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FARKAS–TYPE RESULTS FOR GENERAL COMPOSED CONVEX OPTIMIZATION PROBLEMS WITH INEQUALITY CONSTRAINTS

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Abstract. In this paper, we consider a general composed convex optimization problem with inequality systems involving a finite number of convex constraints. We establish the strong duality between the primal problem and the Fenchel-Lagrange dual problem by a conjugate duality approach. Moreover, we obtain some new Farkas-type results for this problem by using weak and strong duality theorems. Our results contain some recent results as special cases.

1. Introduction

It is well-known that the Farkas lemma [7] plays an important role in the development of linear programming and optimization theory. During the last two decades, a number of Farkas-type results have been given in the literature with application to more general nonlinear programming problems and nonsmooth optimization problems (see, for example, [8]–[12], [15], [17], [21]).

Recently, Bot and Wanka [5] obtained some new Farkas-type results for inequality systems involving a finite as well as infinite number of convex constraints using the epi-sum formula (see Lemma 2.2 below) and the theory of conjugate duality (see [6]) for convex problems. Also recently, by using the epi-sum formula, Bot et al. [2] presented some Farkas-type results for composed convex optimization problems with inequality systems involving finitely many functions. In deriving the Farkas-type duality results, the epi-sum formula plays an important role. Recently, some useful new sufficient conditions in terms of separability (see Lemma 3.1 of [14]) and interesting extensions (see [15]) for the epi-sum formula have been derived. Some relevant discussion of the epi-sum formula can be found in the new comprehensive monograph [18].

Motivated and inspired by the research works mentioned above, in this paper, we consider a general composed convex optimization problems with inequality systems involving a finite number of convex constraints. We give a Fenchel-Lagrange dual for this problem and establish weak and strong duality assertions. Moreover, we obtain some new Farkas-type results for general composed convex optimization problems. Our results extend and improve some corresponding results in [2, 3, 5].

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2. Preliminaries

In this section, we describe the notations and present preliminary results. Throughout this paper, all vectors will be column vectors. A column vector will be transposed to a row vector by an upper index T. The inner product of two vectors $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ in the n-dimensional real space $\mathbb{R}^n$ will be denoted by $x^T y = \sum_{i=1}^n x_i y_i$. A nonempty subset K of $\mathbb{R}^m$ is said to be a cone if $\lambda K \subseteq K$ for all $\lambda \geqslant 0$. Also K is said to be a convex cone if K is a cone and $K + K \subseteq K$.

Let K be a nonempty closed convex cone with

$$K^* = \{\alpha \in R^m : \ \alpha^T x \geqslant 0, \, \forall\, x \in K\}$$

its dual cone. Consider the ordering $\leqslant_K$ in $\mathbb{R}^m$ induced by K as

$$y \leqslant_K x \quad \text{iff} \quad x - y \in K, \, \forall\, x, y \in \mathbb{R}^m.$$

Let $X \subseteq \mathbb{R}^n$ be a nonempty subset. The relative interior and the convex hull of the set X are denoted by $\mathrm{ri}\,(X)$ and $co(X)$, respectively. Furthermore, the cone and convex cone generalized by the set X are denoted by $cone(X) = \bigcup_{\lambda \geqslant 0} \lambda X$ and $coneco(X) = \bigcup_{\lambda \geqslant 0} \lambda co(X)$, respectively. Let $X \subseteq \mathbb{R}^n$ be a given subset. We consider the following two functions, respectively, the support function σ_X given by

$$\sigma_X(u) = \sup_{x \in X} u^T x$$

and the indictor function δ_X given by

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we denote by $\mathrm{dom}\,(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ its effective domain and by $epi(f) = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \ f(x) \leqslant r\}$ its epigraph, respectively. We say that f is proper if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

Now, we define for the function f the conjugate relative to X by

$$f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f_X^*(p) = \sup_{x \in X} \{p^T x - f(x)\}.$$

