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SOBOLEV EMBEDDINGS FOR UNBOUNDED DOMAIN WITH VARIABLE EXPONENT HAVING VALUES ACROSS N

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Abstract. We study Sobolev embeddings for unbounded domain with variable exponent having values across N. A main result of this paper is the following theorem: Let Ω be an unbounded domain in $\mathbb{R}^N$ satisfying the uniform cone condition. Suppose that $p:\overline{\Omega}\to\mathbb{R}$ is Lipschitz and $1<\inf_\Omega p\leqslant\sup_\Omega p<\infty$. Then there holds a continuous embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q\in L^\infty(\Omega)$ satisfying condition $p(x)\leqslant q(x)\leqslant p^*(x)$ for a.e. $x\in\Omega$, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ if $p(x)<N$ and $p^*(x)=\infty$ if $p(x)\geqslant N$. In this theorem the usual condition $\sup p(x)<N$ is not required.

1. Introduction

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent. For a survey we refer to [5, 9, 19, 32]. For the application backgrounds in nonlinear elasticity and electrorheological fluids we refer to [1, 4, 21, 29].

The theory of variable exponent Sobolev spaces is an important theoretical tool to study the variable exponent problems. For the study of the variable exponent Lebesgue-Sobolev spaces we refer to [5–9, 11–13, 15–20, 22–24, 26–32]. From the point of differential equations the Sobolev embedding theorems are very important. For brevity, in this paper, we only consider the embeddings in the space $W^{1,p(\cdot)}(\Omega)$ since the situation in the space $W^{k,p(\cdot)}(\Omega)$ is similar, where Ω is an open domain in $\mathbb{R}^N$. It is well-known that, in the constant exponent case, the embeddings in $W^{1,p}(\Omega)$ are qualitatively different according as $1\leqslant p<N$ (Lebesgue space), $p=N$ (exponential Orlicz space) or $p>N$ (Hölder space). For an improvement of the classical Sobolev embedding theorems when $1\leqslant p<N$ or $p=N$ we refer to [25] and references therein. In the variable exponent case things are more complicated, especially in the case when $p_-<N<p_+$, i.e. when the variable exponent $p(\cdot)$ has the values across N, where

$$p_-:=\operatorname{essinf}_{x\in\Omega}p(x)\ \text{ and }\ p_+:=\operatorname{esssup}_{x\in\Omega}p(x).$$

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The Sobolev embedding theorems in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ have been studied by many authors (see e.g. [6–9, 11–13, 15–20, 23, 24, 26, 27, 30, 31]). Roughly speaking, when $p_+ < N$, it is proved that, under appropriate assumptions, there holds the critical Sobolev embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$. Edmunds and Rákosník [11,12] have established the embedding theorems when $p(x) < N$ for $x \in \Omega$ and $p_+ = N$, and when $p(x) > N$ for $x \in \Omega$, respectively. Recently Harjulehto and Hästö [20] have studied the Sobolev embedding in $W^{1,p(\cdot)}(\Omega)$ in the case that Ω is a bounded John domain and $p : \overline{\Omega} \to [1,N]$ is log-Hölder continuous. They have introduced "a slightly modified scale of variable exponent function spaces, $L^{p(\cdot),*}(\Omega)$, with the property $L^{p(\cdot),*}(\Omega) \cong L^{p^*(\cdot)}(\Omega)$ if $p_+ < N$ and $L^{p(\cdot),*}(\Omega_N) \cong \exp L^{N'}(\Omega_N)$ (where $\Omega_N = p^{-1}(N)$)" and proved that there holds an embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),*}(\Omega)$. Although these embedding results are very interesting, however, the following question is still open.

An open question: What is the space Y such that there holds an embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow Y$ in the case when $p_- < N < p_+$ and this embedding can include the classical Sobolev embeddings as its special cases? (A further question is : What is the sharp embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow Y$ in the case when $p_- < N < p_+$?)

In this paper we consider the embeddings in $W^{1,p(\cdot)}(\Omega)$ when Ω is an unbounded domain in $\mathbb{R}^N$ and $p_- < N < p_+$, but do not consider the open question. We only give some embedding results of type $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. Such results are not an answer to the open question, however they are useful in some practical problems. For example, in some papers studying the $p(x)$-Laplacian equations in an unbounded domain Ω, the assumption $p_+ < N$, a strong restrictive condition, is required (see e.g. [3, 10, 14, 33]), but in fact, by the results obtained in the present paper the assumption $p_+ < N$ can be canceled, and this just the purpose of the present paper.

