] and Harjulehto-Hästö [5].

2. Continuity of Riesz potentials in the case of constant exponent

Throughout this paper, let C, C_1, C_2,... denote various constants independent of the variables in question.

Suppose φ is a positive monotone function on the interval $[0, \infty)$ satisfying

$(\varphi 1)$ $\qquad\qquad c^{-1}\varphi(r) \leqslant \varphi(r^2) \leqslant c\varphi(r) \qquad \text{whenever } r > 0$

with a positive constant c. Here we collect several properties which follow from condition $(\varphi 1)$ (see [8] and [11]).

$(\varphi 2)$ φ satisfies the doubling condition, that is, there exists $c > 1$ such that

$$c^{-1}\varphi(r) \leqslant \varphi(2r) \leqslant c\varphi(r) \qquad \text{whenever } r > 0.$$

$(\varphi 3)$ For each $\gamma > 0$, there exists $c = c(\gamma) \geqslant 1$ such that

$$c^{-1}\varphi(r) \leqslant \varphi(r^\gamma) \leqslant c\varphi(r) \qquad \text{whenever } r > 0.$$

($\varphi4$) If $\gamma > 0$, then there exists $c = c(\gamma) \geqslant 1$ such that

$$s^\gamma \varphi(s) \leqslant ct^\gamma \varphi(t) \qquad \text{whenever } 0 < s < t.$$

($\varphi5$) If $\gamma > 0$, then there exists $c = c(\gamma) \geqslant 1$ such that

$$t^{-\gamma} \varphi(t) \leqslant cs^{-\gamma} \varphi(s) \qquad \text{whenever } 0 < s < t.$$

($\varphi6$) If $\gamma > 0$ and $r_0 > 0$, then there exists $c = c(\gamma, r_0) \geqslant 1$ such that

$$\int_r^{r_0} t^{-\gamma} \varphi(t) dt/t \leqslant cr^{-\gamma} \varphi(r) \qquad \text{whenever } 0 < r < r_0/2.$$

Let $\varphi_1(r)$ be a positive nonincreasing function on $(0, \infty)$ satisfying $(\varphi1)$, and $\varphi_2(r)$ be a positive nondecreasing function on $(0, \infty)$ satisfying $(\varphi1)$. Our examples of φ_1 are

$$(\log(2 + r^{-1}))^{\delta_1}, (\log(2 + r^{-1}))^{\delta_1} \log(2 + (\log(2 + r^{-1})))^{\delta_2}, ...$$

with positive constants δ_j; examples of φ_2 are given similarly. If $1 < p < \infty$ and $0 \leqslant v < n$, then we consider

$$\Phi_+(r_1, r_2) = \left(\int_{r_1}^{r_2} [t^{n-\alpha p-v} \varphi_1(t) \varphi_2(t^{-1})]^{-p'/p} \frac{dt}{t} \right)^{1/p'}$$

for $0 \leqslant r_1 < r_2$ and

$$\Phi^+(r_1, r_2) = \left(\int_{r_1}^{r_2} [t^{n-\alpha p-v+p} \varphi_1(t) \varphi_2(t^{-1})]^{-p'/p} \frac{dt}{t} \right)^{1/p'},$$

for $0 < r_1 < r_2$, where $1/p + 1/p' = 1$. Further we set

$$\Phi(r) = \Phi_+(0, r) + r\Phi^+(r, 1)$$

for $0 < r < 1$. We see from $(\varphi5)$ and $(\varphi6)$ that if $v < n - \alpha p$, then

$$C^{-1}[r^{n-\alpha p-v} \varphi_1(r) \varphi_2(r^{-1})]^{-1/p} \leqslant \Phi_+(r, 1) \leqslant C[r^{n-\alpha p-v} \varphi_1(r) \varphi_2(r^{-1})]^{-1/p} \quad (2.1)$$

for $0 < r < 1/2$, so that

$$\Phi_+(0, 1) = \infty.$$

Similarly, if $v < n - \alpha p + p$, then

$$C^{-1}[r^{n-\alpha p+p-v} \varphi_1(r) \varphi_2(r^{-1})]^{-1/p} \leqslant \Phi^+(r, 1) \leqslant C[r^{n-\alpha p+p-v} \varphi_1(r) \varphi_2(r^{-1})]^{-1/p}$$

$$(2.2)$$

whenever $0 < r < 1/2$, so that $\Phi^+(0, 1) = \infty$.

If f is a locally integrable function on $\mathbf{R}^n$, then we set

$$F(x, r_1, r_2) = \left(\int_{r_1}^{r_2} t^{-v} \varphi_1(t) \left(\int_{B(x,2t)} |f(y)|^p \varphi_2(|f(y)|) dy \right) \frac{dt}{t} \right)^{1/p}$$

for $0 \leqslant r_1 < r_2$.

Our first aim in the present paper is to establish continuity properties for Riesz potentials of functions f on $\mathbf{R}^n$ satisfying

$$\sup_{x \in \mathbf{R}^n} F(x, 0, \infty) < \infty, \tag{2.3}$$

as an extension of [9].

THEOREM 2.1. *Let $0 < v < n$ and $n - \alpha p \leqslant v \leqslant n - \alpha p + p$ and f be a locally integrable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (1.2). *If Φ is bounded on the interval* $(0, 1)$, *then $U_\alpha f$ is continuous on $\mathbf{R}^n$ and, moreover, in case $n - \alpha p \leqslant v \leqslant n - \alpha p + p$ and $\Phi^+(0, 1) = \infty$,*

$$|U_\alpha f(x) - U_\alpha f(z)| = o(\Phi(|x - z|))$$

and in case $v = n - \alpha p + p$ and $\Phi^+(0, 1) < \infty$,

$$|U_\alpha f(x) - U_\alpha f(z)| = O(|x - z|)$$

as x runs over a compact set in $\mathbf{R}^n$ and $|x - z| \to 0$.

For a proof of Theorem 2.1, we prepare the following result (cf. [11, Lemma 3.1]):

LEMMA 2.2. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (1.2). *If $\sigma > 0$ and $c > 1$, then*

$$\int_{B(x,r_2) \setminus B(x,r_1)} |x - y|^{\alpha - n} f(y) dy \ \leqslant \ C\Phi_+(r_1, r_2) F(x, r_1, r_2)$$

$$+ \int_{B(x,r_2) \setminus B(x,r_1)} |x - y|^{\alpha - n - \sigma} dy$$

for all $x \in \mathbf{R}^n$ and $r_2 > 2r_1 \geqslant 0$.

Proof. Let $\sigma > 0$. We write $A(x, r_1, r_2) = B(x, r_2) \setminus B(x, r_1)$ and

$$\int_{A(x,r_1,r_2)} |x - y|^{\alpha - n} f(y) dy \ = \ \int_{\{y \in A(x,r_1,r_2): f(y) > |x-y|^{-\sigma}\}} |x - y|^{\alpha - n} f(y) dy$$

$$+ \int_{\{y \in A(x,r_1,r_2): 0 < f(y) \leqslant |x-y|^{-\sigma}\}} |x - y|^{\alpha - n} f(y) dy$$

$$= \ U_1 + U_2.$$

First it is easy to see that

$$U_2 \leqslant \int_{A(x,r_1,r_2)} |x - y|^{\alpha - n - \sigma} dy.$$

Next we treat U_1.

