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TOPOLOGICAL AND GEOMETRICAL STRUCTURE OF CALDERÓN–LOZANOVSKIĬ CONSTRUCTION

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Abstract. We study the structure of general Calderón-Lozanovskiĭ construction. First the problem of order continuity of these spaces is studied. Moreover, we find characterization of strict monotonicity and we try to explain why the obtained criteria are not easy to verify in particular cases.

1. Introduction

The theory of Orlicz spaces L_φ is well known. The Banach lattices called E_φ, generated by the Köthe space E and the Orlicz function φ, are generalizations of Orlicz spaces and Orlicz-Lorentz spaces. The structure of spaces E_φ has been intensively developed during the last 20 years (see for example [6], [7], [10], [12] and [13]) Although E_φ are often called Calderón-Lozanovskiĭ spaces they are only a particular case of Calderón-Lozanovskiĭ construction $\rho(X,Y)$ for $X = L^\infty$. We shall study the structure of general Calderón-Lozanovskiĭ construction $\rho(X,Y)$ which plays the great role in the interpolation theory. First the problem of order continuity of these spaces is studied. The order continuity is a fundamental tool in the theory of Banach lattices (see [11], [15], [19]). The criteria for this property in the spaces E_φ can be found in [6] and [7] (see also [14] for the local point of view). Some particular sufficient conditions for order continuity of $\rho(X,Y)$ have been presented in [22]. We shall discuss this problem more precisely, finding several sufficient and necessary conditions for order continuity of $\rho(X,Y)$. We also present an alternative proof for characterization of order continuity of spaces E_φ which might be useful in studying this property in the general construction $\rho(X,Y)$, where the problem remains open. In the second part of the paper we shall consider strict monotonicity of $\rho(X,Y)$. The monotonicity properties play a great role in the theory of Banach lattices. They are of importance in the best dominated approximation problems in Banach lattices (see [9], [16]). They are also strongly applicable in the ergodic theory (see [1]). It is worth mentioning that monotonicity property of E is a restriction of an appropriate rotundity property to pairs of compatible elements on the positive cone E_+ (see [8]). We shall find a full criterion for strict monotonicity of $\rho(X,Y)$ getting a generalization of a respective result for E_φ from

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[12]. The paper is ended with a discussion concerning difficulties of these studies and open problems.

2. Preliminaries

Let $S(X)$ (resp. $B(X)$) be the unit sphere (resp. the closed unit ball) of a real Banach space $(X, \|\cdot\|_X)$.

Let (T, Σ, μ) be a σ-finite and complete measure space. By $L^0 = L^0(T)$ we denote the set of all μ-equivalence classes of real valued measurable functions defined on T.

A Banach space $E = (E, \|\cdot\|_E)$ is said to be a *Köthe space* if E is a linear subspace of L^0 and:

(i) if $x \in E, y \in L^0$ and $|y| \leqslant |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \leqslant \|x\|_E$;

(ii) for all $A \in \Sigma$ with $\mu(A) < \infty$ we have $\chi_A \in E$ (see [11] and [19]).

Every *Köthe space* is a Banach lattice under the natural partial order ($x \geqslant 0$ if $x(t) \geqslant 0$ for μ-a.e. $t \in T$). In particular, if we consider the space E over a non-atomic measure, then we shall say that E is a *Köthe function space*. If we specify the measure space (T, Σ, μ) to be the counting measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, m\right)$, then we will say that E is a *Köthe sequence space*. In the last case the symbol $e_i = (0, ..., 0, 1, 0, ...)$ stands for the i-th unit vector. If we consider the symmetric Köthe spaces (rearrangement invariant) we refer to [15] or [19] for the respective definitions.

The set $E_+ = \{x \in E : x \geqslant 0\}$ is called the *positive cone of* E. For any subset $A \subset E$ define $A_+ = A \cap E_+$. For any $A, B \in \Sigma$ we set $A \div B = (A \setminus B) \cup (B \setminus A)$.

A point $x \in E$ is said to have *order continuous norm* if for any sequence (x_m) in E such that $0 \leqslant x_m \leqslant |x|$ and $x_m \to 0$ μ-a.e. we have $\|x_m\|_E \to 0$. A Köthe space E is called *order continuous* ($E \in (OC)$) if every element of E has order continuous norm (see [11], [19] and [23]). As usual, E_a stands for the subspace of order continuous elements of E. It is known that $x \in E_a$ iff $\|x\chi_{A_n}\|_E \downarrow 0$ for any sequence $\{A_n\}$ satisfying $A_n \searrow \varnothing$ (that is $A_n \supset A_{n+1}$ and $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$). Clearly, $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$ iff $\chi_{A_n} \to 0$ μ-a.e..

E is said to be *strictly monotone* ($E \in (SM)$) if $\|y\|_E < \|x\|_E$ for each $0 \leqslant y \leqslant x$ with $y \neq x$ (see [2], [8]).

We will say that E has the *Fatou property* if conditions $0 \leqslant x_n \uparrow x \in L^0$ with $(x_n)_{n=1}^{\infty}$ in E and $\sup_n \|x_n\|_E < \infty$ imply that $x \in E$ and $\|x\|_E = \lim_n \|x_n\|_E$.

We say that φ is an *Orlicz function* whenever $\varphi : \mathbb{R}_+ \cup \{0\} \to [0, \infty]$, φ is convex, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero. Additionally, if φ vanishes only at zero, takes only finite values and

$$\lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty \quad \text{and} \quad \lim_{u \to 0} \frac{\varphi(u)}{u} = 0,$$

then φ is called an $\mathcal{N}$-function. Denote the class of $\mathcal{N}$ functions by O. *In the whole paper we shall assume that* $\varphi \in O$.

It is known that the growth condition Δ_2 for the Orlicz function φ is a fundamental tool in the theory of Orlicz spaces L_φ and Calderón-Lozanovskiĭ spaces E_φ. In particular it is necessary and sufficient for order continuity of E_φ (L_φ). Recall that an Orlicz function φ satisfies *condition* $\Delta_2(0)$ $(\varphi \in \Delta_2(0))$ if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$ and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \in [0, u_0]$. We say an Orlicz function φ satisfies *condition* $\Delta_2(\infty)$ $(\varphi \in \Delta_2(\infty))$ if there exist $K > 0$, $u_0 > 0$ such that $\varphi(u_0) < \infty$ and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \geqslant u_0$. If there exists $K > 0$ such that $\varphi(2u) \leqslant K\varphi(u)$ for all $u \geqslant 0$, then we say that φ satisfies *condition* $\Delta_2(\mathbb{R}_+)$ $(\varphi \in \Delta_2(\mathbb{R}_+))$.

DEFINITION 1. We will say that a function $\rho : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ belongs to the class U provided:

(i) ρ is positively homogenous, that is, $\rho(au, av) = a\rho(u, v)$ for each $a, u, v \geqslant 0$.

(ii) $\rho(0, v) = \rho(u, 0) = 0$ for each $u, v \geqslant 0$.

(iii) $\rho(\cdot, v)$ and $\rho(u, \cdot)$ are continuous, concave functions of one variable for any $u, v \geqslant 0$.

(iv) $\lim_{u\to\infty} \rho(u, v) = \lim_{u\to\infty} \rho(v, u) = \infty$ for each $v > 0$.

It is easy to see that each function $\rho \in U$ is concave on $\mathbb{R}_+^2$.

REMARK 2. For each $\varphi \in O$ one can associate a function $\rho_{\varphi,R} \in U$ by

$$\rho_{\varphi,R}(u, v) = \begin{cases} u\varphi^{-1}\left(\frac{v}{u}\right) & \text{if} \quad u > 0, \\ 0 & \text{for} \quad u = 0. \end{cases}$$

Conversely, given any $\rho \in U$, if we set $\varphi_R(v) = \rho^{-1}(1, v)$, then $\varphi_R \in O$ (φ_R -from the right hand). Analogously, given $\varphi \in O$ we define

$$\rho_{\varphi,L}(u, v) = \begin{cases} v\varphi^{-1}\left(\frac{u}{v}\right) & \text{if} \quad v > 0, \\ 0 & \text{for} \quad v = 0 \end{cases}$$

with $\rho_{\varphi,L} \in U$. Finally, for $\rho \in U$ setting $\varphi_L(u) = \rho^{-1}(u, 1)$, we get $\varphi_L \in O$ (φ_L -from the left hand).