For a variable exponent $p(\cdot)$, define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ \infty, & \text{if } p(x) \geqslant N. \end{cases}$$

The main results of this paper are the following theorems.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain satisfying the uniform cone condition (see [2] for the definition). Suppose that $p : \overline{\Omega} \to R$ is Lipschitz and $1 < p_- \leqslant p_+ < \infty$. Then there holds a continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q \in L^\infty(\Omega)$ satisfying condition*

$$p(x) \leqslant q(x) \leqslant p^*(x) \text{ for a.e. } x \in \Omega. \tag{1.1}$$

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain satisfying the uniform cone condition. Suppose that $p : \overline{\Omega} \to R$ is uniformly continuous and $1 < p_- \leqslant p_+ < \infty$. Then there holds a continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q \in L^\infty(\Omega)$ satisfying condition*

$$p(x) \leqslant q(x) \leqslant q(x) + \varepsilon \leqslant p^*(x) \text{ for a.e. } x \in \Omega, \tag{1.2}$$

where ε *is a positive constant.*

THEOREM 1.3. *Suppose that* $p:\mathbb{R}^N\to\mathbb{R}$ *is uniformly continuous and* $1<p_-\leqslant p_+<\infty$. *Then there holds a compact embedding* $W_r^{1,p(\cdot)}(\mathbb{R}^N)\hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$ *for any* $q\in L^\infty(\mathbb{R}^N)$ *satisfying condition*

$$p(x)+\varepsilon\leqslant q(x)\leqslant q(x)+\varepsilon\leqslant p^*(x)\ \text{for a.e. } x\in\mathbb{R}^N,$$

where ε *is a positive constant and*

$$W_r^{1,p(\cdot)}(\mathbb{R}^N):=\left\{u\in W^{1,p(\cdot)}(\mathbb{R}^N):u\ \text{ is radially symmetric }\right\}.$$

Theorems 1.1, 1.2 and 1.3 are a generalization of Theorem 1.1 in [15], Theorem 1.2 in [15] and Theorem 3.1 in [17] respectively, there the assumption $p_+<N$ is required.

In the case when Ω is bounded, the corresponding theorems 1.1 and 1.2 are clear and known, and in the proof of them a simple fact that

$$p_1(\cdot)\leqslant p_2(\cdot)\Longrightarrow L^{p_2(\cdot)}(\Omega)\hookrightarrow L^{p_1(\cdot)}(\Omega)\ \text{ and }\ W^{1,p_2(\cdot)}(\Omega)\hookrightarrow W^{1,p_1(\cdot)}(\Omega) \tag{1.3}$$

is used (see Section 2 below). However, for an unbounded domain Ω, the fact is not true. Hence the proof of Theorems 1.1 and 1.2 is essentially different from the proof of the corresponding results in the case when Ω is bounded.

This paper is in three sections. In Section 2, we give some preliminaries. In Section 3, we give the proof of Theorems 1.1–1.3.

2. Preliminaries

Let Ω be an open set in $\mathbb{R}^N$. Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω. Two measurable functions defined on Ω are regarded as the same element of $S(\Omega)$ when they are equal almost everywhere in Ω. Let $p\in S(\Omega)$. For a measurable subset E of Ω, denote $p_-(E)=\operatorname{essinf}_{x\in E}p(x)$ and $p_+(E)=\operatorname{esssup}_{x\in E}p(x)$. $p_-(\Omega)$ and $p_+(\Omega)$ are written simply by p_- and p_+ respectively. $|E|$ denotes the N-Lebesgue measure of E. We denote by $C^{0,1}(\overline{\Omega})$ the set of all Lipschitz functions defined on $\overline{\Omega}$.

Now let $p\in S(\Omega)$ be given such that $1\leqslant p_-\leqslant p_+<\infty$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega)=\left\{u\in S(\Omega):\int_\Omega|u|^{p(x)}\,dx<\infty\right\}$$

with the norm

$$|u|_{L^{p(\cdot)}(\Omega)}=|u|_{p(\cdot)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u}{\lambda}\right|^{p(x)}dx\leqslant 1\right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega):\ |\nabla u|\in L^{p(\cdot)}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}.$$

The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable Banach spaces. We refer to [5, 6, 8, 15, 16, 24, 28] for the elementary properties of the space $W^{1,p(\cdot)}(\Omega)$.