From Hölder's inequality, we obtain

$$
U_1 \leqslant \left(\int_{\{y \in A(x,r_1,r_2); f(y) > |x-y|^{-\sigma}\}} |x-y|^{(\alpha-n+\nu/p)p'} [\varphi_1(|x-y|)\varphi_2(f(y))]^{-p'/p} dy \right)^{1/p'}
$$
$$
\times \left(\int_{\{y \in A(x,r_1,r_2); f(y) > |x-y|^{-\sigma}\}} |x-y|^{-\nu} \varphi_1(|x-y|) f(y)^p \varphi_2(f(y)) dy \right)^{1/p}.
$$

In view of $(\varphi 3)$, we see that if $f(y) > |x-y|^{-\sigma}$, then

$$
\varphi_2(f(y)) \geqslant \varphi_2(|x-y|^{-\sigma}) \geqslant C\varphi_2(|x-y|^{-1}),
$$

so that

$$
\left(\int_{\{y \in A(x,r_1,r_2); f(y) > |x-y|^{-\sigma}\}} |x-y|^{(\alpha-n+\nu/p)p'} [\varphi_1(|x-y|)\varphi_2(f(y))]^{-p'/p} dy \right)^{1/p'}
$$
$$
\leqslant C \left(\int_{A(x,r_1,r_2)} |x-y|^{(\alpha-n+\nu/p)p'} [\varphi_1(|x-y|)\varphi_2(|x-y|^{-1})]^{-p'/p} dy \right)^{1/p'}
$$
$$
\leqslant C\Phi_+(r_1,r_2).
$$

Further, integrating by parts and changing to polar coordinates, we find

$$
\int_{A(x,r_1,r_2)} |x-y|^{-\nu} \varphi_1(|x-y|) f(y)^p \varphi_2(f(y)) dy
$$
$$
\leqslant \int_{r_1}^{r_2} \left(\int_{B(x,t)} f(y)^p \varphi_2(f(y)) dy \right) d(-t^{-\nu}\varphi_1(t))
$$
$$
+ \left[t^{-\nu}\varphi_1(t) \int_{B(x,t)} f(y)^p \varphi_2(f(y)) dy \right]_{r_1}^{r_2}
$$

since $t^{-\nu}\varphi_1(t)$ is nonincreasing by $(\varphi 5)$. By the doubling property of φ_1, we note that

$$
\int_{r/2}^{r} |d(-t^{-\nu}\varphi_1(t))| \leqslant Cr^{-\nu}\varphi_1(r)
$$

for $r > 0$ and

$$
r_2^{-\nu}\varphi_1(r_2) \int_{B(x,r_2)} f(y)^p \varphi_2(f(y)) dy \leqslant C \int_{r_2/2}^{r_2} t^{-\nu}\varphi_1(t) \left(\int_{B(x,2t)} f(y)^p \varphi_2(f(y)) dy \right) \frac{dt}{t}.
$$

Hence

$$
\int_{A(x,r_1,r_2)} |x-y|^{-\nu} \varphi_1(|x-y|) f(y)^p \varphi_2(f(y)) dy
$$
$$
\leqslant C \int_{r_1}^{r_2} t^{-\nu}\varphi_1(t) \left(\int_{B(x,2t)} f(y)^p \varphi_2(f(y)) dy \right) \frac{dt}{t}. \tag{2.4}
$$

Consequently it follows that

$$U_1 \leqslant C\Phi_+(r_1, r_2)F(x, r_1, r_2).$$

Thus the present lemma is proved. □

Proof of Theorem 2.1. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (1.1) and (1.2). We show that the conclusion holds for $x \in B(0, R)$. For this purpose we may assume that $f = 0$ outside $B(0, 2R)$, since $\int_{\mathbf{R}^n \backslash B(0,2R)} |x-y|^{\alpha-n} f(y)dy$ is infinitely differentiable in $B(0, R)$ by (1.1).

Let $x \in B(0, R)$. For $z \in B(x, r)$ and $0 < r < R$, write

$$\begin{aligned}
U_\alpha f(z) &= \int_{B(x,2r)} |z - y|^{\alpha-n} f(y)dy + \int_{B(x,3R) \backslash B(x,2r)} |z - y|^{\alpha-n} f(y)dy \\
&= u_1(z) + u_2(z).
\end{aligned}$$

If $0 < \sigma < \alpha$, then we have by Lemma 2.2 and $(\varphi 2)$,

$$\begin{aligned}
|u_1(z)| &\leqslant \int_{B(z,3r)} |z - y|^{\alpha-n} f(y)dy \\
&\leqslant C\Phi_+(0, 3r)F(z, 0, 3r) + Cr^{\alpha-\sigma} \\
&\leqslant C\Phi_+(0, r)F(x, 0, 4r) + Cr^{\alpha-\sigma}.
\end{aligned}$$

Here note from the doubling property of φ that

$$\begin{aligned}
\Phi_+(0, r) &\geqslant \left(\int_{r/2}^r [t^{n-\alpha p-v} \varphi_1(t) \varphi_2(t^{-1})]^{-p'/p} \frac{dt}{t} \right)^{1/p'} \\
&\geqslant Cr^{\alpha-(n-v)/p} [\varphi_1(r) \varphi_2(r^{-1})]^{-1/p}.
\end{aligned}$$

Hence, if $0 < \sigma < (n - v)/p$, then $(\varphi 4)$ gives

$$r^{\alpha-\sigma} = \Phi_+(0, r)o(1),$$

so that

$$|u_1(z)| \leqslant C\Phi_+(0, r)(o(1) + F(x, 0, 4r))$$

for $z \in B(x, r)$.

On the other hand, using the mean value theorem for analysis, we have

$$\begin{aligned}
|u_2(x) - u_2(z)| &\leqslant \int_{B(x,3R) \backslash B(x,2r)} ||x - y|^{\alpha-n} - |z - y|^{\alpha-n}| f(y)dy \\
&\leqslant Cr \int_{B(x,3R) \backslash B(x,2r)} |x - y|^{\alpha-n-1} f(y)dy \\
&= Cr \int_{B(x,\varepsilon_0) \backslash B(x,2r)} |x - y|^{\alpha-n-1} f(y)dy \\
&\quad + Cr \int_{B(x,3R) \backslash B(x,\varepsilon_0)} |x - y|^{\alpha-n-1} f(y)dy \\
&= Cr(I_1 + I_2)
\end{aligned}$$

for $0 < \varepsilon_0 < R$. First consider the case $\alpha < (n - v)/p + 1$. In this case take $\alpha - 1 < \sigma < (n - v)/p$. Then, applying Hölder's inequality and Lemma 2.2 again, we find

$$
\begin{aligned}
I_1 &\leqslant C \left(\int_{2r}^{\varepsilon_0} t^{(\alpha - n - 1 + v/p)p'} [\varphi_1(t)\varphi_2(t^{-1})]^{-p'/p} t^{n-1} dt \right)^{1/p'} \\
&\quad \times \left(\int_{2r}^{\varepsilon_0} t^{-v} \varphi_1(t) \left(\int_{B(x,2t)} f(y)^p \varphi_2(f(y)) dy \right) \frac{dt}{t} \right)^{1/p} + Cr^{\alpha - \sigma - 1}.
\end{aligned}
$$

Noting by (2.2) that

$$
r^{\alpha - \sigma - 1} = \Phi^+(2r, \varepsilon_0) o(1)
$$

when $4r < \varepsilon_0$, we obtain

$$
I_1 \leqslant C\Phi^+(2r, \varepsilon_0)(o(1) + F(x, 2r, \varepsilon_0)).
$$

Since I_2 is dominated by a positive constant $C(\varepsilon_0)$, using (2.2), we find

$$
\begin{aligned}
|u_2(x) - u_2(z)| &\leqslant Cr\Phi^+(2r, \varepsilon_0)(o(1) + F(x, 2r, \varepsilon_0)) + C(\varepsilon_0)r \\
&\leqslant Cr\Phi^+(2r, \varepsilon_0)(C(\varepsilon_0)o(1) + F(x, 2r, \varepsilon_0)).
\end{aligned}
$$

Now it follows that

$$
\begin{aligned}
|U_\alpha f(x) - U_\alpha f(z)| &\leqslant C\Phi_+(0, r)(o(1) + F(x, 0, 4r)) \\
&\quad + Cr\Phi^+(2r, \varepsilon_0)(C(\varepsilon_0)o(1) + F(x, 2r, \varepsilon_0)) \tag{2.5}
\end{aligned}
$$

for $z \in B(x, r)$ and $0 < r < \varepsilon_0$. This gives

$$
\limsup_{r \to 0} \sup_{z \in B(x,r), x \in B(0,R)} \frac{|U_\alpha f(x) - U_\alpha f(z)|}{\Phi(r)} \leqslant CF(x, 0, \varepsilon_0),
$$

where C does not depend on ε_0.