DEFINITION 3. We will say that ρ satisfies the Δ_2 condition from the left side for all values $u \in \mathbb{R}_+$ ($\rho \in \Delta_2(L, \mathbb{R}_+)$ shortly) if there is $K > 0$ such that

$$\rho(u, v) \leqslant \rho\left(\frac{K}{2}u, \frac{1}{2}v\right)$$

for all $(u, v) \in \mathbb{R}_+^2$. Analogously, ρ satisfies the Δ_2 condition from the right side for all values ($\rho \in \Delta_2(R, \mathbb{R}_+)$ shortly) if there is $K > 0$ such that

$$\rho(u, v) \leqslant \rho\left(\frac{1}{2}u, \frac{K}{2}v\right)$$

for all $(u, v) \in \mathbb{R}_+^2$.

It is easy to see that

$$\rho_{\varphi,L} \in \Delta_2(L,\mathbb{R}_+) \quad \text{if and only if} \quad \varphi \in \Delta_2(\mathbb{R}_+).$$

NOTATION 4. We shall write $\rho \in \Delta_2(L,R,\mathbb{R}_+)$ if $\rho \in \Delta_2(L,\mathbb{R}_+)$ and $\rho \in \Delta_2(R,\mathbb{R}_+)$.

We can define, by the analogy to Orlicz functions, the appropriate Δ_2 conditions for ρ at zero and at infinity.

DEFINITION 5. We will say that ρ satisfies the Δ_2 condition from the left side at infinity [at zero] ($\rho \in \Delta_2(L,\infty)$ $[\rho \in \Delta_2(L,0)]$ shortly) if there are numbers $K > 0$ and $u_0 > 0$ such that

$$\rho(u,v) \leqslant \rho\left(\frac{K}{2}u, \frac{1}{2}v\right)$$

for all $(u,v) \in \mathbb{R}^2_+$ with $\frac{u}{v} > u_0$ $[\frac{u}{v} < u_0]$.

Like above it is easy to see, that $\rho_{\varphi,L} \in \Delta_2(L,\infty)$ $[\rho_{\varphi,L} \in \Delta_2(L,0)]$ if and only if $\varphi \in \Delta_2(\infty)$ $[\varphi \in \Delta_2(0)]$, respectively. The case $\rho \in \Delta_2(R,\infty)$ $[\rho \in \Delta_2(R,0)]$ is understood analogously but with the quotient $\frac{v}{u}$ in place of $\frac{u}{v}$.

Moreover, the appropriate lemmas from the theory of Orlicz functions remain true (see [4] for the respective proofs for Orlicz $\mathscr{N}$ functions).

LEMMA 6. *Let $\rho \in U$. The following conditions are equivalent:*

(i) $\rho \in \Delta_2(L,\infty)$.
(ii) for each $u_0 > 0$ there is $K > 0$ such that $\rho(u,v) \leqslant \rho\left(\frac{K}{2}u, \frac{1}{2}v\right)$ for all $(u,v) \in \mathbb{R}^2_+$ with $\frac{u}{v} > u_0$.
(iii) for each $K > 1$ and any $u_0 > 0$ there is $1 < \sigma < K$ such that $\rho(u,v) \leqslant \rho\left(\frac{K}{\sigma}u, \frac{1}{\sigma}v\right)$ for all $(u,v) \in \mathbb{R}^2_+$ with $\frac{u}{v} > u_0$.

NOTATION 7. Let X, Y be Köthe spaces and $1 \leqslant p \leqslant \infty$. We denote by $X \oplus_p Y$ the p-product of X and Y with the norm

$$\begin{aligned} \|(x,y)\|_p &= \|(x,y)\|_{X\oplus_p Y} = \left(\|x\|_X^p + \|y\|_Y^p\right)^{\frac{1}{p}} \quad \text{if} \quad 1 \leqslant p < \infty, \\ \|(x,y)\|_\infty &= \|(x,y)\|_{X\oplus_\infty Y} = \max\{\|x\|_X, \|y\|_Y\} \quad \text{if} \quad p = \infty. \end{aligned}$$

DEFINITION 8. Suppose that $\rho \in U$ and X, Y are Köthe spaces over the same measure space (we will also say in this situation that a couple (X,Y) is compatible). Let $1 \leqslant p \leqslant \infty$. By the Calderón-Lozanovskiĭ construction $\rho(X,Y)$ we mean the space

$$\rho(X,Y) = \left\{z \in L^0 : |z| \leqslant \rho(x,y) \quad \text{for some} \quad x \in X_+, y \in Y_+\right\}$$

equipped with the norm

$$_p\|z\|_{\rho(X,Y)} = \inf\left\{\|(x,y)\|_p : x \in X_+, y \in Y_+ \quad \text{with} \quad |z| \leqslant \rho(x,y)\right\}.$$

REMARK 9. (i) *The case* $p=\infty$. It is easy to see that

$$\rho(X,Y)=\left\{z\in L^0:|z|\leqslant\lambda\rho(x,y)\ \text{for some}\ x\in B(X)_+,y\in B(Y)_+,\ \lambda>0\right\}$$

and ${}_\infty\|z\|_{\rho(X,Y)}=\inf\lambda,$ where the infimum is extended over all possible $\lambda>0$ for which one can find $x\in B(X)_+$ and $y\in B(Y)_+$ satisfying the inequality $|z|\leqslant\lambda\rho(x,y)$ (the space $\rho(X,Y)$ with the norm considered in this way has been considered by Lozanovskiĭ, see [20]). This construction plays an important role in the theory of interpolation spaces and for the special functions $\phi(s,t)=s^{1-\theta}t^{\theta}\,(0<\theta<1)$ it is related to the complex interpolation method of Calderón (see [3]). If E_0 and E_1 have the Fatou property (or both E_0 and E_1 are separable), Ovchinnikov proved in [21] that $\phi(E_0,E_1)$ is an interpolation space between E_0 and E_1.

(ii) *The case of spaces* E_φ. If $p=\infty,$ $X=L^\infty,$ $Y=E$ and $\rho\in U$, then $\rho(L^\infty,E)=E_\varphi$ and $\|w\|_\rho=\|w\|_\varphi,$ where $\varphi(v)=\rho^{-1}(1,v)$ (note that $\varphi=\varphi_R$ according to Remark 2),

$$I_\varphi(x)=\begin{cases}\|\varphi\circ x\|_E & \text{if}\quad \varphi\circ x\in E\\ \infty & \text{otherwise}\end{cases},$$

$$E_\varphi=\{x\in L^0:I_\varphi(cx)<\infty\ \ \text{for some}\ c>0\}$$

and

$$\|x\|_\varphi=\inf\left\{\lambda>0:I_\varphi(x/\lambda)\leqslant1\right\}.$$

If $E=L^1$ $(E=l^1)$, then E_φ is the Orlicz function (sequence) space equipped with the Luxemburg norm. If E is a Lorentz function (sequence) space Λ_ω (λ_ω), then E_φ is the corresponding Orlicz-Lorentz function (sequence) space $(\Lambda_\omega)_\varphi$ ($(\lambda_\omega)_\varphi$) equipped with the Luxemburg norm (see [6], [7], [8], [10], [12]).

THEOREM 10. ([22, Proposition 1]) *Let* (X,Y) *be couple of Köthe spaces with Fatou property and* $1\leqslant p\leqslant\infty$. *Then for any* $z\in\rho(X,Y)$ *we have*

$$\begin{aligned}{}_p\|z\|_{\rho(X,Y)} &= \inf\left\{\|(u,v)\|_{X\oplus_pY}:|z|=\rho(u,v),(u,v)\in(X\oplus Y)_+\right\}\\ &= \min\left\{\|(u,v)\|_{X\oplus_pY}:|z|=\rho(u,v),(u,v)\in(X\oplus Y)_+\right\}.\end{aligned}$$

In the following, considering a couple of Köthe spaces (X,Y) we will understand that both X and Y have Fatou property and are compatible.

3. Results

3.1. Order continuity

Since order continuity is topologically invariant and all norms ${}_p\|\cdot\|_{\rho(X,Y)}$ are mutually equivalent we may consider only the space $\left(\rho(X,Y),{}_\infty\|\cdot\|_{\rho(X,Y)}\right)$. We shall write shortly $\left(\rho(X,Y),\|\cdot\|_\rho\right)$ and $\|(x,y)\|=\|(x,y)\|_{X\oplus_\infty Y}=\max\{\|x\|_X,\|y\|_Y\}$. The proof of the following lemma is immediate.

LEMMA 11. *Let* (X, Y) *be a couple of order continuous Köthe spaces. Then* $\rho(X, Y) \in (OC)$.

It is possible that the space $\rho(X, Y)$ can be order continuous even when neither X nor Y is order continuous. In order to discuss the case we introduce the following notion.