We denote $p'(x) = \frac{p(x)}{p(x)-1}$ for $p(x) \in (1,\infty)$. We will use the Young inequality

$$ab \leqslant \frac{a^{p'(x)}}{p'(x)} + \frac{b^{p(x)}}{p(x)}, \quad \forall a,b \geqslant 0.$$

For any $r > 0$ we define

$$r_* = \frac{Nr}{N+r}. \tag{2.1}$$

Then $r_* < N$ and $(r_*)^* = \frac{Nr_*}{N-r_*} = r$. Note that $r_* > 1$ if and only if $r > \frac{N}{N-1}$.

For $p,q \in S(\Omega)$, the notation "$p(\cdot) \leqslant q(\cdot)$ on Ω" denotes that $p(x) \leqslant q(x)$ for a.e. $x \in \Omega$, and the notation "$p(\cdot) \ll q(\cdot)$ on Ω" denotes that there exists $\varepsilon > 0$ such that $p(x) + \varepsilon \leqslant q(x)$ for a.e. $x \in \Omega$.

It is well-known that the classical Sobolev embedding theorems depend on regularity properties of Ω. For convenience of readers, we write some definitions stemming from [2].

(The Cone Condition) (see [2, p. 82]) Ω satisfies the cone condition if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. Note that C_x need not be obtained from C by parallel translation, but simply by rigid motion.

(The Uniform Cone Condition) (see [2, p. 83]) Ω satisfies the uniform cone condition if there exists a locally finite open cover $\{U_j\}$ of the boundary of Ω and a corresponding sequence $\{C_j\}$ of finite cones, each congruent to some fixed finite cone C, such that

1) There exists $M < \infty$ such that every U_j has diameter less than M.

2) $\Omega_\delta \subset \cup_{j=1}^{\infty} U_j$ for some $\delta > 0$, where $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$.

3) $Q_j \equiv \cup_{x \in \Omega \cap U_j}(x + C_j) \subset \Omega$ for every j.

4) ord$\{Q_j\} \leqslant K$ for some finite K, namely, every collection of $K+1$ of the sets Q_j has empty intersection.

In [2, p. 83] it is also defined that Ω satisfies the strong local Lipschitz condition, which, when Ω is bounded, is just that Ω has a locally Lipschitz boundary. Between these conditions there holds the following relation (see [2, p. 84]):

the strong local Lipschitz condition $\Longrightarrow$ the uniform cone condition
$\Longrightarrow$ the cone condition.

Note that the classical Sobolev embedding theorems from $W^{1,p}(\Omega)$ into some Lebesgue space $L^q(\Omega)$ are obtained under the cone condition (see [2, p. 85]).

Let us first consider the case when Ω is bounded.

Let Ω be a bounded open set in $\mathbb{R}^N$. A function $p:\overline{\Omega}\to\mathbb{R}$ is said to be log-Hölder continuous on $\overline{\Omega}$, denoted by $p\in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, if there exists a positive constant L such that

$$|p(x)-p(y)|\leqslant\frac{L}{|\log|x-y||},\quad\forall x,y\in\overline{\Omega}\text{ with }|x-y|<\frac{1}{2}.\tag{2.2}$$

PROPOSITION 2.1. (see [8]) *Let $\Omega\subset R^N$ be a bounded domain with Lipschitz boundary. Suppose that $p\in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ and*

$$1<p_-\leqslant p_+<N.$$

Then there holds a continuous embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p^*(\cdot)}(\Omega)$, and consequently for any $q\in L^\infty(\Omega)$ satisfying condition

$$1\leqslant q(x)\leqslant p^*(x)\text{ for a.e. }x\in\Omega,\tag{2.3}$$

there holds a continuous embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$.

Based on Proposition 2.1 we can easily obtain the following proposition in which the restrictive condition $p_+<N$, required in Proposition 2.1, is canceled.