Next consider the case $\alpha = (n - v)/p + 1$. In this case, taking $0 < \sigma < \alpha - 1$, we find

$$
\begin{aligned}
I_1 &\leqslant C \left(\int_{2r}^{\varepsilon_0} t^{(\alpha - n - 1 + v/p)p'} [\varphi_1(t)\varphi_2(t^{-1})]^{-p'/p} t^{n-1} dt \right)^{1/p'} \\
&\quad \times \left(\int_{2r}^{\varepsilon_0} t^{-v} \varphi_1(t) \left(\int_{B(x,2t)} f(y)^p \varphi_2(f(y)) dy \right) \frac{dt}{t} \right)^{1/p} + C\varepsilon_0^{\alpha - \sigma - 1}.
\end{aligned}
$$

If $\Phi^+(0, 1) = \infty$, then we derive

$$
I_1 \leqslant C\Phi^+(2r, \varepsilon_0)(o(1) + F(x, 2r, \varepsilon_0)).
$$

Since I_2 is dominated by a positive constant $C(\varepsilon_0)$, we find

$$
\begin{aligned}
|u_2(x) - u_2(z)| &\leqslant Cr\Phi^+(2r, \varepsilon_0)(o(1) + F(x, 2r, \varepsilon_0)) + C(\varepsilon_0)r \\
&\leqslant Cr\Phi^+(2r, \varepsilon_0)(C(\varepsilon_0)o(1) + F(x, 2r, \varepsilon_0)),
\end{aligned}
$$

so that

$$|U_\alpha f(x) - U_\alpha f(z)| \leqslant C\Phi_+(0,r)(o(1) + F(x,0,4r))$$
$$+ Cr\Phi^+(r,\varepsilon_0)(C(\varepsilon_0)o(1) + F(x,0,\varepsilon_0))$$

for $z \in B(x,r)$. Thus (2.5) holds in this case, too.

If $\Phi^+(0,1) < \infty$, then the above arguments imply that

$$|U_\alpha f(x) - U_\alpha f(z)| \leqslant C|x-z| \qquad \text{when } |x-z| \leqslant 1,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 2.3. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and*

$$\sup_{x \in \mathbf{R}^n} \int_0^\infty r^{-\nu}(\log(2+r^{-1}))^{\delta_1} \left(\int_{B(x,r)} f(y)^p (\log(2+|f(y)|))^{\delta_2} dy \right) \frac{dr}{r} < \infty$$

for $\nu = n - \alpha p > 0$, $\delta_1 \geqslant 0$ and $\delta_2 \geqslant 0$. If $\delta_1 + \delta_2 > p - 1$, then

$$|U_\alpha f(x) - U_\alpha f(z)| = o((\log(1/|x-z|))^{1-(\delta_1+\delta_2+1)/p})$$

when x runs over a compact set in $\mathbf{R}^n$ and $|x-z| \to 0$.

Corollary 2.4. (cf. Sawano-Tanaka [17, Theorem 5.4]) *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and*

$$\sup_{x \in \mathbf{R}^n} \int_0^\infty r^{-\nu} \left(\int_{B(x,r)} f(y)^p dy \right) \frac{dr}{r} < \infty.$$

Then, in case $n - \alpha p < \nu < n - \alpha p + p$,

$$|U_\alpha f(x) - U_\alpha f(z)| = o(|x-z|^{(\nu-n+\alpha p)/p})$$

and, in case $\nu = n - \alpha p + p$,

$$|U_\alpha f(x) - U_\alpha f(z)| = o(|x-z| \log(1/|x-z|))^{1/p'})$$

when x runs over a compact set in $\mathbf{R}^n$ and $|x-z| \to 0$.

Corollary 2.5. *Let $\nu = n - \alpha p > 0$ and f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and*

$$\sup_{x \in \mathbf{R}^n, r>0} r^{\alpha p - n}(\log(2+r^{-1}))^{\delta_1} \int_{B(x,r)} f(y)^p (\log(2+f(y)))^{\delta_2} \, dy < \infty$$

for $\delta_1 \geqslant 0$ and $\delta_2 \geqslant 0$. If $\delta_1 + \delta_2 > p$, then $U_\alpha f$ is continuous on $\mathbf{R}^n$ and

$$|U_\alpha f(x) - U_\alpha f(z)| \leqslant C(\log(1/|x-z|))^{1-(\delta_1+\delta_2)/p}$$

when $|x-z| < 1/2$.

For this, let $\varphi_1(t) = (\log(2 + t^{-1}))^\delta$ with $p - 1 - \delta_2 < \delta < \delta_1 - 1$, $\varphi_2(t) = (\log(2 + t))^{\delta_2}$ and $0 < \sigma < \alpha$; then we have

$$\Phi_+(0, r) = \left(\int_0^r (\log(2 + t^{-1}))^{-p'(\delta + \delta_2)/p} t^{-1} dt \right)^{1/p'} \leqslant C(\log(1/r))^{1/p' - (\delta + \delta_2)/p}$$

and

$$F(x, 0, r) \leqslant C \left(\int_0^r (\log(2 + t^{-1}))^{\delta - \delta_1} t^{-1} dt \right)^{1/p} \leqslant C(\log(1/r))^{(1 + \delta - \delta_1)/p}.$$

Lemma 2.2 gives

$$\int_{B(x,2r)} |x - y|^{\alpha - n} f(y) dy \leqslant C\Phi_+(0, r) F(x, 0, r) + Cr^{\alpha - \sigma}$$

$$\leqslant C(\log(1/r))^{1 - (\delta_1 + \delta_2)/p}$$

for $0 < r < 1/2$.

Next, letting $n - \alpha p < \nu_1 < n - \alpha p + p$, $\varphi_1(t) = 1$, $\varphi_2(t) = (\log(2 + t))^{\delta_2}$ and $\alpha - 1 < \sigma < \alpha$, we find

$$\Phi^+(r, \infty) = \left(\int_r^\infty [t^{n - (\alpha - 1)p - \nu_1} \varphi_2(t^{-1})]^{-p'/p} t^{-1} dt \right)^{1/p'}$$

$$\leqslant Cr^{(\alpha - 1) - (n - \nu_1)/p} (\log(1/r))^{-\delta_2/p}$$

and

$$F(x, r, \infty) \leqslant C \left(\int_r^\infty t^{-\nu_1 + n - \alpha p} (\log(2 + t^{-1}))^{-\delta_1} \frac{dt}{t} \right)^{1/p}$$

$$\leqslant Cr^{-\nu_1/p + (n - \alpha p)/p} (\log(1/r))^{-\delta_1/p}.$$

Hence it follows from Lemma 2.2 again that

$$\int_{\mathbf{R}^n \setminus B(x,2r)} |x - y|^{\alpha - n - 1} f(y) dy \leqslant C\Phi^+(2r, \infty) F(x, 2r, \infty) + Cr^{\alpha - 1 - \sigma}$$

$$\leqslant Cr^{-1} (\log(1/r))^{-(\delta_1 + \delta_2)/p} + Cr^{\alpha - 1 - \sigma}$$

$$\leqslant Cr^{-1} (\log(1/r))^{-(\delta_1 + \delta_2)/p}$$

for $0 < r < 1/2$.

Now, letting $r = |x - z|$ and using (2.5), we establish

$$|U_\alpha f(x) - U_\alpha f(z)| \leqslant C(\log(1/|x - z|))^{1 - (\delta_1 + \delta_2)/p}$$

whenever $|x - z| < 1/2$, which proves the corollary.

REMARK 2.6. In Corollary 2.5, if $n - \alpha p = 0$ and $\delta_1 = 0$, then $U_\alpha f$ is seen to be continuous on $\mathbf{R}^n$ whenever $\delta_2 > p - 1$, in view of the first author [9]. In case $n - \alpha p > 0$, we need the condition $\delta_2 > p$ for the continuity of $U_\alpha f$.

For this, we first note from (2.1) that $\Phi_+(0,1) < \infty$ if and only if $\delta_2 > p$. If $v = n - \alpha p > 0$ and $\delta_2 \leqslant p$, then consider the function

$$f(y) = |y|^{-\alpha}(\log|y|^{-1})^{-\delta_2/p}\chi_{B(0,1/2)}(y),$$

where χ_E denotes the characteristic function of E. Since $\delta_2 \leqslant p$ by our assumption, we see that

(1) $\displaystyle\int |y|^{\alpha-n}f(y)dy = \infty$;

(2) $\displaystyle\int_{B(x,r)} f(y)^p(\log(1+f(y)))^{\delta_2}dy \leqslant C\int_{B(x,r)}|y|^{-\alpha p}dy \leqslant Cr^{n-\alpha p} = Cr^v$
for all $x \in \mathbf{R}^n$ and $0 < r < 1/2$.