DEFINITION 12. We shall say that a couple of Köthe spaces (X, Y) is jointly order discontinuous ($(X, Y) \in (JOD)$ shortly) if there exist elements $x \in X \setminus X_a$, $y \in Y \setminus Y_a$ and a sequence of measurable sets $A_n \searrow \varnothing$ such that for any sequence (B_n) in Σ with $B_n \subset A_n$ $(n \in \mathbb{N})$ there are a number $a > 0$ and a subsequence (n_k) in $\mathbb{N}$ such that either

$$\left\|x\chi_{B_{n_k}}\right\|_X \geqslant a \text{ and } \left\|y\chi_{B_{n_k}}\right\|_Y \geqslant a \text{ for all } k$$

or

$$\left\|x\chi_{B'_{n_k}}\right\|_X \geqslant a \text{ and } \left\|y\chi_{B'_{n_k}}\right\|_Y \geqslant a \text{ for all } k,$$

where $B'_n := A_n \setminus B_n$.

THEOREM 13. *If* $\rho \in U$, *then:*
(*i*) $(X, Y) \notin (JOD)$ *whenever* $\rho(X, Y) \in (OC)$.
(*ii*) $\rho(X, Y) \in (OC)$ *provided* $(X, Y) \notin (JOD)$ *and* $\rho \in \Delta_2(R, L, \mathbb{R}_+)$.

Proof. (*i*) Suppose that $(X, Y) \in (JOD)$. Take elements $x \in (X \setminus X_a)_+$, $y \in (Y \setminus Y_a)_+$ and sequence (A_n) from Definition 12. Define $z = \rho(x, y)$ and the sequence

$$z_n = \rho(x, y)\chi_{A_n} = \rho(x\chi_{A_n}, y\chi_{A_n}).$$

By Theorem 10 we conclude that for any n there exists $(u_n, v_n) \in (X \oplus Y)_+$ such that

$$\|z_n\|_\rho = \|(u_n, v_n)\| \text{ and } z_n = \rho(u_n, v_n).$$

Put

$$B_n = \{t \in A_n : u_n(t) > x(t)\} \text{ and } B'_n = A_n \setminus B_n.$$

We have therefore $v_n\chi_{B'_n} \geqslant y\chi_{B'_n}$. From assumed condition (JOD) we know that there exist subsequence (n_k) of $\mathbb{N}$ and constant $a > 0$ satisfying

$$\left(\left\|x\chi_{B_{n_k}}\right\|_X \geqslant a \text{ and } \left\|y\chi_{B_{n_k}}\right\|_Y \geqslant a\right) \text{ or } \left(\left\|x\chi_{B'_{n_k}}\right\|_X \geqslant a \text{ and } \left\|y\chi_{B'_{n_k}}\right\|_Y \geqslant a\right). \tag{1}$$

Because of symmetry we are allowed to consider just one case, say $\left\|x\chi_{B_{n_k}}\right\|_X \geqslant a$ and $\left\|y\chi_{B_{n_k}}\right\|_Y \geqslant a$ for all k. Then

$$\|z_{n_k}\|_\rho = \|(u_{n_k}, v_{n_k})\| \geqslant \left\|(u_{n_k}, v_{n_k})\chi_{B_{n_k}}\right\| \geqslant \left\|u_{n_k}\chi_{B_{n_k}}\right\|_X \geqslant \left\|x\chi_{B_{n_k}}\right\|_X \geqslant a$$

for all k and therefore $z \notin \rho(X, Y)_a$.

(*ii*) Take any $z \in \rho(X, Y)$. By Theorem 10 there is $(x, y) \in (X \oplus Y)_+$ with $|z| = \rho(x, y)$ (see [22]). We can consider only the case where $x \in X \setminus X_a$, $y \in Y \setminus Y_a$ (since the other case can be found in [22]). Take any sequence (A_n) with $A_n \searrow \varnothing$.

Without loss of generality we can assume that $\mu\left(\operatorname{supp} x \div \operatorname{supp} y\right)=0$ and $A_n \subset \operatorname{supp} x$ for any n. Since $(X,Y) \notin (JOD)$ there exists a sequence (B_n) such that $B_n \subset A_n$ for any n and

$$\left(\|x\chi_{B_n}\|_X \to 0 \text{ or } \|y\chi_{B_n}\|_Y \to 0\right) \text{ and } \left(\left\|x\chi_{B_n'}\right\|_X \to 0 \text{ or } \left\|y\chi_{B_n'}\right\|_Y \to 0\right),$$

where $B_n' = A_n \setminus B_n$. If ($\|x\chi_{B_n}\|_X \to 0$ and $\left\|x\chi_{B_n'}\right\|_X \to 0$) or ($\|y\chi_{B_n}\|_Y \to 0$ and $\left\|y\chi_{B_n'}\right\|_Y \to 0$) the proof would follow as that of Proposition 4 in [22]. Therefore, assume that

$$\|x\chi_{B_n}\|_X \to 0 \text{ and } \left\|y\chi_{B_n'}\right\|_Y \to 0. \tag{2}$$

If instead of (2) we would have $\left\|x\chi_{B_n'}\right\|_X \to 0$ and $\|y\chi_{B_n}\|_Y \to 0$, the proof is similar. From $\rho \in \Delta_2\left(R, L, \mathbb{R}_+\right)$ there exist constants $K, L > 0$ such that for any $u, v > 0$ and any $m \in \mathbb{N}$

$$\rho(u,v) \leqslant \rho\left(\left(\frac{K}{2}\right)^m u, \left(\frac{1}{2}\right)^m v\right),$$
$$\rho(u,v) \leqslant \rho\left(\left(\frac{1}{2}\right)^m u, \left(\frac{L}{2}\right)^m v\right).$$

Moreover, there exist sequences $m(n) \to \infty$ and $k(n) \to \infty$ satisfying

$$\left(\frac{K}{2}\right)^{m(n)} \sqrt{\|x\chi_{B_n}\|_X} \leqslant M,$$

$$\left(\frac{L}{2}\right)^{k(n)} \sqrt{\left\|y\chi_{B_n'}\right\|_Y} \leqslant N,$$

for some constants $M, N > 0$. Put $i(n) = \min(m(n), k(n)) \to \infty$. Setting $z_n = z\chi_{A_n}$, we get

$$\begin{aligned} z_n &= \rho(x,y)\chi_{A_n} = \rho(x,y)\chi_{B_n} + \rho(x,y)\chi_{B_n'} \\ &\leqslant \rho\left(\left(\frac{K}{2}\right)^{i(n)} x, \left(\frac{1}{2}\right)^{i(n)} y\right)\chi_{B_n} + \rho\left(\left(\frac{1}{2}\right)^{i(n)} x, \left(\frac{L}{2}\right)^{i(n)} y\right)\chi_{B_n'}. \end{aligned}$$

Finally

$$\begin{aligned} \|z_n\|_\rho &\leqslant \left(\frac{K}{2}\right)^{i(n)} \|x\chi_{B_n}\|_X + \left(\frac{1}{2}\right)^{i(n)} \|y\chi_{B_n}\|_Y + \left(\frac{L}{2}\right)^{i(n)} \left\|y\chi_{B_n'}\right\|_Y + \left(\frac{1}{2}\right)^{i(n)} \left\|x\chi_{B_n'}\right\|_X \\ &\leqslant M\sqrt{\|x\chi_{B_n}\|_X} + \left(\frac{1}{2}\right)^{i(n)} \|y\chi_{B_n}\|_Y + N\sqrt{\left\|y\chi_{B_n'}\right\|_Y} + \left(\frac{1}{2}\right)^{i(n)} \left\|x\chi_{B_n'}\right\|_X \to 0. \end{aligned}$$

Since $z \in \rho(X,Y)$ is arbitrary, we conclude that $\rho(X,Y) \in (OC)$. □

REMARK 14. Obviously, when X is isomorphic to Y and they are not order continuous, then $(X,Y) \in (JOD)$. However, there may exist spaces X, Y both of them not being order continuous such that the couple $(X,Y) \notin (JOD)$, and in view of the above theorem, the space $\rho(X,Y)$ can be order continuous. The following example presents such a case.