PROPOSITION 2.2. *Let $\Omega\subset R^N$ be a bounded domain with Lipschitz boundary. Suppose $p\in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ and $1<p_-$. Then for any $q\in L^\infty(\Omega)$ satisfying (2.3), there holds a continuous embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$.*

Proof. Let $q\in L^\infty(\Omega)$ satisfy (2.3). Take $r\geqslant q_+$ such that $r_*>1$. Define $\widetilde{p}:\overline{\Omega}\to\mathbb{R}$ by

$$\widetilde{p}(x)=\begin{cases}p(x), & \text{if } p(x)<r_*\\ r_*, & \text{if } p(x)\geqslant r_*.\end{cases}$$

Then $\widetilde{p}\in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, $\widetilde{p}_->1$, $\widetilde{p}_+<N$ and $\widetilde{p}(x)\leqslant p(x)$ for all $x\in\Omega$. For $x\in\Omega$ with $p(x)<r_*$, we have $\widetilde{p}^*(x)=p^*(x)\geqslant q(x)$. For $x\in\Omega$ with $p(x)\geqslant r_*$, we have $\widetilde{p}^*(x)=(r_*)^*=r\geqslant q_+\geqslant q(x)$. Thus $1\leqslant q(x)\leqslant\widetilde{p}^*(x)$ for a.e. $x\in\Omega$. By Proposition 2.1, $W^{1,\widetilde{p}(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ and hence $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ because $W^{1,p(\cdot)}(\Omega)\hookrightarrow W^{1,\widetilde{p}(\cdot)}(\Omega)$. $\square$

REMARK 2.1. There are some variants of Proposition 2.1 asserting the critical embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p^*(\cdot)}(\Omega)$ with $p_+<N$. Such embedding has been obtained by Edmunds and Rákosník [11], Edmunds and Rákosník [12], Fan, Shen and Zhao [15], and Harjulehto and Hästö [20] under the hypotheses that $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $p\in C^{0,1}(\overline{\Omega})$ and $1\leqslant p_-$ (see [11]); that $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $p\in W^{1,\sigma}(\Omega)$ with $\sigma>N$ and $1\leqslant p_-$ (see [12]); that $\Omega\subset\mathbb{R}^N$ is a bounded (or unbounded) domain satisfying the cone condition, $p\in C^{0,1}(\overline{\Omega})$ and $1<p_-$ (see [15]); and that $\Omega\subset\mathbb{R}^N$ be a bounded John domain with Lipschitz boundary $p\in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ and $1\leqslant p_-$ (see [20]); respectively. Based on such variants of Proposition 2.1 we can obtain the corresponding variants of Proposition 2.2.

Now let us turn to consider the case when Ω is unbounded. A basis of the main theorems of the present paper is the following proposition asserting the critical embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ in the case when Ω is unbounded and $p_+ < N$.

PROPOSITION 2.3. (see [15]) *Let $\Omega \subset R^N$ be a bounded or an unbounded domain satisfying the cone condition, $p \in C^{0,1}(\overline{\Omega})$ with Lipschitz constant L and $1 < p_- \leqslant p_+ < N$. Then there holds a continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$, where the embedding constant c depends only on N, p_-, p_+, L and the dimensions of the cone C in the cone condition. Furthermore, for any $q \in L^\infty(\Omega)$ satisfying condition*

$$p(x) \leqslant q(x) \leqslant p^*(x) \text{ for a.e. } x \in \Omega,$$

there holds a continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

We want prove Theorem 1.1 by using Proposition 2.3. However, the methods used in the proof of Proposition 2.2 are not completely suitable for the unbounded domain case because in the proof of Proposition 2.2 the fact $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,\widetilde{p}(\cdot)}(\Omega)$ is used but it is not true in the unbounded domain case. To prove Theorem 1.1 we need some new methods, in particular the following result is useful.

PROPOSITION 2.4. *Let Q be a bounded open set in R^N with $|Q| \leqslant 1$. Suppose that $p_1, p_2 \in L^\infty(Q)$ and*

$$1 \leqslant p_1(x) \leqslant p_2(x) \text{ for a.e. } x \in Q.$$

Then $|u|_{L^{p_1(\cdot)}(Q)} \leqslant 2\,|u|_{L^{p_2(\cdot)}(Q)}$ for every $u \in L^{p_2(\cdot)}(Q)$, and $\|u\|_{W^{1,p_1(\cdot)}(Q)} \leqslant 2\,\|u\|_{W^{1,p_2(\cdot)}(Q)}$ for every $u \in W^{1,p_2(\cdot)}(Q)$.