Clearly, if $X = \mathbb{R}^n$, then the conjugate function relative to X becomes the classical conjugate function of f (the Fenchel-Moreau conjugate)

$$f^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f^*(p) = \sup_{x \in \mathbb{R}^n} \{p^T x - f(x)\}.$$

In addition, it can be easily proved that

$$f_X^* = (f + \delta_X)^* \text{ and } \delta_X^* = \sigma_X.$$

We consider also the linear operator

$$M:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}^n,\ \ M(x,r)=(r,x).$$

DEFINITION 2.1. A function $f:\mathbb{R}^m\to\overline{\mathbb{R}}$ is said to be K-increasing if, for any $x,y\in\mathbb{R}^m$ with $x\leqslant_K y$, we have $f(x)\leqslant f(y)$.

DEFINITION 2.2. A function $F:\mathbb{R}^n\to\mathbb{R}^m$ is said to be K-convex if, for all $x,y\in\mathbb{R}^n$ and $\lambda\in[0,1]$, we have

$$\lambda F(x)+(1-\lambda)F(y)\in F(\lambda x+(1-\lambda)y)+K.$$

DEFINITION 2.3. [16] Let $f_1,\cdots,f_m:\mathbb{R}^n\to\overline{\mathbb{R}}$ be given functions. The function $f_1\Box\cdots\Box f_m:\mathbb{R}^n\to\overline{\mathbb{R}}$, defined by

$$(f_1\Box\cdots\Box f_m)(x):=\inf\{\sum_{i=1}^m f_i(x_i):\sum_{i=1}^m x_i=x\},$$

is called the infimal convolution function of $f_1,\cdots,f_m$.

The following lemmas will be used in the sequel.

LEMMA 2.1. [16] *Let $f_1,\cdots,f_m:\mathbb{R}^n\to\overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i))$ is nonempty, then*

$$(\sum_{i=1}^m f_i)^*(p)=(f_1^*\Box\cdots\Box f_m^*)(p)=\inf\{\sum_{i=1}^m f_i^*(p_i):\sum_{i=1}^m p_i=p\}$$

and for each $p\in\mathbb{R}^n$ the infimum is attained.

LEMMA 2.2. [2] *Let $f_1,\cdots,f_m:\mathbb{R}^n\to\overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i))$ is nonempty, then*

$$\operatorname{epi}((\sum_{i=1}^m f_i)^*)=\sum_{i=1}^m \operatorname{epi}(f_i^*).$$

LEMMA 2.3. [2] *Let $f:\mathbb{R}^k\to\overline{\mathbb{R}}$ be a proper convex function and $\alpha>0$ be a real number. Then*

$$\operatorname{epi}((\alpha f)^*)=\alpha\operatorname{epi}(f^*).$$

3. Duality for general composed convex optimization problems

Let X be a nonempty convex subset of $\mathbb{R}^n$ and K be a nonempty closed convex cone in $\mathbb{R}^k$. Assume that $f:\mathbb{R}^k\to\overline{\mathbb{R}}$ is a proper, convex and K-increasing function, $h:\mathbb{R}^n\to\overline{\mathbb{R}}$ is a proper convex function, $F:\mathbb{R}^n\to\mathbb{R}^k$ is a K-convex function and $g=(g_1,\cdots,g_m)^T:\mathbb{R}^n\to\mathbb{R}^m$ is a vector-valued function, where g_i is convex for $i=1,\cdots,m$. Moreover, we assume that

$$X\cap\operatorname{dom}(h)\cap F^{-1}(\operatorname{dom}(f))\neq\emptyset, \tag{3.1}$$

where $F^{-1}(\operatorname{dom}(f))=\{x\in\mathbb{R}^n:F(x)\in\operatorname{dom}(f)\}$.

In this section, we consider the following optimization problem:

$$\text{(P)} \qquad \inf_{x\in X,\, g(x)\leqslant 0} \quad f(F(x))+h(x).$$

Denote by $v(P)$ the optimal objective value of the optimization problem (P).