This implies that $U_\alpha f$ is not continuous at the origin, in spite of the fact that f satisfies conditions (1.1) and (1.2) required in Theorem 2.1.

The following is a consequence of Nakai [13, Theorem 2.2].

THEOREM 2.7. *Let $v = n - \alpha p > 0$ and G be a bounded open set in $\mathbf{R}^n$. Then there exist positive constants A_1, A_2 such that*

$$\sup_{x\in G, 0<r<R_G} r^{-v}(\log(2+r^{-1}))^p \int_{B(x,r)} \exp(\exp(A_1|U_\alpha f(x)|))dx < A_2$$

whenever f is a locally integrable function on G satisfying

$$\sup_{x\in G, 0<r<R_G} r^{-v}(\log(2+r^{-1}))^{\delta_1} \int_{B(x,r)} |f(y)|^p(\log(2+|f(y)|))^{\delta_2}\,dy \leqslant 1, \qquad (2.6)$$

where $\delta_2 = p - \delta_1 \geqslant 0$ and R_G denotes the diameter of G.

Here, in view of condition (2.6), we find a positive constant C such that

$$\sup_{x\in G, r>0} r^{-v}(\log(2+r^{-1}))^p \int_{B(x,r)} |f(y)|^p\,dy \leqslant C. \qquad (2.7)$$

In fact, taking $0 < a < (n-v)/p = \alpha$, we have

$$\int_{B(x,r)} |f(y)|^p dy \leqslant \int_{B(x,r)} r^{-ap}dy + \int_{B(x,r)} |f(y)|^p\left(\frac{\log(2+|f(y)|)}{\log(2+r^{-a})}\right)^{\delta_2} dy$$

$$\leqslant Cr^{n-ap} + C\left(\log(2+r^{-1})\right)^{-\delta_2}\int_{B(x,r)} |f(y)|^p\left(\log(2+|f(y)|)\right)^{\delta_2} dy,$$

which gives (2.7).

Proof of Theorem 2.7. Letting $\varphi_1(t) = (\log(1+t^{-1}))^{-1+\delta_1}$ and $\varphi_2(t) = (\log(1+t))^{\delta_2}$, we have by Lemma 2.2

$$\int_{G\setminus B(x,r)} |x-y|^{\alpha-n}|f(y)|dy \leqslant C + C\left(\int_r^{R_G} [\log(1+t^{-1})]^{-1}t^{-1}dt\right)^{1/p'}$$

$$\times \left(\int_r^{R_G} [\log(1+t^{-1})]^{-1+\delta_1}t^{-v}\left(\int_{B(x,2t)} |f(y)|^p(\log(2+|f(y)|))^{\delta_2}dy\right)\frac{dt}{t}\right)^{1/p}$$

$$\leqslant C\log(\log(1+r^{-1}))$$

for $x \in G$ and $0 < r < 1/4$. Hence it follows that

$$
\begin{aligned}
|U_\alpha f(x)| &\leqslant \int_{B(x,r)} |x-y|^{\alpha-n}|f(y)|dy + \int_{G\setminus B(x,r)} |x-y|^{\alpha-n}|f(y)|dy \\
&\leqslant Cr^\alpha Mf(x) + C\log(\log(1/r)),
\end{aligned}
$$

where Mf denotes the maximal function of f, that is,

$$
Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{G\cap B(x,r)} |f(y)|dy
$$

with $|B(x,r)|$ denoting the Lebesgue measure of $B(x,r)$; here we set $f = 0$ outside G, as before. Taking $r > 0$ such that $r = Mf(x)^{-1/\alpha}(\log(\log Mf(x)))^{1/\alpha}$ when $Mf(x)$ is large enough,

$$
|U_\alpha f(x)| \leqslant C + C\log(\log(4 + Mf(x))) \leqslant C_1 \log(\log(4 + Mf(x))).
$$

Consequently, we find

$$
\int_{B(z,r)} \exp(p\exp(C_1^{-1}|U_\alpha f(x)|))dx \leqslant \int_{B(z,r)} (4 + Mf(x))^p dx.
$$

Applying Lemma 2.1 in [12], we obtain by (2.7)

$$
\int_{B(z,r)} Mf(x)^p dx \leqslant Cr^\nu(\log(2 + 1/r))^{-p},
$$

which yields

$$
\int_{B(z,r)} \exp(p\exp(C_1^{-1}|U_\alpha f(x)|))dx \leqslant Cr^\nu(\log(2 + 1/r))^{-p}.
$$

Thus the proof is completed. $\qquad\square$

3. Variable exponents

In this section, we deduce the continuity of $U_\alpha f$ at a fixed point $x_0 \in \mathbf{R}^n$, when $p(\cdot)$ satisfies

(p1) $\inf_{\mathbf{R}^n\setminus B_0} p(x) > p_0 = (n-\nu)/\alpha > 1$ and $\sup_{\mathbf{R}^n} p(x) < \infty$;

(p2) $\left| p(x) - \left\{ p_0 + \dfrac{a\log(\log(1/|x_0 - x|))}{\log(1/|x_0 - x|)} \right\} \right| \leqslant \dfrac{b}{\log(1/|x_0 - x|)}$ for $x \in B_0$,

where $a > 0$, $b > 0$ and $B_0 = B(x_0, r_0)$ with $0 < r_0 < 1/4$. In this case,

$$
p(x) \geqslant p_0 + \omega_{a,-b}(|x - x_0|) \tag{3.1}
$$

for $x \in B_0$, where $\omega_{a,c}(r) = \dfrac{a\log(\log(1/r))}{\log(1/r)} + \dfrac{c}{\log(1/r)}$ for $0 < r \leqslant r_0$ and $\omega_{a,c}(r) = \omega_{a,c}(r_0)$ for $r > r_0$. If r_0 is small enough, then $\omega_{a,-b}(r)$ is increasing on $[0, r_0]$; here we set $\omega(0) = 0$.

For $0 \leqslant v \leqslant n$ and a real number β, we define

$$\|f\|_{p(\cdot),v,\beta} = \inf \left\{ \lambda > 0 : \sup_{x \in \mathbf{R}^n} \int_0^\infty r^{-v}(\log(2+r^{-1}))^\beta \left(\int_{B(x,r)} \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \right) \frac{dr}{r} \leqslant 1 \right\}.$$

Our final goal is to establish the following result, which deals with the continuity of α-potentials at x_0.

THEOREM 3.1.   *If f is a nonnegative measurable function on B_0 satisfying $\|f\|_{p(\cdot),v,\beta} \leqslant 1$ for $v = n - \alpha p_0 > 0$ and $\beta > p_0 - 1 - a\alpha$. Then*

$$|U_\alpha f(x) - U_\alpha f(z)| = o((\log(1/|x-z|))^{-A})$$

as $x \to z \in B_0$, where $A = (a\alpha + \beta + 1)/p_0 - 1 > 0$.

Before the proof we prepare the following results.
Let $1/p'(x) = 1 - 1/p(x)$.

LEMMA 3.2.   (cf. [3, Lemma 2.1])   *There exists a positive constant c such that*

$$p'(y) \leqslant p_0' - \omega_{\tilde{a},c}(|x_0 - y|) \qquad \text{whenever } y \in \mathbf{R}^n, \tag{3.2}$$

where $\tilde{a} = a/(p_0-1)^2$.

LEMMA 3.3.   *Let f be a nonnegative measurable function on B_0 satisfying $\|f\|_{p(\cdot),v,\beta} \leqslant 1$. Then, for each $\varepsilon > 0$, there exists $\delta_0 > 0$ such that*

$$\int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \leqslant C\varepsilon (\log(1/\delta))^{-A}$$

for all $x \in B_0$ and $0 < \delta < \delta_0$.