Proof. Take any $u \in L^{p_2(\cdot)}(Q)$ with $|u|_{p_2(\cdot)} = 1$. Obviously, $u \in L^{p_1(\cdot)}(Q)$. Put

$$Q_0 = \{x \in Q : p_1(x) = p_2(x)\} \text{ and } Q_1 = Q \setminus Q_0.$$

Then, by the Young inequality,

$$\begin{aligned}
\int_Q |u|^{p_1(x)}\,dx &= \int_{Q_0} |u|^{p_1(x)}\,dx + \int_{Q_1} 1 \cdot |u|^{p_1(x)}\,dx \\
&= \int_{Q_0} |u|^{p_2(x)}\,dx + \int_{Q_1} \frac{1}{\left(\frac{p_2(x)}{p_1(x)}\right)'}\,dx + \int_{Q_1} \frac{|u|^{p_2(x)}}{\frac{p_2(x)}{p_1(x)}}\,dx \\
&\leqslant \int_{Q_0} |u|^{p_2(x)}\,dx + |Q_1| + \int_{Q_1} |u|^{p_2(x)}\,dx \\
&\leqslant 1 + \int_Q |u|^{p_2(x)}\,dx = 1 + 1 = 2,
\end{aligned}$$

which implies $|u|_{p_1(\cdot)} \leqslant 2$. This shows that $|u|_{p_1(\cdot)} \leqslant 2\,|u|_{p_2(\cdot)}$ for every $u \in L^{p_2(\cdot)}(Q)$. When $u \in W^{1,p_2(\cdot)}(Q)$, we have that $u \in W^{1,p_1(\cdot)}(Q)$ and

$$\begin{aligned}
\|u\|_{W^{1,p_1(\cdot)}(Q)} &= |u|_{L^{p_1(\cdot)}(Q)} + |\nabla u|_{L^{p_1(\cdot)}(Q)} \\
&\leqslant 2\,|u|_{L^{p_2(\cdot)}(Q)} + 2\,|\nabla u|_{L^{p_2(\cdot)}(Q)} = 2\,\|u\|_{W^{1,p_2(\cdot)}(Q)}. \qquad \square
\end{aligned}$$

3. Proof of main results

In this section we prove Theorems 1.1–1.3.

Proof of Theorem 1.1. Let the hypotheses of Theorem 1.1 hold and let $q \in L^{\infty}(\Omega)$ satisfy (1.1). Analogously to the proof of Proposition 2.2, take $r > q_+$ such that $r_* > 1$, and define $\widetilde{p} : \overline{\Omega} \to \mathbb{R}$ by

$$\widetilde{p}(x) = \begin{cases} p(x), & \text{if } p(x) < r_* \\ r_*, & \text{if } p(x) \geqslant r_*. \end{cases} \tag{3.1}$$

Then $\widetilde{p} \in C^{0,1}(\overline{\Omega})$, $\widetilde{p}_- > 1$, $\widetilde{p}_+ \leqslant r_* < N$ and $\widetilde{p}(x) \leqslant p(x)$ for every $x \in \Omega$. For $x \in \Omega$ with $p(x) < r_*$, we have that $\widetilde{p}(x) = p(x)$, $\widetilde{p}^*(x) = p^*(x) \geqslant q(x)$. For $x \in \Omega$ with $p(x) \geqslant r_*$, we have $\widetilde{p}^*(x) = (r_*)^* = r > q_+ \geqslant q(x) \geqslant p(x)$. Thus

$$\widetilde{p}(\cdot) \leqslant p(\cdot) \leqslant q(\cdot) \leqslant \widetilde{p}^*(\cdot) \text{ on } \Omega \text{ and } p(\cdot) \ll \widetilde{p}^*(\cdot) \text{ on } \Omega. \tag{3.2}$$

Below we shall prove that there holds a continuous embedding

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\widetilde{p}^*(\cdot)}(\Omega). \tag{3.3}$$

It is obvious that the assertion of Theorem 1.1, $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, follows from (3.3) and (3.2).

Let any $u \in W^{1,p(\cdot)}(\Omega)$ be given. We shall prove that $u \in L^{\widetilde{p}^*(\cdot)}(\Omega)$.