Since condition (3.1) holds, it is easy to see that $v(P)<+\infty$. By Proposition 2 in [20], the function $f\circ F$ is a convex function. Thus, the problem (P) is a convex optimization problem. In order to give a dual problem for (P), we consider the following convex optimization problem:

$$\text{(P}_1\text{)} \qquad \inf_{x\in X,\, g(x)\leqslant 0,\, y\in\mathbb{R}^k,\, F(x)-y\in -K} \quad f(y)+h(x).$$

PROPOSITION 3.1. *For the optimal objective values of* (P) *and* (P_1)*, we have* $v(P)=v(P_1)$.

Proof. Let x be feasible to (P). If $x\notin \operatorname{dom}(h)\bigcap F^{-1}(\operatorname{dom}(f))$, then either $h(x)=+\infty$ or $f(F(x))=+\infty$ or both, such that

$$f(F(x))+h(x)=+\infty\geqslant v(P_1).$$

If $x\in \operatorname{dom}(h)\bigcap F^{-1}(\operatorname{dom}(f))$, let $y=F(x)$. It follows that $F(x)-y=0\in -K$. Thus, (x,y) is feasible to (P_1) and

$$f(F(x))+h(x)=f(y)+h(x)\geqslant v(P_1).$$

Conversely, let us consider (x,y) feasible to (P_1). Then x is feasible to (P) and $F(x)-y\in -K$. It follows that $F(x)\leqslant_K y$. Since f is K-increasing,

$$v(P)\leqslant f(F(x))+h(x)\leqslant f(y)+h(x).$$

Taking the infimum on the right side over (x,y) feasible to (P_1), we obtain $v(P)\leqslant v(P_1)$. Therefore, $v(P)=v(P_1)$. This completes the proof. □

This result allows us to affirm that any dual problem of (P_1) is automatically a dual problem of (P).

For the problem (P_1), we consider its Lagrange dual problem:

$$\text{(D}_L\text{)} \qquad \sup_{\alpha\geqslant 0,\, \beta\in K^*}\ \inf_{x\in X,\, y\in\mathbb{R}^k}\{f(y)+h(x)+\alpha^T g(x)+\beta^T(F(x)-y)\}.$$

By the definition of the conjugate relative to a set and Lemma 2.1, we have

$$\begin{aligned}
&\inf_{x\in X,\, y\in\mathbb{R}^k}\{f(y)+h(x)+\alpha^T g(x)+\beta^T(F(x)-y)\}\\
&\qquad =\inf_{x\in X}\{h(x)+\alpha^T g(x)+\beta^T F(x)\}+\inf_{y\in\mathbb{R}^k}\{f(y)-\beta^T y\}\\
&\qquad =-\sup_{x\in X}\{-h(x)-\alpha^T g(x)-\beta^T F(x)\}-\sup_{y\in\mathbb{R}^k}\{\beta^T y-f(y)\}\\
&\qquad =-(h+\alpha^T g+\beta^T F)^*_X(0)-f^*(\beta)\\
&\qquad =-f^*(\beta)-\inf_{p\in\mathbb{R}^n,\, q\in\mathbb{R}^n}\{h^*(p)+(\alpha^T g)^*_X(q)+(\beta^T F)^*(-p-q)\},
\end{aligned}$$

and the last infimum is attained for some $p,\ q\in\mathbb{R}^n$.

Therefore, the Fenchel-Lagrange dual problem (more information regarding this type of dual is to be found in [4], [19] and [20]) to (P_1) is

$$\text{(D)} \qquad \sup_{\alpha\geqslant 0,\, \beta\in K^*,\, p\in\mathbb{R}^n,\, q\in\mathbb{R}^n} \{-f^*(\beta)-h^*(p)-(\alpha^T g)^*_X(q)-(\beta^T F)^*(-p-q)\}.$$

As a direct consequence of our construction of (D), the following weak duality holds.

THEOREM 3.1. $v(P)\geqslant v(D)$.