EXAMPLE 15. Let $T = \langle 0,\infty)$ and φ,ψ be Orlicz $\mathcal{N}$ functions such that

$$\varphi \in \Delta_2(0) \text{ and } \varphi \notin \Delta_2(\infty) \text{ as well as } \psi \notin \Delta_2(0) \text{ and } \psi \in \Delta_2(\infty).$$

We will show that $\left(L_\varphi, L_\psi\right) \notin (JOD)$. Of course $L_\varphi, L_\psi \notin (OC)$ (see for example [17]). Take any $0 \leqslant x \in L_\varphi \setminus \left(L_\varphi\right)_a$, $0 \leqslant y \in L_\psi \setminus \left(L_\psi\right)_a$ and arbitrary sequence (A_n) in Σ such that $A_n \searrow \varnothing$ (i.e. $\chi_{A_n} \searrow 0$ μ-a.e.). Notice that for a set $A = \{t \in T : x(t) \geqslant 1\}$ we have $\mu(A) < \infty$. We claim that

$$x\chi_{T\setminus A} \in L_a^\varphi.$$

We have $I_\varphi(\lambda x) = \|\varphi \circ (\lambda x)\|_{L^1} < \infty$ for some $\lambda > 0$ because $x \in L_\varphi$. Suppose that $0 \leqslant z_n \leqslant x\chi_{T\setminus A}$ and $z_n \to 0$ μ-a.e.. Then $I_\varphi(\lambda z_n) \to 0$ since $\varphi \circ (\lambda x) \in L^1 \in (OC)$. Fix $k \in \mathbb{N}$. By $\varphi \in \Delta_2(0)$ and $\varphi \in O$ we conclude that there is a number $K_0 > 0$ such that $\varphi\left(\frac{2^k}{\lambda}u\right) \leqslant K_0\varphi(u)$ for any $u \in [0,\lambda]$. Thus

$$I_\varphi\left(2^k z_n\right) = I_\varphi\left(\frac{2^k}{\lambda}\lambda z_n\right) \leqslant K_0 I_\varphi(\lambda z_n) \to 0,$$

as $n \to \infty$. This means that $\|z_n\|_\varphi \to 0$ which proves the claim. On the other hand

$$y\chi_A \in L_a^\psi,$$

since $L_\psi(A, \Sigma\cap A, \mu|_A) \in (OC)$. For

$$\begin{aligned} B_n &= A_n \cap A, \\ B_n' &= A_n \setminus B_n, \end{aligned}$$

we have $\left\|x\chi_{B_n'}\right\|_E \to 0$ and $\|y\chi_{B_n}\|_F \to 0$. Finally $(L^\varphi, L^\psi) \notin (JOD)$ and taking any $\delta \in (0,1)$, $\rho(u,v) = u^\delta v^{1-\delta}$ we have by Theorem 13 (ii) that

$$\rho(L^\varphi, L^\psi) = (L^\varphi)^\delta (L^\psi)^{1-\delta} \in (OC).$$

REMARK 16. This should be pointed out that if one of X or Y is isomorphically equal to L^∞ and the second is not order continuous, then the couple $(X,Y) \in (JOD)$. Clearly, $(L^\infty)_a = \{0\}$ but there are Köthe spaces X with $X_a = \{0\}$ and X is not isomorphically equal to L^∞. It is enough to take $X = L^1 \cap L^\infty([0,\infty])$ with the norm $\|x\| = \|x\|_{L^1} + \|x\|_{L^\infty}$. This leads to the following

REMARK 17. Does $(X,Y) \in (JOD)$ when $X_a = \{0\}$ and $Y \notin (OC)$? The answer is "no" in general. Indeed, let $X = L^1(\mathbb{R},\Sigma,\mu) \cap L^\infty(\mathbb{R},\Sigma,\mu)$ with the norm $\|x\|_X = \|x\|_1 + \|x\|_\infty$ and $Y = L^\varphi(\mathbb{R},\Sigma,\mu)$ be the Orlicz space with the Luxemburg norm such that $\varphi \in \Delta_2(\infty)$, $\varphi \notin \Delta_2(0)$ and φ is an $\mathcal{N}$-function. We have $X_a = \{0\}$ and $Y \notin (OC)$. We shall show that for any couple $(x,y) \in (X\setminus X_a)_+ \oplus (Y\setminus Y_a)_+$

and any sequence $(A_n) \subset \Sigma$ satisfying $A_n \searrow \varnothing$ and $A_n \subset A_m$ for $n > m$, there exists sequence (B_n) with $B_n \subset A_n$ and

$$\left\| x\chi_{B_n'} \right\|_X \to 0 \text{ and } \left\| y\chi_{B_n} \right\|_Y \to 0,$$

where $B_n' = A_n \setminus B_n$. Let $0 \leqslant x \in X \setminus X_a$, $0 \leqslant y \in Y \setminus Y_a$ and (A_n) be like above. There are two possibilities: $\mu(A_n) \to 0$ or $\mu(A_n) = \infty$ for any n. Since $L_\varphi(A, \Sigma \cap A, \mu|_A) \in (OC)$ while $\mu(A) < \infty$, only the second case need to be considered. Define

$$C_k = \left\{ t \in \mathbb{R} : x(t) \geqslant \frac{1}{k} \right\},$$

$$B_k^n = \left\{ t \in A_n : x(t) \geqslant \frac{1}{k} \right\} \text{ and } (B_k^n)' = A_n \setminus B_k^n.$$

Then $\mu(B_k^n) < \infty$ and $\mu(C_k) < \infty$ for any $k, n \in \mathbb{N}$. Moreover, $\mu(B_k^n) \overset{n\to\infty}{\to} 0$ for any k since $B_k^n \subset C_k$ and $\chi_{B_k^n} \leqslant \chi_{A_n} \searrow 0$ μ-a.e.. Therefore, $\left\| y\chi_{B_k^n} \right\|_Y \overset{n\to\infty}{\to} 0$ (since $L_\varphi(C_k, \Sigma \cap C_k, \mu|_{C_k}) \in (OC)$) for any k and consequently we can find a nondecreasing sequence of natural numbers $i(n)$ such that $\left\| y\chi_{B_{i(n)}^n} \right\|_Y \overset{n\to\infty}{\to} 0$ and $i(n) \to \infty$. Finally, we get

$$\begin{aligned} \left\| x\chi_{\left(B_{i(n)}^n\right)'} \right\|_X &= \left\| x\chi_{\left(B_{i(n)}^n\right)'} \right\|_1 + \left\| x\chi_{\left(B_{i(n)}^n\right)'} \right\|_\infty \\ &\leqslant \left\| x\chi_{A_n} \right\|_1 + \frac{1}{i(n)} \to 0 \end{aligned}$$

as $n \to \infty$. So we conclude that $(X, Y) \notin (JOD)$.

It is proved in [22] that the appropriate (left or right) Δ_2 condition for all values is enough for Calderón-Lozanovskiĭ space $\rho(X, Y)$ to be order continuous even if one of X or Y is not. However, the theory of Orlicz spaces and generalized Orlicz spaces E_φ shows that Δ_2 condition on the whole $\mathbb{R}_+$ is not necessarily in some cases. It is sometimes enough to assume Δ_2 condition at zero or at infinity as the following theorem shows.

THEOREM 18. *(i)* $\rho(X, Y) \in (OC)$ *if* $X \subset Y$, $X \in (OC)$ *and* $\rho \in \Delta_2(L, 0)$.
(ii) $\rho(X, Y) \in (OC)$ *if* $Y \subset X$, $X \in (OC)$ *and* $\rho \in \Delta_2(L, \infty)$.
(iii) $\rho(X, Y) \in (OC)$ *if* $X \subset Y$, $Y \in (OC)$ *and* $\rho \in \Delta_2(R, \infty)$.
(iv) $\rho(X, Y) \in (OC)$ *if* $Y \subset X$, $Y \in (OC)$ *and* $\rho \in \Delta_2(R, 0)$.

Proof. (i) Take any $z \in \rho(X, Y)$. By Theorem 10 there is $(x, y) \in (X \oplus Y)_+$ with $|z| = \rho(x, y)$ (see [22]). We have $|z| \leqslant \rho(x, x \vee y) \in \rho(X, Y)$ because $x \vee y \in Y$. Thus, by Lemma 6, there exist $K_1 > 0$ such that for any $t \in \operatorname{supp} x \vee y$

$$\rho(x(t), (x \vee y)(t)) \leqslant \rho\left(\frac{K_1}{2} x(t), \frac{1}{2}(x \vee y)(t) \right), \tag{3}$$

since $\frac{x(t)}{(x\vee y)(t)} \leqslant 1$. By the induction, applying Lemma 6, we can define a sequence K_i satisfying for any n

$$\rho\left(x\left(t\right),\left(x\vee y\right)\left(t\right)\right)\leqslant\rho\left(\frac{\prod_{i=1}^{n}K_i}{2^n}x\left(t\right),\frac{1}{2^n}\left(x\vee y\right)\left(t\right)\right).$$

Take any sequence (A_n) in Σ with $A_n \searrow \varnothing$ and define $z_n = \rho\left(x\chi_{A_n}, y \vee x\chi_{A_n}\right)$. It is enough to check that $\|z_n\|_\rho \to 0$ because $\rho\left(x\chi_{A_n}, y\chi_{A_n}\right) \leqslant z_n$. Since $X \in (OC)$ so $\|x\chi_{A_n}\|_X \searrow 0$. Moreover, there are a nondecreasing sequence (i_n) with $i_n \to \infty$ and a number $M > 0$ such that

$$\frac{\prod_{i=1}^{i_n}K_i}{2^{i_n}}\sqrt{\|x\chi_{A_n}\|_X}\leqslant M$$

for each n. We can continue our proof like in Theorem 13.