Since $\widetilde{p}$ and p are uniformly continuous on $\overline{\Omega}$, by (3.2), there exists $\delta_1 > 0$ small enough such that

$$(\widetilde{p}^*)_-(E) \geqslant p_+(E) \text{ for any } E \subset \overline{\Omega} \text{ with } \operatorname{diam} E \leqslant \delta_1. \tag{3.4}$$

Since $u \in W^{1,p(\cdot)}(\Omega)$, there exists $\delta_2 \in (0,1)$ small enough such that

$$\int_E \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx \leqslant 1 \text{ for any } E \subset \Omega \text{ with } |E| \leqslant \delta_2. \tag{3.5}$$

Note that Ω satisfies the uniform cone condition. Let $\{U_j\}$, C, δ, and K be as in the definition of the uniform cone condition. From the definition we can see that there exists a locally finite open cover $\{V_i\}_{i=1}^{\infty}$ of $\partial\Omega$, being a refinement of $\{U_j\}$, such that for each i, $\operatorname{diam} V_i \leqslant \delta_1$, $|V_i| \leqslant \delta_2$, $V_i \cap \Omega$ satisfies the cone condition with a finite cone $\varepsilon_1 C$, where $\varepsilon_1 \in (0,1)$ is independent of i, $\cup_{i=1}^{\infty} V_i \supset \Omega_{\delta_3}$ for some $\delta_3 \in (0,\delta)$, and $\operatorname{ord}\{V_i\} \leqslant K_1$ for some positive integer $K_1 \geqslant K$. Without loss of generality, we may assume that the aperture angle of the cone C is less than $\frac{\pi}{2}$. Take $\delta_4 > 0$ small enough such that for a N-cube Q with edge length δ_4, there hold $|Q| \leqslant \delta_2$ and $\operatorname{diam} Q < \min\{\delta_1, \delta_3\}$. We can find a closed-cube over $\{\overline{Q_i}\}_{i=1}^{\infty}$ of $\Omega \setminus \Omega_{\delta_3}$ such that each Q_i is an open N-cube with edge length δ_4, $Q_i \cap (\Omega \setminus \Omega_{\delta_3}) \neq \emptyset$ for every i, and $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Note that $\overline{Q_i} \subset \Omega$ because $\operatorname{diam} Q_i < \delta_3$, and each N-cube with edge length δ_4 satisfies the cone condition with a finite cone $\varepsilon_2 C$ for some sufficiently small $\varepsilon_2 \in (0,1)$. Put $W_i = V_i \cap \Omega$. We renumber $\{W_i\}_{i=1}^{\infty} \cup \{Q_i\}_{i=1}^{\infty}$ and denote it by $\{G_j\}_{j=1}^{\infty}$, where $G_j = W_i$ or Q_i. Then $\{G_j\}_{j=1}^{\infty}$ satisfies the following conditions:

1) Each $G_j \subset \Omega$ is open and $\left|\Omega \setminus \left(\cup_{j=1}^{\infty} G_j\right)\right| = 0$.

2) Each G_j satisfies the cone condition with a finite cone C_0 independent of j, where $C_0 = \varepsilon C$ with $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

3) For each j, $\operatorname{diam} G_j \leqslant \delta_1$ and $|G_j| \leqslant \delta_2 < 1$.

4) $\operatorname{ord}\{G_j\} \leqslant K_2$ for some positive integer K_2.

For each bounded open set G_j, since $\widetilde{p}(x) \leqslant p(x)$ for all $x \in \Omega$ and $|G_j| < 1$, we have that $W^{1,p(\cdot)}(G_j) \hookrightarrow W^{1,\widetilde{p}(\cdot)}(G_j)$, and by Proposition 2.4,

$$\|u\|_{W^{1,\widetilde{p}(\cdot)}(G_j)} \leqslant 2\,\|u\|_{W^{1,p(\cdot)}(G_j)} \quad \text{for every } u \in W^{1,p(\cdot)}(G_j). \tag{3.6}$$

Since $\operatorname{diam} G_j \leqslant \delta_1$,

$$(\widetilde{p}^*)_-(G_j) \geqslant p_+(G_j) \text{ for every } j. \tag{3.7}$$

Applying Proposition 2.3 to $W^{1,\widetilde{p}(\cdot)}(G_j)$, we know that there holds the embedding

$$W^{1,\widetilde{p}(\cdot)}(G_j) \hookrightarrow L^{\widetilde{p}^*(\cdot)}(G_j) \tag{3.8}$$

and the embedding constant c_j depends only on N, $\widetilde{p}_-$, $\widetilde{p}_+$ and the dimensions of the cone C_0, and hence we can take the embedding constant $c_j = c_0$ which is independent of j (note that this is a key of the proof). We may assume $c_0 \geqslant 1$.