Proof. Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot),v,\beta} \leqslant 1$. First consider the case $0 < \delta \leqslant |x_0 - x|/2$ and $x \in B_0$. In this case we see that

$$p(y) \geqslant p_0 + \omega_{a,-b}(|x-y|) \equiv p_1(y)$$

for $y \in B(x,\delta)$. Then note that

$$\sup_{x \in \mathbf{R}^n} \int_0^\infty r^{-v}(\log(2+r^{-1}))^\beta \left(\int_{B(x,2r)} f(y)^{p_1(y)} dy \right) \frac{dr}{r} \leqslant C.$$

Set

$$F_1(x,\delta) = \int_0^\delta r^{-v}(\log(2+r^{-1}))^\beta \left(\int_{B(x,2r)} f(y)^{p_1(y)} dy \right) \frac{dr}{r}.$$

If $0 < k < 1$ and $\sigma = av/p_0 - \beta$, then we have by Young's inequality

$$\int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) dy$$

$$\leqslant k \left\{ \int_{B(x,\delta)} (|x-y|^{\alpha-n+v/p_0} (\log(1/|x-y|))^{\sigma/p_1(y)}/k)^{p_1'(y)} dy \right.$$

$$\left. + \int_{B(x,\delta)} |x-y|^{-vp_1(y)/p_0} (\log(1/|x-y|))^{-\sigma} f(y)^{p_1(y)} dy \right\}.$$

In view of Lemma 3.2, we find

$$p'_1(y) \leqslant p'_0 - \omega(|x-y|)$$

for $y \in B(x,\delta)$, where $\omega(r) = \omega_{\bar{a},c}(r)$. Hence it follows that

$$\int_{B(x,\delta)} (|x-y|^{\alpha-n+v/p_0}(\log(1/|x-y|))^{\sigma/p_1(y)}/k)^{p'_1(y)}dy$$

$$\leqslant k^{-p'_0}\int_{B(x,\delta)} |x-y|^{-np'_1(y)/p'_0}(\log(1/|x-y|))^{\sigma p'_1(y)/p_1(y)}dy$$

$$\leqslant k^{-p'_0}\int_{B(x,\delta)} |x-y|^{-n(p'_0-\omega(|x-y|))/p'_0}(\log(1/|x-y|))^{(p'_0-\omega(|x-y|))\sigma/p_1(y)}dy$$

$$\leqslant Ck^{-p'_0}\int_0^\delta (\log(1/t))^{-n\bar{a}/p'_0+\sigma p'_0/p_0}t^{-1}dt$$

$$\leqslant Ck^{-p'_0}(\log(1/\delta))^{-Ap'_0},$$

since $-anp'_0/p_0^2 + \sigma p'_0/p_0 + 1 = -Ap'_0 < 0$.

On the other hand, as in (2.4), integration by parts gives

$$\int_{B(x,\delta)} |x-y|^{-vp_1(y)/p_0}(\log(1/|x-y|))^{-\sigma}f(y)^{p_1(y)}dy$$

$$\leqslant C\int_{B(x,\delta)} |x-y|^{-v}(\log(1/|x-y|))^{av/p_0-\sigma}f(y)^{p_1(y)}dy$$

$$\leqslant C\int_0^\delta t^{-v}(\log(1/t))^\beta \left(\int_{B(x,2t)} f(y)^{p_1(y)}dy\right)\frac{dt}{t}$$

$$\leqslant CF_1(x,\delta).$$

Thus we establish

$$\int_{B(x,\delta)} |x-y|^{\alpha-n}f(y)dy \leqslant Ck\left(k^{-p'_0}(\log(1/\delta))^{-Ap'_0} + F_1(x,\delta_1)\right).$$

Now, considering $k = (\log(1/\delta))^{-A}F(x,\delta_1)^{-1/p'_0}$ for small $\delta > 0$, say $0 < \delta < \delta_2 (< \delta_1)$, we see that

$$\int_{B(x,\delta)} |x-y|^{\alpha-n}f(y)dy \leqslant C(\log(1/\delta))^{-A}F_1(x,\delta_1)^{1/p_0}$$

whenever $0 < \delta < \delta_2$ and $\delta \leqslant |x_0-x|/2$.

Next consider the case $|x_0-x|/2 < \delta < \delta_2/3$. Note from (3.1) that

$$p(y) \geqslant p_0 + \omega_{a,-b}(|x_0-y|) \equiv p_2(y)$$

and

$$\sup_{x\in\mathbf{R}^n}\int_0^\infty r^{-v}(\log(2+r^{-1}))^\beta \left(\int_{B(x,2r)} f(y)^{p_2(y)}dy\right)\frac{dr}{r} \leqslant C.$$

Set

$$F_2(x, \delta) = \int_0^\delta r^{-\nu} (\log(2 + r^{-1}))^\beta \left(\int_{B(x,2r)} f(y)^{p_2(y)} dy \right) \frac{dr}{r}.$$

If $y \in B(x, \delta) \setminus B(x, |x_0 - x|/2)$, then $|x_0 - y| \leqslant 3|x - y|$ and $|x_0 - y| \leqslant |x_0 - x| + |x - y| < 3\delta$, so that

$$\int_{B(x,\delta) \setminus B(x,|x_0-x|/2)} |x - y|^{\alpha-n} f(y) dy \leqslant C \int_{B(x_0,3\delta)} |x_0 - y|^{\alpha-n} f(y) dy.$$

In view of the above discussions we can find $0 < \delta_3 < \delta_2$ such that

$$\int_{B(x_0,3\delta)} |x_0 - y|^{\alpha-n} f(y) dy \leqslant C (\log(1/\delta))^{-A} F_2(x, \delta_1)^{1/p_0}$$

whenever $0 < \delta < \delta_3$. Therefore, applying polar coordinates, we obtain

$$\int_{B(x,\delta)} |x - y|^{\alpha-n} f(y) dy$$

$$= \int_{B(x,|x_0-x|/2)} |x - y|^{\alpha-n} f(y) + \int_{B(x,\delta) \setminus B(x,|x_0-x|/2)} |x - y|^{\alpha-n} f(y)$$

$$\leqslant C(\log(1/|x_0 - x|))^{-A} F_1(x, \delta_1)^{1/p_0} + C(\log(1/\delta))^{-A} F_2(x, \delta_1)^{1/p_0}$$

$$\leqslant C(\log(1/\delta))^{-A} \{F_1(x, \delta_1)^{1/p_0} + F_2(x, \delta_1)^{1/p_0}\}$$

for $0 < \delta < \delta_3$, which proves the present lemma. $\qquad\square$

LEMMA 3.4. *Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot),\nu,\beta} \leqslant 1$. Then*

$$\int_{B_0 \setminus \{B(x_0,\delta) \cup B(x,\delta)\}} |x - y|^{\alpha-n-1} f(y) dy \leqslant C\delta^{-1} (\log(1/\delta))^{-A-1/p_0'}$$

for $x \in B_0$ and $0 < \delta < 1/2$.

Proof. Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot),\nu,\beta} \leqslant 1$. Note that

$$p(y) \geqslant p_0 + \omega_{a,-b}(\delta) \equiv p_1$$

for $x \in B_0$ and $y \in B_0 \setminus \{B(x_0, \delta) \cup B(x, \delta)\}$, and

$$\sup_{x \in \mathbf{R}^n} \int_0^\infty r^{-\nu} (\log(2 + r^{-1}))^\beta \left(\int_{B(x,2r)} f(y)^{p_1} dy \right) \frac{dr}{r} \leqslant C.$$

Here, if $\delta_0 > 0$ is small enough, then

$$\alpha(p_2 - p_0)/p_0 - 1 < 0,$$

where $p_2 = p_0 + \omega_{a,-b}(\delta_0)$. Then note that

$$\alpha - n - 1 + \nu/p_1 + n/p_1' = \alpha - 1 - (n - \nu)/p_1 \leqslant \alpha(p_2 - p_0)/p_0 - 1 < 0$$

whenever $0 < \delta \leqslant \delta_0$. Hence, using Hölder's inequality and integration by parts, we have

$$\int_{B_0 \setminus \{B(x_0, \delta) \cup B(x, \delta)\}} |x - y|^{\alpha - n - 1} f(y) dy$$

$$\leqslant \left(\int_{B_0 \setminus \{B(x_0, \delta) \cup B(x, \delta)\}} (|x - y|^{\alpha - n - 1 + v/p_1} (\log(1/|x - y|))^{-\beta/p_1})^{p_1'} dy \right)^{1/p_1'}$$

$$\times \left(\int_{B_0 \setminus \{B(x_0, \delta) \cup B(x, \delta)\}} |x - y|^{-v} (\log(1/|x - y|))^{\beta} f(y)^{p_1} dy \right)^{1/p_1}$$

$$\leqslant C \left(\int_{\delta}^{\infty} t^{(-1 + \alpha(p_1 - p_0)/p_1)p_1'} (\log(1/t))^{-p_1'\beta/p_1} t^{-1} dt \right)^{1/p_1'}$$

$$\times \left(\int_{\delta}^{\infty} t^{-v} (\log(1/t))^{\beta} \left(\int_{B(x, 2t)} f(y)^{p_1} dy \right) \frac{dt}{t} \right)^{1/p_1}$$

$$\leqslant C \delta^{-1 + \alpha(p_1 - p_0)/p_1} (\log(1/\delta))^{-\beta/p_1}$$

$$\leqslant C \delta^{-1} (\log(1/\delta))^{-(a\alpha + \beta)/p_0}$$

whenever $0 < \delta \leqslant \delta_0$, which yields the required inequality. $\qquad \square$