It is well-known that strong duality may not hold in general. In order to obtain strong duality, many constraint qualifications have been extensively studied (see, for example, [1, 2, 3, 13, 14, 15]*). In this paper, we consider the following constraint qualification:*

$$(CQ) \quad \exists\, x_0 \in \operatorname{ri}(X) \cap \operatorname{ri}(\operatorname{dom}(h)) \ \textit{such that} \begin{cases} F(x_0)\in \operatorname{ri}(\operatorname{dom}(f))-\operatorname{ri}(K),\\ g_i(x_0)\leqslant 0,\ \ i\in L,\\ g_i(x_0)<0,\ \ i\in N,\end{cases}$$

where $L:=\{i\in\{1,\cdots,m\}: g_i \textit{ is an affine function}\}$ *and* $N:=\{i\in\{1,\cdots,m\}\}\setminus L$.

THEOREM 3.2. *Assume that* $v(P)$ *is finite. If* (CQ) *is satisfied then* $v(P)=v(D)$ *and the dual problem has an optimal solution.*

Proof. Since $v(P)=v(P_1)$, we need only to prove that $v(P_1)=v(D)$. Now, we consider the Lagrange dual to the problem (P_1)

$$(\text{D}_L) \qquad \sup_{\alpha\geqslant 0,\, \beta\in K^*}\ \inf_{x\in X,\, y\in\mathbb{R}^k}\{f(y)+h(x)+\alpha^T g(x)+\beta^T(F(x)-y)\}.$$

Since condition (CQ) is fulfilled and all the involved functions are convex, it is well-known from the literature (see, [16]) that strong duality between (D_L) and (P_1) holds, i.e., $v(D_L)=v(P_1)$. Therefore, there exist $\overline{\alpha}\geqslant 0$ and $\overline{\beta}\in K^*$ such that

$$\begin{aligned} v(P_1) &= \sup_{\alpha\geqslant 0,\, \beta\in K^*}\ \inf_{x\in X,\, y\in\mathbb{R}^k}\{f(y)+h(x)+\alpha^T g(x)+\beta^T(F(x)-y)\}\\ &= \inf_{x\in X,\, y\in\mathbb{R}^k}\{f(y)+h(x)+\overline{\alpha}^T g(x)+\overline{\beta}^T(F(x)-y)\}\\ &= \inf_{x\in X}\{h(x)+\overline{\alpha}^T g(x)+\overline{\beta}^T F(x)\}+\inf_{y\in\mathbb{R}^k}\{f(y)-\overline{\beta}^T y\}\\ &= -\sup_{x\in X}\{-h(x)-\overline{\alpha}^T g(x)-\overline{\beta}^T F(x)\}-\sup_{y\in\mathbb{R}^k}\{\overline{\beta}^T y-f(y)\}\\ &= -f^*(\overline{\beta})-(h+\overline{\alpha}^T g+\overline{\beta}^T F)^*_X(0). \end{aligned}$$

Now since

$$\operatorname{ri}(\operatorname{dom}(h))\cap\operatorname{ri}(F^{-1}(\operatorname{dom}(f)))\cap\operatorname{ri}(\operatorname{dom}(g))=\operatorname{ri}(X)\neq\emptyset,$$

by Lemma 2.1, we get

$$v(P_1) = -f^*(\overline{\beta}) - \inf_{p\in\mathbb{R}^n,\, q\in\mathbb{R}^n}\{h^*(p) + (\overline{\alpha}^T g)^*_X(q) + (\overline{\beta}^T F)^*(-p-q)\}$$

and there exist p^*, $q^* \in \mathbb{R}^n$ such that the infimum is attained, i.e.,

$$v(P_1) = -f^*(\overline{\beta}) - h^*(p^*) - (\overline{\alpha}^T g)^*_X(q^*) - (\overline{\beta}^T F)^*(-p^* - q^*).$$

This with Proposition 3.1 yields $v(P) = v(D)$ and $(p^*, q^*, \overline{\alpha}, \overline{\beta})$ is an optimal solution for (D). This completes the proof. □

To illustrate Theorem 3.2, we give the following example.