(ii) Let $z \in \rho(X,Y)$. Then $|z| = \rho(x,y)$ for some $x \in X_+, y \in Y_+$. If $y \in Y_a$ then the proof is obvious. Suppose that $y \in (Y \setminus Y_a)_+$. We have

$$z \leqslant \rho\left(x \vee y, y\right)$$

and $x \vee y \in X$. By Lemma 6(ii) for $u_0 = 1$ there is $K > 1$ such that $\rho(u,v) \leqslant \rho\left(\frac{K}{2}u, \frac{1}{2}v\right)$ for all $(u,v) \in \mathbb{R}^2_+$ with $\frac{u}{v} \geqslant 1$. Since the inequality $K^n\frac{x\vee y}{y}\chi_{\operatorname{supp} y} \geqslant \chi_{\operatorname{supp} y}$ is true for any n, so $\rho(x\vee y, y) \leqslant \rho\left(\left(\frac{K}{2}\right)^n x\vee y, \frac{1}{2^n}y\right)$. We follow as in proof of (i).

The proofs of (iii) and (iv) are similar. □

Recall that given Köthe spaces X and Z the generalized dual X^Z is defined by

$$X^Z = \left\{x \in L^0\left(T,\Sigma,\mu\right) : xy \in Z \text{ for each } y \in X\right\}$$

(see [18]). For $Z = L^1$ the space $X^{L^1} = X'$ is the Köthe dual of X. We shall need the following generalization.

DEFINITION 19. Let X and Z be a real Köthe function spaces over measure space (T,Σ,μ) and take an unbounded set $K \subset R_+$. Set

$$X_K = \{x \in X : x[T] \subset K \cup \{0\}\}$$

and

$$(X_K)^Z = \left\{x \in L^0\left(T,\Sigma,\mu\right) : xy \in Z \text{ for each } y \in X_K\right\}.$$

If $K = \mathbb{R}$, then clearly $X_K = X$.

LEMMA 20. (see [18]) *If X is a Köthe space, then $X^X = L^\infty$.*

We will need a generalization of the above lemma.

LEMMA 21. *Let X be an order continuous Köthe function space such that $X \not\subset L^\infty$. Suppose that the set $K \subset \mathbb{R}_+$ is unbounded. Then we have the equality of sets*

$$(X_K)^X = L^\infty.$$

Proof. The inclusion $\supset$ is evident. Suppose that there is $x \in (X_K)^X \setminus L^\infty$. Define

$$A_n = \left\{ t \in T : n^3 \leqslant |x(t)| < (n+1)^3 \right\}.$$

Since $x \notin L^\infty$, we have $\mu(A_n) > 0$ for infinitely many n. Set $M = \{n \in \mathbb{N} : \mu(A_n) > 0\}$. If $X \in (OC)$, then the function $\gamma : \Sigma \to \mathbb{R}_+$ defined $\gamma(A) = \|\chi_A\|_X$ has Darboux property (see [5]). Consequently there exist sets $B_n \subset A_n$ with $\frac{1}{n^2 \|\chi_{B_n}\|_X} \in K$ for each $n \in \mathbb{N}$. Put

$$y = \sum_{n \in M} \frac{1}{n^2 \|\chi_{B_n}\|_X} \chi_{B_n}.$$

We have of course $y \in X$ by completeness and $y \in X_K$. On the other hand

$$\|xy\| \geqslant \left\| \frac{n^3}{n^2 \|\chi_{B_n}\|} \chi_{B_n} \right\| = n,$$

for any $n \in M$. The contradiction proves the lemma. □

REMARK 22. The above Lemma is also true if we assume, in place of $X \in (OC)$, that X is a symmetric Köthe function space and $\mu(T) < \infty$. In order to go analogously in the proof it is enough to show that $\chi_A \in X_a$ for any $A \in \Sigma$. First, notice that, since $X \not\subset L^\infty$, there exist $x \in (X \setminus X_a)_+$ with $x \notin L^\infty$. Furthermore, we shall show that $\sqrt{x} \in X_a$. Defining $A_n = \{t \in T : x(t) \geqslant n\}$ we get $M \geqslant \|x\chi_{A_n}\| \geqslant a$ for some $M, a > 0$ and any n (such a choice of (A_n) is optimal because of symmetry). However, $\left\|\sqrt{x}\chi_{A_n}\right\| \to 0$ because $\sqrt{x}\chi_{A_n} \leqslant \frac{1}{\sqrt{n}} x\chi_{A_n}$. But $\sqrt{x}\chi_{A_1} \geqslant \chi_{A_1}$ and consequently $\chi_{A_1} \in X_a$. Since X is symmetric we conclude that $\chi_A \in X_a$ for any set $A \in \Sigma$. Lemma 21 can be proved as before.

It is known that, under certain assumptions, the Δ_2 condition is necessary for order continuity of E_φ (see [7]). We shall present an independent proof of this fact applying the above Lemma.

THEOREM 23. ([7]) *Let $\varphi \in O$ and $L^\infty \subset E$ with E being a Köthe function space . Suppose that $E \in (OC)$ or E is a symmetric space over the finite measure space. If $\varphi \notin \Delta_2(\infty)$ (equivalently $\rho \overset{def}{=} \rho_{\varphi,R} \notin \Delta_2(R,\infty)$), then $\rho(L^\infty, E) \cong E_\varphi \notin (OC)$.*

Proof. Since $\rho \notin \Delta_2(R,\infty)$ so

$$\limsup_{t \to \infty} K(t) = \infty,$$

where K is the function defined by the equality

$$\rho(1,t) = \rho\left(\frac{1}{2}, \frac{K(t)}{2} t\right). \tag{4}$$

Since K is continuous, there exists a nonempty set $H \subset \mathbb{R}_+$ such that $K(s) \leqslant K(t)$ whenever $s \leqslant t$ for any $s, t \in H$. Moreover, H may be chosen to be unbounded, $0 \in H$ and such that $K[H]$ is unbounded too. Define

$$E_H = \{x \in E : x[T \setminus A] \subset H \text{ for some } A \in \Sigma \text{ with } \mu(A) = 0\}.$$

Suppose for the contrary that $\rho(L^\infty, E) \in (OC)$. Notice that $E \not\subset L^\infty$. Indeed, otherwise $L^\infty \subset E \subset L^\infty$ and E is isomorphic to L^∞, whence, by Theorem 13 and Remark 16, $\rho(L^\infty, E) \notin (OC)$. Let $x \in E_H$ be such that the sets

$$A_n = \{t : x(t) > n\}$$

have positive measure for all $n \in \mathbb{N}$. Of course $\mu(A_n) \to 0$. Setting

$$z_n = \rho(\chi_T, x)\chi_{A_n}$$

we get $\|z_n\|_\rho \searrow 0$. Moreover, in view of Theorem 10, for any n there exists a couple (u_n, v_n) with

$$\|z_n\|_\rho = \|(u_n, v_n)\| \text{ and } z_n = \rho(u_n, v_n).$$

Therefore, there exists n_0 satisfying $u_{n_0} \leqslant \frac{1}{2}\chi_{A_{n_0}}$. Applying equality (4) we have

$$\rho(1, x(t)) = \rho\left(\frac{1}{2}, \frac{K(x(t))}{2}x(t)\right)$$

for μ-a.e. $t \in A_{n_0}$, and consequently

$$\frac{K \circ x}{2}x\chi_{A_{n_0}} \leqslant v_{n_0} \in E.$$

Define the function

$$M = \frac{K \circ x}{2}\chi_{A_{n_0}}.$$

Of course M is unbounded. We would like to show that $M \in (E_H)^E$ to get the contradiction with Lemma 21. It is enough to show that for any $y \in E_H$ with $x \leqslant y$, $yM \in E$. Let y be arbitrary, $x \leqslant y$ and put

$$\begin{aligned} f &= \rho(\chi_T, y), \\ f_n &= \rho(\chi_{A_n}, y\chi_{A_n}). \end{aligned}$$