Proof of Theorem 3.1. Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot), v, \beta} \leqslant 1$. The proof will be carried out along the same lines as in the proof of Theorem 2.1. Write

$$U_\alpha f(x) - U_\alpha f(z) = \int_{B(x, 2|x-z|)} |x - y|^{\alpha - n} f(y) dy - \int_{B(x, 2|x-z|)} |z - y|^{\alpha - n} f(y) dy$$

$$+ \int_{B_0 \setminus B(x, 2|x-z|)} (|x - y|^{\alpha - n} - |z - y|^{\alpha - n}) f(y) dy.$$

By Lemma 3.3, we have

$$\int_{B(x, 2|x-z|)} |x - y|^{\alpha - n} f(y) dy \leqslant C\varepsilon (\log(1/|x - z|))^{-A} \qquad (3.3)$$

and

$$\int_{B(x, 2|x-z|)} |z - y|^{\alpha - n} f(y) dy \leqslant \int_{B(z, 3|x-z|)} |z - y|^{\alpha - n} f(y) dy$$

$$\leqslant C\varepsilon (\log(1/|x - z|))^{-A} \qquad (3.4)$$

for $x, z \in B_0$ and $|x - z| < \delta_0$. On the other hand, by the mean value theorem for analysis, we have for $x, z \in B_0$ with $|x - z| < \delta_0$,

$$\int_{B_0 \setminus B(x,2|x-z|)} \left| |x - y|^{\alpha-n} - |z - y|^{\alpha-n} \right| f(y) dy$$

$$\leqslant C|x - z| \int_{B_0 \setminus B(x,2|x-z|)} |x - y|^{\alpha-n-1} f(y) dy$$

$$\leqslant C|x - z| \left\{ \int_{B_0 \setminus \{B(x_0,2|x-z|) \cup B(x,2|x-z|)\}} |x - y|^{\alpha-n-1} f(y) dy \right.$$

$$\left. + \int_{B(x_0,2|x-z|) \setminus B(x,2|x-z|)} |x - y|^{\alpha-n-1} f(y) dy \right\}.$$

It follows from Lemma 3.4 that

$$\int_{B_0 \setminus \{B(x_0,2|x-z|) \cup B(x,2|x-z|)\}} |x - y|^{\alpha-n-1} f(y) dy \leqslant C|x - z|^{-1} (\log(1/|x - z|))^{-A-1/p_0'}.$$

Moreover we see from Lemma 3.3 that

$$\int_{B(x_0,2|x-z|) \setminus B(x,2|x-z|)} |x - y|^{\alpha-n-1} f(y) dy$$

$$\leqslant C|x - z|^{-1} \int_{B(x_0,2|x-z|)} |x_0 - y|^{\alpha-n} f(y) dy$$

$$\leqslant C\varepsilon|x - z|^{-1} (\log(1/|x - z|))^{-A}.$$

Therefore, we obtain

$$\int_{B_0 \setminus B(x,2|x-z|)} \left| |x - y|^{\alpha-n} - |z - y|^{\alpha-n} \right| f(y) dy$$

$$\leqslant C\varepsilon (\log(1/|x - z|))^{-A} + C(\log(1/|x - z|))^{-A-1/p_0'},$$

which together with (3.3) and (3.4) gives

$$|U_\alpha f(x) - U_\alpha f(z)| \leqslant C\varepsilon (\log(1/|x - z|))^{-A} + C(\log(1/|x - z|))^{-A-1/p_0'}$$

for $x, z \in B_0$ and $|x - z| < \delta_0$, as required.　　　　□

REMARK 3.5.　Let $b > (a\alpha^2 + \alpha)/(n - v) > 1$, $0 < r_0 < 1/e$ and

$$p(y) = \frac{n - v}{\alpha} + \frac{a \log(\log(1/|y|))}{\log(1/|y|)}$$

for $y \in B(0, r_0)$. Consider the function

$$f(y) = |y|^{-\alpha} (\log(1/|y|))^{-b} \chi_{B_-(0,r_0)}(y),$$

where $B_-(0, r_0) = \{y \in B(0, r_0) : y_n < 0\}$ and χ_E denotes the characteristic function of E. Set $F = \{(t, 0) \in \mathbf{R} \times \mathbf{R}^{n-1} : t > 0\}$. Then we easily see that

$$\int_{B(0,r_0)} |y|^{\alpha-n} f(y) dy - \int_{B(0,r_0)} |x - y|^{\alpha-n} f(y) dy \geqslant C(\log(1/|x|))^{1-b}$$

for $x \in F$ and

$$\int_{B(0,r_0)} f(y)^{p(y)} dy \leqslant \int_{B(0,r_0)} |y|^{-(n-v)} (\log(1/|y|))^{-b(n-v)/\alpha+a\alpha} dy$$

$$\leqslant Cr^v (\log(1/r))^{-b(n-v)/\alpha+a\alpha},$$

so that

$$\int_0^{r_0} r^{-v} \left(\int_{B(0,r_0)} f(y)^{p(y)} dy \right) \frac{dr}{r} < \infty$$

since $-b(n - v)/\alpha + a\alpha + 1 < 0$ by our assumption. This means that the exponent A in Theorem 3.1 is best possible.

We define

$$\|f\|_{p(\cdot),v,\beta} = \inf \left\{ \lambda > 0 : \sup_{x \in \mathbf{R}^n, r>0} r^{-v} (\log(2 + r^{-1}))^\beta \left(\int_{B(x,r)} \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \right) \leqslant 1 \right\}.$$

As in the proof of Corollary 2.5, Theorem 3.1 gives the following result.

THEOREM 3.6. *Assume that $\tilde{A} = (a\alpha + \beta)/p_0 - 1 > 0$. If f is a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot),v,\beta} \leqslant 1$, then $U_\alpha f$ is continuous at x_0 and satisfies*

$$|U_\alpha f(x) - U_\alpha f(x_0)| \leqslant C(\log(1/|x - x_0|))^{-\tilde{A}}$$

for all $x \in B_0$.

4. Continuity II

In this section, let $p(\cdot)$ be a nondecreasing function on $(0, \infty)$ such that

(p3)
$$p(r) = p_0 + \frac{a \log(\log(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)}$$

for $0 < r \leqslant r_0$ and $p(r) = p(r_0)$ for $r > r_0$, where p_0, a and b are real numbers such that $p_0 > 1$ and $a > 0$. Set

$$p(x) = p(\rho(x))$$

with $\rho(x) = |1 - |x||$.

In this section we give a version of the results in the previous section. Since the proof of Theorem 3.6 is a little more complicated, we show a version of this theorem.

THEOREM 4.1.   *Assume that $\tilde{A} = (a\alpha + \beta)/p_0 - 1 > 0$ and $\nu = n - \alpha p_0 > 0$. If f is a nonnegative measurable function on the unit ball $\mathbf{B}$ with $\|f\|_{p(\cdot),\nu,\beta} \leqslant 1$, then $U_\alpha f$ is continuous on $\mathbf{B}$ and satisfies*

$$|U_\alpha f(x) - U_\alpha f(z)| \leqslant C(\log(1/|x - z|))^{-\tilde{A}}$$

for $x, z \in \mathbf{B}$ with $|x - z| < 1/2$.