Since $|G_j| \leqslant \delta_2$, by (3.5),

$$\int_{G_j} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx \leqslant 1 \text{ for every } j. \tag{3.9}$$

Thus, for each G_j, by (3.6)-(3.9), we have that

$$\begin{aligned}
\int_{G_j} |u|^{\widetilde{p}^*(x)}\, dx &= \left(|u|_{L^{\widetilde{p}^*(\cdot)}(G_j)}\right)^{\widetilde{p}^*(\xi_j)} \leqslant \left(c_0\,\|u\|_{W^{1,\widetilde{p}(\cdot)}(G_j)}\right)^{\widetilde{p}^*(\xi_j)} \\
&\leqslant \left(2c_0\,\|u\|_{W^{1,p(\cdot)}(G_j)}\right)^{\widetilde{p}^*(\xi_j)} \\
&\leqslant 2^r c_0^r \left(\int_{G_j} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx\right)^{\frac{\widetilde{p}^*(\xi_j)}{p(\eta_j)}} \\
&\leqslant 2^r c_0^r \int_{G_j} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx,
\end{aligned} \tag{3.10}$$

where ξ_j and η_j are some elements in $\overline{G_j}$, and by (3.7), $\frac{\widetilde{p}^*(\xi_j)}{p(\eta_j)} \geqslant 1$.

From (3.10) and $\operatorname{ord}\{G_j\} \leqslant K$ it follows that

$$\begin{aligned}
\int_{\Omega} |u|^{\widetilde{p}^*(x)}\, dx &\leqslant \sum_{j=1}^{\infty} \int_{G_j} |u|^{\widetilde{p}^*(x)}\, dx \leqslant 2^r c_0^r \sum_{j=1}^{\infty} \int_{G_j} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx \\
&\leqslant 2^r c_0^r K_2 \int_{\Omega} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx,
\end{aligned} \tag{3.11}$$

which shows that (3.3) holds. Theorem 1.1 is proved. □

Proof of Theorem 1.2. Let the hypotheses of Theorem 1.2 hold and let $q \in L^{\infty}(\Omega)$ satisfy (1.2). Take $r > q_+$ such that $r_* > 1$. Define $\widetilde{p}_1 : \overline{\Omega} \to \mathbb{R}$ by

$$\widetilde{p}_1(x) = \begin{cases} p(x), & \text{if } p(x) < r_* \\ r_*, & \text{if } p(x) \geqslant r_*. \end{cases}$$

Then $\widetilde{p}_1 : \overline{\Omega} \to \mathbb{R}$ is uniformly continuous, $(\widetilde{p}_1)_- > 1$, $(\widetilde{p}_1)_+ < N$ and $\widetilde{p}_1(x) \leqslant p(x)$ for every $x \in \Omega$. It is easy to verify that

$$\widetilde{p}_1(\cdot) \leqslant p(\cdot) \leqslant q(\cdot) \ll (\widetilde{p}_1)^*(\cdot) \text{ on } \Omega. \tag{3.12}$$

Since $\widetilde{p}_1 : \overline{\Omega} \to \mathbb{R}$ is uniformly continuous, given any $\varepsilon_1 > 0$, there exists a Lipschitz function $\widetilde{p} : \overline{\Omega} \to \mathbb{R}$ such that

$$\widetilde{p}(x) \leqslant \widetilde{p}_1(x) \leqslant \widetilde{p}(x) + \varepsilon_1 \text{ for every } x \in \Omega.$$

Noting that $q(\cdot) \ll (\widetilde{p}_1)^*(\cdot)$ on Ω, we can take $\varepsilon_1 > 0$ small enough such that

$$\widetilde{p}(\cdot) \leqslant p(\cdot) \leqslant q(\cdot) \ll (\widetilde{p})^*(\cdot) \text{ on } \Omega. \tag{3.13}$$

Note that (3.13) implies (3.2). Using the same arguments as were done in the proof of Theorem 1.1, we can prove that

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\widetilde{p}^*(\cdot)}(\Omega), \tag{3.14}$$

and then the assertion of Theorem 1.2, $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, follows from (3.14) and (3.13). □

We omit the proof of Theorem 1.3 because it can be carried out in the same way as the proof of Theorem 3.1 in [17]. Indeed, the only difference between the two theorems is that in Theorem 1.3 the assumption $p_+ < N$ is not required. The only difference between the proofs of the two theorems is that in the proof of Theorem 1.3 we need use Theorem 1.2 instead of Theorem 1.2 in [15] used in the proof of Theorem 3.1 in [17].