Take couples (r_n, s_n) satisfying

$$\|f_n\|_\rho = \|(r_n, s_n)\| \text{ and } f_n = \rho(r_n, s_n).$$

Consequently $\|f_n\|_\rho \to 0$ and therefore, $\|r_n\|_\infty \to 0$. We conclude as before that there exists $n_1 > n_0$ such that $r_{n_1} \leqslant \frac{1}{2}\chi_{A_{n_1}}$. But

$$\rho(1, y(t)) = \rho\left(\frac{1}{2}, \frac{K(y(t))}{2}y(t)\right)$$

for μ-a.e. $t \in A_{n_1}$ and finally

$$\frac{K \circ y}{2}y\chi_{A_{n_1}} \leqslant s_{n_1} \in E.$$

Furthermore, by the monotonicity of K in H, we get

$$\frac{1}{2}K \circ x\chi_{A_{n_1}} \leqslant \frac{1}{2}K \circ y\chi_{A_{n_1}}.$$

Therefore

$$M\chi_{A_{n_1}}y \leqslant \frac{K\circ y}{2}y\chi_{A_{n_1}} \in E.$$

Moreover, $x\chi_{T\setminus A_{n_1}} \leqslant n_1\chi_{T\setminus A_{n_1}}$ and

$$M\chi_{T\setminus A_{n_1}} = \frac{K\circ x}{2}\chi_{T\setminus A_{n_1}} \leqslant \frac{K(n_1)}{2}\chi_{T\setminus A_{n_1}} \in L^{\infty}.$$

Thus we get $My = My\chi_{T\setminus A_{n_1}} + My\chi_{A_{n_1}} \in E$ and because y was arbitrary, $M \in (E_H)^E$ which contradicts Lemma 21. □

3.2. Strict monotonicity

When we consider the space ${}_p\rho(X,Y)$ with $p=\infty$ we shall write $\rho(X,Y)$ instead of ${}_\infty\rho(X,Y)$ and $\|z\|_\rho$ instead of ${}_\infty\|z\|_\rho$ for short.

Given an $\mathcal{N}$ function φ we set

$$W_\varphi^R = \left\{w\in L^0 : z\varphi\circ\left(\frac{\lambda|w|}{z}\right)\in Y \text{ for some } \lambda>0, z\in B(X)_+ \text{ with } \operatorname{supp} w\subset \operatorname{supp} z\right\},$$

$$I_\varphi^R(w) = \inf\left\{\left\|z\varphi\circ\left(\frac{|w|}{z}\right)\right\|_Y : z\in B(X)_+, \operatorname{supp} w\subset\operatorname{supp} z\right\}$$

and

$$\|w\|_\varphi^R = \inf\left\{\lambda>0 : I_\varphi^R(w/\lambda)\leqslant 1\right\}$$

for each $w\in W_\varphi^R$. We define the function $z\varphi\circ\left(\frac{|w|}{z}\right)\in Y$ by

$$z\varphi\circ\left(\frac{|w|}{z}\right)(t) = \begin{cases} z(t)\,\varphi\left(\frac{|w|(t)}{z(t)}\right) & \text{if } t\in\operatorname{supp} z,\\ 0 & \text{if } t\notin\operatorname{supp} z.\end{cases}$$

Analogously one can set

$$W_\varphi^L = \left\{w\in L^0 : z\varphi\circ\left(\frac{\lambda|w|}{z}\right)\in X \text{ for some } \lambda>0, z\in B(Y)_+ \text{ with } \operatorname{supp} w\subset \operatorname{supp} z\right\},$$

$$I_\varphi^L(w) = \inf\left\{\left\|z\varphi\circ\left(\frac{|w|}{z}\right)\right\|_X : z\in B(Y)_+, \operatorname{supp} w\subset\operatorname{supp} z\right\}$$

and

$$\|w\|_\varphi^L = \inf\left\{\lambda>0 : I_\varphi^L(w/\lambda)\leqslant 1\right\}$$

for each $w\in W_\varphi^L$.

REMARK 24. If $\varphi\in O$, then:

(i) $\left(W_\varphi^R, \|\cdot\|_\varphi^R\right) = \left(\rho_{\varphi,R}(X,Y), \|\cdot\|_{\rho_{\varphi,R}}\right)$ and $\|w\|_\varphi^R = \|w\|_{\rho_{\varphi,R}}$ for each $w\in \rho_{\varphi,R}(X,Y)$, where

$$\rho_{\varphi,R}(u,v) = \begin{cases} u\varphi^{-1}\left(\frac{v}{u}\right) & \text{if } u>0,\\ 0 & \text{for } u=0.\end{cases}$$

(ii) $\left(W_{\varphi}^{L}, \|\cdot\|_{\varphi}^{L}\right) = \left(\rho_{\varphi,L}(X,Y), \|\cdot\|_{\rho_{\varphi,L}}\right)$ and $\|w\|_{\varphi}^{L} = \|w\|_{\rho_{\varphi,L}}$ for each $w \in \rho_{\varphi,L}(X,Y)$, where

$$\rho_{\varphi,L}(u,v) = \begin{cases} v\varphi^{-1}\left(\frac{u}{v}\right) & \text{if} \quad v > 0, \\ 0 & \text{for} \quad v = 0. \end{cases}$$

Proof. (i) Let $w \in W_{\varphi}^{R}$ and $\|w\|_{\varphi}^{R} < \lambda$. Then $I_{\varphi}^{R}(w/\lambda) \leqslant 1$, so we find a sequence $(z_n) \subset B(X)_+$ with $\left\|z_n\varphi \circ \left(\frac{|w|}{\lambda z_n}\right)\right\|_Y \to I_{\varphi}^{R}(w/\lambda) \leqslant 1$ and $\operatorname{supp} w \subset \operatorname{supp} z_n$. Denoting $a_n = \max\left\{1, \left\|z_n\varphi \circ \left(\frac{|w|}{\lambda z_n}\right)\right\|_Y\right\}$, we get

$$\begin{aligned} |w| &= \lambda z_n\varphi^{-1} \circ \varphi \circ \left(\frac{|w|}{\lambda z_n}\right) = \lambda z_n\rho_{\varphi,R}\left(\chi_{\operatorname{supp} w}, \varphi \circ \left(\frac{|w|}{\lambda z_n}\right)\right) \\ &= \lambda\rho_{\varphi,R}\left(z_n\chi_{\operatorname{supp} w}, z_n\varphi \circ \left(\frac{|w|}{\lambda z_n}\right)\right) = a_n\lambda\rho_{\varphi,R}\left(\frac{z_n\chi_{\operatorname{supp} w}}{a_n}, \frac{z_n\varphi \circ \left(\frac{|w|}{\lambda z_n}\right)}{a_n}\right) \end{aligned}$$

for each n. Thus $w \in \rho_{\varphi,R}(X,Y)$ and $\|w\|_{\rho_{\varphi,R}} \leqslant a_n\lambda$ for each n. Consequently $\|w\|_{\rho_{\varphi,R}} \leqslant \lim a_n\lambda = \lambda$. Since λ can be taken arbitrarily close to $\|w\|_{\varphi}^{R}$, so $\|w\|_{\rho_{\varphi,R}} \leqslant \|w\|_{\varphi}^{R}$. Suppose that $w \in \rho_{\varphi,R}(X,Y)$ and $\|w\|_{\rho_{\varphi,R}} < \lambda$. Then $|w| \leqslant \lambda\rho_{\varphi,R}(x,y)$ for some $x \in B(X)_+$ and $y \in B(Y)_+$. Without loss of generality we may assume that $\operatorname{supp} x = \operatorname{supp} y = \operatorname{supp} w$. Then

$$|w| \leqslant \lambda\rho_{\varphi,R}(x,y) = \lambda x\rho_{\varphi,R}\left(\chi_{\operatorname{supp} x}, \frac{y}{x}\right) = \lambda x\varphi^{-1} \circ \left(\frac{y}{x}\right).$$

Consequently $x\varphi \circ \left(\frac{|w|}{\lambda x}\right) \leqslant y \in B(Y)_+$. Then $w \in W_{\varphi}^{R}$ and $\|w\|_{\varphi}^{R} \leqslant \lambda$. Thus $\|w\|_{\varphi}^{R} \leqslant \|w\|_{\rho_{\varphi,R}}$.