First note that

$$p'(r) \leqslant p'_0 - \omega(r) \qquad \text{for } y \in \mathbf{B}, \tag{4.1}$$

where $p'_0 = p_0/(p_0 - 1) = (n - \nu)/(n - \nu - \alpha)$ and

$$\omega(r) = \frac{a}{(p_0 - 1)^2} \frac{\log(\log(1/r))}{\log(1/r)} - \frac{C}{\log(1/r)}$$

for $0 < r \leqslant r_0$.

Before the proof we prepare the following results.

LEMMA 4.2.   *For $0 < \delta < 1/2$, set*

$$E_1 = \{y \in \mathbf{B} \cap B(x, \delta) : \rho(y) \geqslant \delta\}.$$

If f is a nonnegative measurable function on $\mathbf{B}$ with $\|f\|_{p(\cdot),\nu,\beta} \leqslant 1$, then

$$\int_{E_1} |x - y|^{\alpha - n} f(y) dy \leqslant C(\log(1/\delta))^{-\tilde{A}}$$

for $x \in \mathbf{B}$.

Proof. For $0 < k < 1$ and $\gamma < \beta - 1$, we have

$$
\begin{aligned}
I_1 &\equiv \int_{E_1} |x - y|^{\alpha - n} f(y) dy \\
&\leqslant k \left\{ \int_{E_1} \left(\frac{|x - y|^{\alpha - n + \nu/p(y)} (\log(1/|x - y|))^{-\gamma/p(y)}}{k} \right)^{p'(y)} dy \right. \\
&\qquad \left. + \int_{E_1} |x - y|^{-\nu} (\log(1/|x - y|))^\gamma f(y)^{p(y)} dy \right\} \\
&\leqslant k \left\{ \int_{E_1} \left(\frac{|x-y|^{\alpha - n + \nu/p(y)} (\log(1/|x-y|))^{-\gamma/p(y)}}{k} \right)^{p'(y)} dy + C(\log(1/\delta))^{\gamma - \beta + 1} \right\}.
\end{aligned}
$$

Let $k = (\log(1/\delta))^{-\sigma}$ with $\sigma > 0$. Since $k^{-p'(y)} \leqslant k^{-p_0'}$ for $y \in E_1$, we find

$$
\begin{aligned}
J & \equiv \int_{E_1} \left(\frac{|x-y|^{\alpha-n+\nu/p(y)}(\log(1/|x-y|))^{-\gamma/p(y)}}{k} \right)^{p'(y)} dy \\
& \leqslant k^{-p_0'} \int_{E_1} \left(|x-y|^{\alpha-n+\nu/p(y)}(\log(1/|x-y|))^{-\gamma/p(y)} \right)^{p'(y)} dy \\
& \leqslant k^{-p_0'} \int_{E_1} |x-y|^{-n}(\log(1/|x-y|))^{-\gamma(p_0'-1)} \\
& \qquad \times \left(|x-y|^{-(\alpha-n+\nu)}(\log(1/|x-y|))^{\gamma} \right)^{\omega(\rho(y))} dy \\
& \leqslant k^{-p_0'} \int_{B(x,\delta)} |x-y|^{-n}(\log(1/|x-y|))^{-\gamma(p_0'-1)} \\
& \qquad \times \left(|x-y|^{\alpha(p_0-1)}(\log(1/|x-y|))^{\gamma} \right)^{\omega(|x-y|)} dy \\
& \leqslant Ck^{-p_0'} \int_0^{\delta} (\log(1/t))^{-\gamma(p_0'-1)-\alpha a/(p_0-1)} t^{-1} dt \\
& \leqslant Ck^{-p_0'} (\log(1/\delta))^{-(\gamma+a\alpha)(p_0'/p_0)+1}
\end{aligned}
$$

when $-(a\alpha + \gamma)p_0'/p_0 + 1 < 0$, that is,

$$
-a\alpha + p_0 - 1 < \gamma < \beta - 1.
$$

Since $\tilde{A} = (a\alpha + \beta)/p_0 - 1 > 0$, we can take such γ. Now it follows that

$$
I_1 \leqslant C(\log(1/\delta))^{-\sigma} \left\{ (\log(1/\delta))^{\sigma p_0' - (\gamma+a\alpha)(p_0'/p_0)+1} + (\log(1/\delta))^{\gamma-\beta+1} \right\}.
$$

By taking

$$
\sigma p_0' - (\gamma + a\alpha)(p_0'/p_0) + 1 = \gamma - \beta + 1,
$$

that is,

$$
\sigma = (\gamma + a\alpha)/p_0 + \gamma/p_0' - \beta/p_0' = a\alpha/p_0 + \gamma - \beta/p_0',
$$

we finally obtain

$$
I_1 \leqslant C(\log(1/\delta))^{-\tilde{A}},
$$

as required.   □

LEMMA 4.3.   *For $0 < \delta < 1/2$, set*

$$
E_2 = \{ y \in \mathbf{B} \cap B(x, \delta) : \rho(y) < \delta \}.
$$

If f is a nonnegative measurable function on $\mathbf{B}$ with $\|f\|_{p(\cdot),\nu,\beta} \leqslant 1$, then

$$
\int_{E_2} |x-y|^{\alpha-n} f(y) dy \leqslant C(\log(1/\delta))^{-\tilde{A}}
$$

for $x \in \mathbf{B}$.

Proof. As in the previous proof, for $k = (\log(1/\delta))^{-\sigma}(< 1)$ with $\sigma > 0$ and $\gamma < \beta - 1$, we have

$$I_2 \equiv \int_{E_2} |x - y|^{\alpha - n} f(y) dy$$

$$\leqslant k \left\{ \int_{E_2} \left(\frac{|x-y|^{\alpha - n + \nu/p(y)} (\log(1/|x - y|))^{-\gamma/p(y)}}{k} \right)^{p'(y)} dy \right.$$

$$\left. + \int_{E_2} |x - y|^{-\nu} (\log(1/|x - y|))^{\gamma} f(y)^{p(y)} dy \right\}$$

$$\leqslant k \left\{ \int_{E_2} k^{-p_0'} \left(|x-y|^{\alpha - n + \nu/p(y)} (\log(1/|x-y|))^{-\gamma/p(y)} \right)^{p'(y)} dy + C(\log(1/\delta))^{\gamma - \beta + 1} \right\}$$

$$= (\log(1/\delta))^{-\sigma} \left\{ (\log(1/\delta))^{\sigma p_0'} \int_{E_2} \left(|x-y|^{\alpha - n + \nu/p(y)} (\log(1/|x - y|))^{-\gamma/p(y)} \right)^{p'(y)} dy \right.$$

$$\left. + C(\log(1/\delta))^{\gamma - \beta + 1} \right\}.$$

For giving an estimate for I_2, we have only to treat the case when $\mathbf{B} \cap B(x, \delta) \subset \{y \in \mathbf{B} : \rho(y) < r_0\}$, since $p(y) \geqslant p(\delta/2) > (n - \nu)/\alpha$ when $\rho(y) \geqslant r_0/2$. In this case,

$$C^{-1}(|x^* - y^*| + |\rho(x) - \rho(y)|) \leqslant |x - y| \leqslant C(|x^* - y^*| + |\rho(x) - \rho(y)|)$$

for $y \in \mathbf{B} \cap B(x, \delta)$, where $x^* = x/|x|$ and $y^* = y/|y|$. Consider the sets

$$E_{21} = \{y \in E_2 : |\rho(x) - \rho(y)| \leqslant \rho(y)\}$$

and

$$E_{22} = \{y \in E_2 : |\rho(x) - \rho(y)| > \rho(y)\}.$$

We find by polar coordinates,

$$J_{21} \equiv \int_{E_{21}} \left(|x - y|^{\alpha - n + \nu/p(y)} (\log(1/|x - y|))^{-\gamma/p(y)} \right)^{p'(y)} dy$$

$$\leqslant C \int_{\{t: |\rho(x) - t| \leqslant t, t < \delta\}} \left(|\rho(x) - t|^{\alpha - n + \nu/p(t)} (\log(1/|\rho(x) - t|))^{-\gamma/p(t)} \right)^{p'(t)}$$

$$\times |\rho(x) - t|^{n-1} dt$$

$$\leqslant C \int_{\{t: |\rho(x) - t| \leqslant t, t < \delta\}} |\rho(x) - t|^{-n} (\log(1/|\rho(x) - t|))^{-\gamma(p_0' - 1)}$$