As was noted in [13], there are some variants of Proposition 2.3. In [13] the following definition was introduced.

DEFINITION 3.1. Let $\gamma \in S(\Omega)$ satisfy $1 \leqslant \gamma_- \leqslant \gamma_+ < \infty$. The space $W^{1,(\infty,\gamma(\cdot))}(\Omega)$ is defined by

$$W^{1,(\infty,\gamma(\cdot))}(\Omega) := \left\{ u \in L^{\infty}(\Omega) : |\nabla u| \in L^{\gamma(\cdot)}(\Omega) \right\}$$

with the norm $\|u\|_{W^{1,(\infty,\gamma(\cdot))}(\Omega)} = \|u\|_{1,(\infty,\gamma(\cdot))} := |u|_{\infty} + |\nabla u|_{\gamma(\cdot)}$.

In [13] the following proposition was proved.

PROPOSITION 3.1. (see [13]) *Let* Ω *be a (bounded or unbounded) domain in* $\mathbb{R}^N$ *satisfying the cone condition and* $p \in W^{1,(\infty,\gamma(\cdot))}(\Omega)$ *with* $1 \leqslant p_- \leqslant p_+ < N$, *where* $\gamma \in S(\Omega)$ *and* $N < \gamma_- \leqslant \gamma_+ < \infty$. *Then there is a continuous embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *provided* $q \in S(\Omega)$ *satisfies condition (1.1).*

Note that the condition $p \in W^{1,(\infty,\gamma(\cdot))}(\Omega)$ is a generalization of the condition $p \in W^{1,\sigma}(\Omega)$ with $\sigma > N$, introduced by Edmunds and Rákosník [12] in the bounded domain case, to the unbounded domain case. As was noted in [13], condition $p \in W^{1,(\infty,\gamma(\cdot))}(\Omega)$ and condition $p \in C^{0,1}(\overline{\Omega})$ can have a uniform form such that $|\nabla p| \in L^{\gamma_1}(\Omega) + L^{\gamma_2}(\Omega)$ with $N < \gamma_1 \leqslant \gamma_2 \leqslant \infty$.

Based on Proposition 3.1, we can obtain a variant of Theorem 1.1 as follows.

THEOREM 3.1. *Let* $\Omega \subset R^N$ *be an unbounded domain satisfying the strong local Lipschitz condition. Suppose that* $p \in W^{1,(\infty,\gamma(\cdot))}(\Omega)$ *with* $1 \leqslant p_- \leqslant p_+ < \infty$, *where* $\gamma \in S(\Omega)$ *and* $N < \gamma_- \leqslant \gamma_+ < \infty$. *Then there holds a continuous embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *for any* $q \in L^\infty(\Omega)$ *satisfying condition (1.1).*

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain. We say that $p : \overline{\Omega} \to \mathbb{R}$ is globally log-Hölder continuous on $\overline{\Omega}$, denoted still by $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, if p satisfies (2.2) and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + |x|)}, \quad \forall x, y \in \Omega \text{ with } |y| \geqslant |x|.$$

It is well-known that, when $p \in C^{0,\frac{1}{|\log t|}}(\mathbb{R}^N)$, there holds the critical embedding $W^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^N)$ (see e.g. [7]). It is also well-known that, in the case when Ω is a bounded domain satisfying the Lipschicz condition and $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, there exists a satisfactory bounded linear extension operator from $W^{1,p(\cdot)}(\Omega)$ into $W^{1,p(\cdot)}(\mathbb{R}^N)$, and consequently, Proposition 2.1 holds (see [8]). If, in the case when Ω is an unbounded domain satisfying the strong local Lipschicz condition and $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, there exists the corresponding extension operator, (the author believes this is true), then the assertion of Theorem 3.1 remains in force if use $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ with $1 < p_- \leqslant p_+ < \infty$ instead of $p \in W^{1,(\infty,\gamma(\cdot))}(\Omega)$ with $1 \leqslant p_- \leqslant p_+ < \infty$.

It is obvious that, for bounded Ω,

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\Omega) \subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}),$$

where $\gamma > N$. However, for unbounded Ω, as was noted in [13], the three conditions that i) $p \in C^{0,1}(\overline{\Omega})$, ii) $p \in W^{1,(\infty,\gamma(\cdot))}(\Omega)$ with $N < \gamma_- \leqslant \gamma_+ < \infty$ and iii) $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ are independent each other.

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