The proof of (ii) is analogous. □

DEFINITION 25. Let $p = \infty$. We shall say that the space $\rho(X,Y)$ satisfies the nm_R (nm_L respectively) condition provided the implication $\|w\|_{\rho} = 1 \Rightarrow I_{\varphi_R}^{R}(w) = 1$ ($\|w\|_{\rho} = 1 \Rightarrow I_{\varphi_L}^{L}(w) = 1$, respectively) holds for each $w \in \rho(X,Y)$ $(w \in \rho(X,Y))$. We shall write $\rho(X,Y) \in (nm_R)$ and $\rho(X,Y) \in (nm_L)$, respectively.

Notice that in the above definition we may equivalently use notations $\rho_{\varphi,R}, \varphi$ for nm_R and $\rho_{\varphi,L}, \varphi$ for nm_L.

REMARK 26. Recall that if $p = \infty$, $X = L^{\infty}$, $Y = E$ and $\rho \in U$, then $\rho(L^{\infty}, E) = E_{\varphi}$ and $\|w\|_{\rho} = \|w\|_{\varphi}$, where $\varphi(v) = \varphi_R(v) = \rho^{-1}(1,v)$ (see Remark 9 (ii) in Section 2). Moreover, $\rho(L^{\infty}, E) \in (nm_R)$ iff $\|x\|_{\varphi} = 1 \Rightarrow I_{\varphi}(x) = 1$ for each $x \in E_{\varphi}$ (the last condition has been considered in [12] as necessary and sufficient for strict monotonicity of E_{φ}).

Proof. Suppose that $\rho(L^\infty, E) \in (nm_R)$ and let $\|x\|_\varphi = 1$. By the assumption

$$1 = I_\varphi^R(x) = \inf_{z \in B(L^\infty)_+} \left\| z\varphi \circ \left(\frac{|x|}{z} \right) \right\|_Y \leqslant \left\| z_0 \varphi \circ \left(\frac{|x|}{z_0} \right) \right\|_Y = I_\varphi(x),$$

where $z_0 = \chi_T$ is the strong unit in L^∞. On the other hand, given any element $z \in B(L^\infty)_+$ we have $z \leqslant \chi_T$, by the convexity of φ, we get

$$\left\| z\varphi \circ \left(\frac{|x|}{z} \right) \right\|_Y \geqslant \left\| z \frac{1}{z} \varphi \circ (|x|) \right\|_Y = \|\varphi \circ (|x|)\|_Y,$$

whence $1 = \inf_{z \in B(L^\infty)_+} \left\| z\varphi \circ \left(\frac{|x|}{z} \right) \right\|_Y \geqslant I_\varphi(x)$. Thus $I_\varphi(x) = 1$ as required. The proof of the second implication is obvious since we have seen above that $I_\varphi(x) = I_\varphi^R(x)$. □

LEMMA 27. *Suppose that X and Y have the Fatou property and $p = \infty$. Then:*
(i) $\rho(X, Y) \in (nm_R)$ if and only if

$$\|z\|_\rho = \|y\|_Y \tag{+}$$

whenever $z \in \rho(X, Y)$, $|z| = \rho(x, y)$ and $\|z\|_\rho = \|(x, y)\|_\infty$.
(ii) $\rho(X, Y) \in (nm_L)$ if and only if $\|z\|_\rho = \|x\|_X$ whenever $z \in \rho(X, Y)$, $|z| = \rho(x, y)$ and $\|z\|_\rho = \|(x, y)\|_\infty$.

Proof. (i) Clearly, condition $(+)$ can be considered equivalently for $z \in S(\rho(X, Y))$. Indeed, suppose that $(+)$ holds on the sphere $S(\rho(X, Y))$ and take $z_1 \in \rho(X, Y) \setminus \{0\}$. We find $x_1 \in X_+, y_1 \in Y_+$ with $|z_1| = \rho(x_1, y_1)$ and $\|z_1\|_\rho = \|(x_1, y_1)\|_\infty$ (see Theorem 10). Setting $z = \frac{z_1}{\|z_1\|_\rho} \in S(\rho(X, Y))$ we get $|z| = \rho\left(\frac{x_1}{\|z_1\|_\rho}, \frac{y_1}{\|z_1\|_\rho} \right)$ and $\|z\|_\rho = \left\| \left(\frac{x_1}{\|z_1\|_\rho}, \frac{y_1}{\|z_1\|_\rho} \right) \right\|_\infty$. By the assumption we get $1 = \|z\|_\rho = \left\| \frac{y_1}{\|z_1\|_\rho} \right\|_Y$, whence $\|z_1\|_\rho = \|y_1\|_Y$.

Necessity. Assume that condition $(+)$ does not hold. Then there is $z \in S(\rho(X, Y))$ with $|z| = \rho(x, y)$, $1 = \|z\|_\rho = \|(x, y)\|_\infty$ and $\|y\|_Y < \|x\|_X = 1$. We have $|z| = \rho(x, y) = x\rho(\chi_{\operatorname{supp} x}, y/x)$. Hence $\frac{|z|}{x} = \rho(\chi_{\operatorname{supp} x}, y/x)$ and consequently $y = x\rho^{-1}\left(1, \frac{|z|}{x}\right)$ with $\|y\|_Y < 1$. Thus

$$I_{\varphi_R}^R(z) = \inf_{w \in B(X)_+} \left\| w\varphi_R \circ \left(\frac{|z|}{w} \right) \right\|_Y < 1,$$

whence $\rho(X, Y) \notin (nm_R)$.

Sufficiency. If $\rho(X, Y) \notin (nm_R)$, then we find an element $w \in \rho(X, Y)$ with $\|w\|_\rho = 1$ and $I_{\varphi_R}^R(w) < 1$. Thus there is an $x \in B(X)_+$ such that $\left\| x\varphi_R \circ \left(\frac{|w|}{x} \right) \right\|_Y < 1$. Setting $y = x\varphi_R \circ \left(\frac{|w|}{x} \right) = x\rho^{-1}\left(1, \frac{|w|}{x}\right)$ we get $|w| = \rho(x, y)$. Then $\|x\|_X = 1$, because $1 = \|w\|_\rho \leqslant \|(x, y)\|_\infty$ and $\|y\|_Y < 1$. Thus condition $(+)$ is not satisfied.

The proof of (ii) is the same. □

COROLLARY 28. *$\rho(X,Y) \in (nm_R)$ and $\rho(X,Y) \in (nm_L)$ if and only if $\|x\|_X = \|y\|_Y$ whenever $z \in \rho(X,Y)$, $|z| = \rho(x,y)$ and $\|z\|_\rho = \|(x,y)\|_\infty$.*

DEFINITION 29. Let $p = \infty$. We shall say that the space $\rho(X,Y)$ satisfies the *$'nm'$* condition ($\rho(X,Y) \in (nm)$ shortly) provided $\|x\|_X = \|y\|_Y$ whenever $\|\rho(x,y)\|_\rho = \|(x,y)\|_\infty$.

REMARK 30. It may happen that $X,Y \in (SM)$, $\rho(X,Y) \in (nm_R)$, $\rho(X,Y) \in (nm_L)$, $|z| = \rho(x,y)$, $\|y\|_Y = \|x\|_X$ and $\|z\|_\rho < \|x\|_X$. To see this take $X = Y = L^1[0,1]$, $\rho(u,v) = \sqrt{uv}$, $x = 2\chi_{[0,\frac{1}{2}]} + \frac{1}{2}\chi_{[\frac{1}{2},1]}$, $x = \frac{1}{2}\chi_{[0,\frac{1}{2}]} + 2\chi_{[\frac{1}{2},1]}$ and $z = \rho(x,y)$. Then $\|y\|_Y = \|x\|_X = 5/4$. However, $z = \chi_{[0,1]} = \rho\left(\chi_{[0,1]}, \chi_{[0,1]}\right)$, so $\|z\|_\rho \leqslant 1$. Obviously, $\rho(X,Y) \in (nm_R)$, $\rho(X,Y) \in (nm_L)$, because $\rho \in \Delta_2(R,\mathbb{R}_+)$ and $\rho \in \Delta_2(L,\mathbb{R}_+)$ (see Lemma 31 below). The same is possible when $X = Y = L^2[0,1]$.

LEMMA 31. *If $p = \infty$, then:*
(i) If $\rho \in \Delta_2(L,\mathbb{R}_+)$, then $\rho(X,Y) \in (nm_L)$.
(ii) If $\rho \in \Delta_2(R,\mathbb{R}_+)$, then $\rho(X,Y) \in (nm_R)$.
(iii) If $X \in (SM)$, then $\rho(X,Y) \in (nm_R)$.
(iv) If $Y \in (SM)$, then $\rho(X,Y) \in (nm_L)$.
Moreover, the converse of any above implication is not true in general.