EXAMPLE 3.1. Let $m = k = n = 1$, $X = [0,1]$ and $K = \mathbb{R}_+$. Let $F(x) = x^2$, $g(x) = x^2 - 1$ and $h(x) = |x|$ for all $x \in \mathbb{R}$. Let

$$f(x) = \begin{cases} x, & \text{if } x \in [0, +\infty), \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly the conditions of Theorem 3.2 are satisfied. It is easy to verify that $v(P) = v(D) = 0$.

4. Some Farkas-type results via weak and strong duality

In this section, we obtain some Farkas-type results for a general composed convex optimization problem with finitely many constraints by using the weak and strong duality obtained in the previous section. Also some special cases are discussed.

THEOREM 4.1. *Suppose that* (CQ) *holds. Then the following statements are equivalent:*

(i) $x \in X$, $g(x) \leqslant 0 \Rightarrow f(F(x)) + h(x) \geqslant 0$;

(ii) there exist p, $q \in \mathbb{R}^n$, $\alpha \geqslant 0$ *and* $\beta \in K^*$ *such that*

$$f^*(\beta) + h^*(p) + (\alpha^T g)^*_X(q) + (\beta^T F)^*(-p-q) \leqslant 0.$$

Proof. (i)⇒ (ii). Suppose that (i) holds. It follows that $v(P) \geqslant 0$. Since the assumptions of Theorem 3.2 are satisfied, strong duality holds, i.e., $v(P) = v(D) \geqslant 0$ and the dual (D) has an optimal solution. Therefore, there exist p, $q \in \mathbb{R}^n$, $\alpha \geqslant 0$ and $\beta \in K^*$ such that

$$f^*(\beta) + h^*(p) + (\alpha^T g)^*_X(q) + (\beta^T F)^*(-p-q) \leqslant 0.$$

(ii)⇒(i). Choose p, $q \in \mathbb{R}^n$, $\alpha \geqslant 0$ and $\beta \in K^*$ such that

$$f^*(\beta) + h^*(p) + (\alpha^T g)^*_X(q) + (\beta^T F)^*(-p-q) \leqslant 0.$$

It follows that

$$v(D) \geqslant -f^*(\beta) - h^*(p) - (\alpha^T g)^*_X(q) - (\beta^T F)^*(-p-q) \geqslant 0.$$

Since weak duality between (P) and (D) holds, $v(P) \geqslant 0$ and so (i) holds. This completes the proof. □

COROLLARY 4.1. *Suppose (CQ) holds. Then either the inequality system*

$$\text{(I)}\ x \in X,\ g(x) \leqslant 0 \Rightarrow f(F(x)) + h(x) < 0$$

has no solution or the system

$$\text{(II)}\ f^*(\beta) + h^*(p) + (\alpha^T g)_X^*(q) + (\beta^T F)^*(-p-q) \leqslant 0,\ p,\ q \in \mathbb{R}^n,\ \alpha \geqslant 0,\ \beta \in K^*.$$

has a solution but never both.

REMARK 4.1. A slight modification of the proof in Theorem 4.3 in [1] guarantees the following:

Statement (ii) in Theorem 4.1 is equivalent to

$$(0,0,0) \in \{0\} \times M(\operatorname{epi}(f^*)) + \bigcup_{\beta \in K^*} (\operatorname{epi}((\beta^T F)^*)) \times \{-\beta\} + \operatorname{epi}(\sigma_X) \times \{0\}$$
$$+ \operatorname{coneco}(\bigcup_{i=1}^{m} \operatorname{epi}(g_i^*)) \times \{0\} + \operatorname{epi}(h^*) \times \{0\}. \tag{4.1}$$

EXAMPLE 4.1. Let X, K, F, g, h and f be the same as in Example 3.1. It is easy to verify that

$$\operatorname{epi}(f^*) = [0,1] \times \mathbb{R}_+, \quad \operatorname{epi}(h^*) = [0,1] \times \mathbb{R}_+,$$

$$\operatorname{epi}(\sigma_X) = \{0\} \times \mathbb{R}_+, \quad \operatorname{epi}(g^*) = \{0\} \times [1, +\infty)$$

and

$$\operatorname{epi}((\beta F)^*) = \{0\} \times \mathbb{R}_+, \quad \forall \beta \geqslant 0.$$

Note that $(0,0) \in \operatorname{coneco}(\operatorname{epi}(g^*))$. Thus, we can find $\beta = 0$ such that (4.1) holds.