$$\times \left(|\rho(x) - t|^{\alpha(p_0 - 1)} (\log(1/|\rho(x) - t|))^{\gamma} \right)^{\omega(|\rho(x) - t|)} |\rho(x) - t|^{n-1} dt$$

$$\leqslant C \int_0^{\delta} (\log(1/s))^{-a\alpha/(p_0 - 1) - \gamma(p_0' - 1)} s^{-1} ds$$

$$\leqslant C(\log(1/\delta))^{-(\gamma + a\alpha)(p_0'/p_0) + 1},$$

as in the pervious proof. Further we find

$$J_{22} \equiv \int_{E_{22}} \left(|x-y|^{\alpha-n+v/p(y)} (\log(1/|x-y|))^{-\gamma/p(y)} \right)^{p'(y)} dy$$

$$\leqslant C \int_{E_{22}} \left(|x^*-y|^{\alpha-n+v/p(y)} (\log(1/|x^*-y|))^{-\gamma/p(y)} \right)^{p'(y)} dy,$$

where $x^* = x/|x|$. Hence

$$J_{22} \leqslant C \int_0^{\delta} (\log(1/t))^{-a\alpha/(p_0-1)-\gamma(p_0'-1)} t^{-1} dt$$

$$\leqslant C(\log(1/\delta))^{-(\gamma+a\alpha)(p_0'/p_0)+1},$$

as before. Now it follows that

$$I_2 \leqslant C(\log(1/\delta))^{-\sigma} \left\{ (\log(1/\delta))^{\sigma p_0' - (\gamma+a\alpha)(p_0'/p_0)+1} + (\log(1/\delta))^{\gamma-\beta+1} \right\},$$

which yields the required result. □

Finally, according to the proof of Lemma 3.4, we obtain the following result.

LEMMA 4.4. *If f is a nonnegative measurable function on $\mathbf{B}$ with $\|f\|_{p(\cdot),v,\beta} \leqslant 1$, then*

$$\int_{\mathbf{B}\setminus B(x,\delta)} |x-y|^{\alpha-n-1} f(y) dy \leqslant C\delta^{-1}(\log(1/\delta))^{-\tilde{A}-1/p_0'}$$

for $x \in \mathbf{B}$.

Proof. Write

$$\int_{\mathbf{B}\setminus B(x,\delta)} |x-y|^{\alpha-n-1} f(y) dy = \int_{\{y\in\mathbf{B}\setminus B(x,\delta):\rho(y)\geqslant\delta\}} |x-y|^{\alpha-n-1} f(y) dy$$

$$+ \int_{\{y\in\mathbf{B}\setminus B(x,\delta):\rho(y)<\delta\}} |x-y|^{\alpha-n-1} f(y) dy$$

$$= I_1 + I_2.$$

As in the proof of Lemma 3.4, we prove

$$I_1 \leqslant C\delta^{-1}(\log(1/\delta))^{-\tilde{A}-1}.$$

For I_2, letting $k > 1$, we have by Young's inequality

$$I_2 \leqslant k \left\{ \int_{\{y\in\mathbf{B}\setminus B(x,\delta):\rho(y)<\delta\}} \left(\frac{|x-y|^{\alpha-1-n+v/p(y)} (\log(1/|x-y|))^{-\gamma/p(y)}}{k} \right)^{p'(y)} dy \right.$$

$$\left. + \int_{\{y\in\mathbf{B}\setminus B(x,\delta):\rho(y)<\delta\}} |x-y|^{-v} (\log(1/|x-y|))^{-\gamma/p(y)} f(y)^{p(y)} dy \right\}$$

with γ given before. Note that for $y \in \{\mathbf{B} \setminus B(x, \delta) : \rho(y) < \delta\}$, we find

$$k^{-p'(y)} \leqslant k^{-p_0' + \omega_1(\delta)},$$

where

$$\omega_1(r) = \frac{a}{(p_0 - 1)^2} \frac{\log(\log(1/r))}{\log(1/r)}.$$

Hence we obtain

$$I_2 \leqslant k \left\{ C k^{-p_0' + \omega_1(\delta)} \int_{\{y \in \mathbf{B} \setminus B(x, \delta) : \rho(y) < \delta\}} \left\{ |x - y|^{\alpha - 1 - n + \nu/p(y)} (\log(1/|x - y|))^{-\gamma/p(y)} \right\}^{p'(y)} dy \right.$$
$$\left. + C(\log(1/\delta))^{\gamma - \beta + 1} \right\}.$$

In a way similar to the estimates of J_{21} and J_{22} in the previous proof, we establish

$$\int_{\{y \in \mathbf{B} \setminus B(x, \delta) : \rho(y) < \delta\}} \left(|x - y|^{\alpha - 1 - n + \nu/p(y)} (\log(1/|x - y|))^{-\gamma/p(y)} \right)^{p'(y)} dy$$
$$\leqslant C \int_0^\delta \left(\delta^{\alpha - 1 - n + \nu/p(t)} (\log(1/\delta))^{-\gamma/p(t)} \right)^{p'(t)} \delta^{n-1} dt$$
$$= C \delta^{-p_0' - 1} (\log(1/\delta))^{-\gamma(p_0' - 1)} \int_0^\delta \delta^{-(\alpha - 1 - n + \nu)\omega(t)} (\log(1/\delta))^{\gamma \omega(t)} dt$$
$$\leqslant C \delta^{-p_0' - 1} (\log(1/\delta))^{-\gamma(p_0' - 1)} \int_0^\delta \delta^{-(\alpha - 1 - n + \nu)\omega(t)} dt$$
$$\leqslant C \delta^{-p_0'} (\log(1/\delta))^{-a\alpha/(p_0 - 1) - a/(p_0 - 1)^2 - \gamma(p_0' - 1)}.$$

Consequently it follows that

$$I_2 \leqslant C k \left\{ k^{-p_0' + \omega_1(\delta)} \delta^{-p_0'} (\log(1/\delta))^{-a\alpha/(p_0 - 1) - a/(p_0 - 1)^2 - \gamma(p_0' - 1)} + C(\log(1/\delta))^{\gamma - \beta + 1} \right\}.$$

Considering $k = \delta^{-1} (\log(1/\delta))^{-(a\alpha + \beta)/p_0 - \gamma + \beta - 1/p_0'}$, we finally find

$$I_2 \leqslant C \delta^{-1} (\log(1/\delta))^{-(a\alpha + \beta)/p_0 + 1/p_0} = C \delta^{-1} (\log(1/\delta))^{-\tilde{A} - 1/p_0'},$$

which proves the required result. $\square$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let f be a nonnegative measurable function on $\mathbf{B}$ with $\|f\|_{p(\cdot), \nu, \beta} \leqslant 1$. By Lemmas 4.2 and 4.3, we have

$$\int_{\mathbf{B} \cap B(x, 2|x - z|)} |x - y|^{\alpha - n} f(y) dy \leqslant C(\log(1/|x - z|))^{-\tilde{A}} \qquad (4.2)$$

for $x, z \in \mathbf{B}$ and $|x - z| < 1/2$. Similarly,

$$\int_{\mathbf{B} \cap B(x,2|x-z|)} |z-y|^{\alpha-n} f(y) dy \leqslant \int_{\mathbf{B} \cap B(z,3|x-z|)} |z-y|^{\alpha-n} f(y) dy \leqslant C(\log(1/|x-z|))^{-\bar{A}}$$

$$(4.3)$$

for $x, z \in \mathbf{B}$ and $|x - z| < 1/2$. On the other hand, by Lemma 4.4, we have for $x, z \in \mathbf{B}$ with $|x - z| < 1/2$,

$$\int_{\mathbf{B} \setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y) dy$$

$$\leqslant C|x-z| \int_{\mathbf{B} \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y) dy$$

$$\leqslant C(\log(1/|x-z|))^{-\bar{A}},$$

which together with (4.2) and (4.3) gives

$$|U_\alpha f(x) - U_\alpha f(z)| \leqslant C(\log(1/|x-z|))^{-\bar{A}}$$

for $x, z \in \mathbf{B}$ and $|x - z| < 1/2$, as required. $\square$

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