Now, we consider some special cases.

If $h(x) = 0$ for all $x \in \mathbb{R}^n$, then the constraint qualification (CQ) becomes

$$(CQ_1) \quad \exists\, x_0 \in \operatorname{ri}(X) \text{ such that } \begin{cases} F(x_0) \in \operatorname{ri}(\operatorname{dom}(f)) - \operatorname{ri}(K), \\ g_i(x_0) \leqslant 0,\ \ i \in L, \\ g_i(x_0) < 0,\ \ i \in N. \end{cases}$$

Therefore, we have the following results.

COROLLARY 4.2. *Suppose that (CQ_1) holds. Then the following statements are equivalent:*

(i) $x \in X,\ g(x) \leqslant 0 \ \Rightarrow\ f(F(x)) \geqslant 0$;

(ii) there exist $p \in \mathbb{R}^n$, $\alpha \geqslant 0$ and $\beta \in K^*$ such that

$$f^*(\beta) + (\alpha^T g)_X^*(p) + (\beta^T F)^*(-p) \leqslant 0.$$

REMARK 4.2. Statement (ii) in Corollary 4.2 is equivalent to

$$(0,0,0) \in \{0\} \times M(\mathrm{epi}(f^*)) + \bigcup_{\beta \in K^*} (\mathrm{epi}((\beta^T F)^*)) \times \{-\beta\}$$
$$+ \mathrm{coneco}(\bigcup_{i=1}^{m} \mathrm{epi}(g_i^*)) \times \{0\} + \mathrm{epi}(\sigma_X) \times \{0\}.$$

REMARK 4.3. If $K = \mathbb{R}_+^m$, then Corollary 4.2 and Remark 4.2 reduce to the corresponding results of Bot et al. in [2].

If $F(x) = x$ and $h(x) = 0$ for all $x \in \mathbb{R}^n$, then the constraint qualification (CQ) becomes

$$(CQ_2) \quad \exists\, x_0 \in \mathrm{ri}(X) \cap \mathrm{ri}(\mathrm{dom}(f)) \text{ such that } \begin{cases} g_i(x_0) \leqslant 0, & i \in L, \\ g_i(x_0) < 0, & i \in N. \end{cases}$$

Thus, it is easy to get the following results.

COROLLARY 4.3. *Suppose that (CQ_2) holds. Then the following statements are equivalent:*

(i) $x \in X,\ g(x) \leqslant 0 \ \Rightarrow\ f(x) \geqslant 0$*;*

(ii) there exist $p \in \mathbb{R}^n$ and $\alpha \geqslant 0$ such that

$$f^*(p) + (\alpha^T g)_X^*(-p) \leqslant 0.$$

REMARK 4.4. Statement (ii) in Corollary 4.3 is equivalent to

$$(0,0) \in \mathrm{epi}(f^*) + \mathrm{coneco}(\bigcup_{i=1}^{m} \mathrm{epi}(g_i^*)) + \mathrm{epi}(\sigma_X).$$

REMARK 4.5. Corollary 4.3 and Remark 4.4 were obtained by Bot and Wanka in [5].

REMARK 4.6. If $g(x) = 0$ for all $x \in \mathbb{R}^n$, then Theorem 4.1 and Remark 4.1 reduce to the corresponding results in [3].

REMARK 4.7. As pointed out by the referee, the analysis in this paper can be carried forward to the cone constraints case with some suitable modifications. We leave it to readers who are interested in this area.

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