Proof. (i) Suppose that $\rho \in \Delta_2(L,\mathbb{R}_+)$. This means that

$$\forall_{K>1} \exists_{1<C<K} \forall_{u,v>0} \quad \rho(u,v) \leqslant \rho\left(\frac{K}{C}u, \frac{1}{C}v\right). \tag{5}$$

If $\rho(X,Y) \notin (nm_L)$, then we find $z \in \rho(X,Y)_+$ such that

$$z = \rho(x,y), \ \|z\|_\rho = \|(x,y)\|_\infty \text{ and } \|x\|_X < \|y\|_Y.$$

Let $\sigma = \|y\|_Y - \|x\|_X$. Take $K > 1$ small enough to satisfy

$$(K-1)\|x\|_X \leqslant \frac{\sigma}{2}.$$

Then

$$\|z\|_\rho \leqslant \left\|\left(\frac{K}{C}x, \frac{1}{C}y\right)\right\|_\infty < \|(x,y)\|_\infty,$$

where $C > 1$ is from (5), a contradiction.

(iii) Let $X \in (SM)$. If $\rho(X,Y) \notin (nm_R)$, then there is $z \in \rho(X,Y)_+$ with

$$z = \rho(x,y), \ \|z\|_\rho = \|(x,y)\|_\infty \text{ and } \|x\|_X > \|y\|_Y.$$

Let $\sigma = \|x\|_X - \|y\|_Y$. One can find a set $A \in \Sigma$, $A \subset \operatorname{supp} z$, of finite measure such that

$$a \leqslant x\chi_A \leqslant b \text{ and } c \leqslant y\chi_A \leqslant d$$

for some $a,b,c,d > 0$. Then $\rho(a,c) \leqslant z\chi_A \leqslant \rho(b,d)$. Consequently, for each $0 < \varepsilon < \frac{a}{2}$ there is $\delta(\varepsilon) > 0$ such that the function $h(\varepsilon,\cdot) \in L^0$ given by the equality

$$z\chi_A = \rho(x,y)\chi_A = \rho(x-\varepsilon, h(\varepsilon,\cdot)+y)\chi_A$$

satisfies the inequality

$$0 \leqslant h(\varepsilon,\cdot) \leqslant \delta(\varepsilon).$$

Clearly $h(\varepsilon,\cdot)\chi_A \in Y$. Moreover, $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0$. We find $\varepsilon > 0$ sufficiently small to satisfy

$$\|h(\varepsilon,\cdot)\chi_A\|_Y \leqslant \delta(\varepsilon)\|\chi_A\|_Y \leqslant \frac{\sigma}{2}.$$

Then

$$\|z\|_\rho \leqslant \|(x-\varepsilon\chi_A, y+h(\varepsilon,\cdot)\chi_A)\| < \|(x,y)\|_\infty.$$

This contradiction finishes the proof. The proof of (ii) and (iv) goes analogously. Furthermore, collecting all implications $(i)-(iv)$ we see that none of them can be reversed in general. □

As an immediate consequence of Lemma 31 we conclude

COROLLARY 32. *Let* $p = \infty$. *If* $X, Y \in (SM)$, *then* $\rho(X,Y) \in (nm)$.

REMARK 33. For $p = \infty$ the implication

$$\rho(X,Y) \in (nm_R) \Rightarrow [\rho \in \Delta_2(R,\mathbb{R}_+) \text{ or } X \in (SM)]$$

does not hold in general. Really, take $X = L^\infty$, Y - a symmetric Köthe function space over a finite measure space such that $Y_a = \{0\}$ and $\rho \in U$ with $\rho \notin \Delta_2(R,\infty)$. Then $\rho(X,Y) \in (nm_R)$ by Remark 26 and Lemma 2.3 from [12]. An easy example of such space Y is $Y_0 = L^1[0,1] \cap L^\infty[0,1]$ with the norm $\|y\|_{Y_0} = \|y\|_{L^1} + \|y\|_{L^\infty}$. Moreover, $\rho(L^\infty, Y_0) \in (SM)$. Indeed, $\rho(L^\infty, Y_0) \in (nm_L)$ because $Y_0 \in (SM)$ (see Lemma 31 (iv)). Since $(Y_0)_a = \{0\}$ then $\rho(L^\infty, Y_0) \in (nm_R)$ (see [12]). Hence it is enough to apply Lemma 36 and Theorem 38 below since the condition $(L^\infty, Y_0) \notin (\text{non}-SM)_{\text{jointly}}$ is fulfilled by the assumption that $Y_0 \in (SM)$.

NOTATION 34. Recall that a point $x \in X_+$ is said to be a point of upper monotonicity provided $\|x+y\|_X > \|x\|_X$ for each $0 \leqslant y \neq 0$. For any Banach lattice X by X_{UM} we denote the set of points of upper monotonicity of X (see [10]).

Clearly, the space $\rho(X,Y)$ can be order continuous even when neither X nor Y is order continuous (this is the case when $(X,Y) \notin (JOD)$). Recall that this can not happen when $X = L^\infty$, because in this case if $Y \notin (OC)$ we have $\rho(X,Y) \notin (OC)$ (see Theorem 13 and Remark 16 above). Similarly it turns out that $\rho(X,Y)$ can be strictly monotone even when neither X nor Y is strictly monotone (see Example 40 below). In order to discuss this case we shall introduce the following notion.

DEFINITION 35. (i) We shall say that the pair (X,Y) is jointly non strictly monotone provided there is a set T_0 such that

$$X(T_0, \Sigma\cap T_0, \mu|_{T_0}) \notin (SM) \text{ and } Y(T_0, \Sigma\cap T_0, \mu|_{T_0}) \notin (SM).$$

We shall write $(X,Y) \in (\text{non} - SM)_{\text{jointly}}$.

(*ii*) Let $1 \leqslant p \leqslant \infty$. We say that $\rho(X,Y)$ satisfies the condition A ($\rho(X,Y) \in (A)$ shortly) provided one of the following conditions is satisfied:

(*a*) there exist elements $x \in X \setminus X_{UM}$, $y \in Y$, and $0 \leqslant u_X \neq 0$ with $\mu(\operatorname{supp} x \div \operatorname{supp} y) = 0$, $\operatorname{supp} u_X \subset \operatorname{supp} x$,

$$_p\|\rho(x,y)\|_\rho = \|(x,y)\|_p \text{ and } \|x+u_X\|_X = \|x\|_X .$$

(*b*) there exist elements $x \in X$, $y \in Y \setminus Y_{UM}$ and $0 \leqslant u_Y \neq 0$ with $\mu(\operatorname{supp} x \div \operatorname{supp} y) = 0$, $\operatorname{supp} u_Y \subset \operatorname{supp} y$,

$$_p\|\rho(x,y)\|_\rho = \|(x,y)\|_p \text{ and } \|y+u_Y\|_Y = \|y\|_Y .$$

(*c*) there exist elements $x \in X \setminus X_{UM}$, $y \in Y \setminus Y_{UM}$ and $0 \leqslant u_X, u_Y \neq 0$ with $\mu(\operatorname{supp} u_X \div \operatorname{supp} u_Y) = 0$,

$$_p\|\rho(x,y)\|_\rho = \|(x,y)\|_p, \|x+u_X\|_X = \|x\|_X \text{ and } \|y+u_Y\|_Y = \|y\|_Y .$$

LEMMA 36. *(i) Let $p = \infty$, $\rho \in U$ and $\rho(X,Y) \in (nm)$. If $\rho(X,Y) \in (A)$ then $(X,Y) \in (non - SM)_{\text{jointly}}$.*

(ii) Let $1 \leqslant p < \infty$ and $\rho \in U$. If $\rho(X,Y) \in (A)$, then $(X,Y) \in (non - SM)_{\text{jointly}}$.

Proof. (*i*) Assume that $(X,Y) \notin (\text{non} - SM)_{\text{jointly}}$. The proof of the case $X \in (SM)$ or $Y \in (SM)$ goes analogously as below. So assume that $X \notin (SM)$, $Y \notin (SM)$ and there is no set T_0 satisfying Definition 35 (*i*). Then there are sets

$$T_1, T_2 \in \Sigma, \qquad T_1 \cap T_2 = \varnothing, \qquad T_1 \cup T_2 = T \tag{6}$$

such that

$$\begin{aligned} &X(T_1, \Sigma \cap T_1, \mu|_{T_1}) \in (SM), \qquad Y(T_1, \Sigma \cap T_1, \mu|_{T_1}) \notin (SM), \\ &X(T_2, \Sigma \cap T_2, \mu|_{T_2}) \notin (SM) \text{ and } Y(T_2, \Sigma \cap T_2, \mu|_{T_2}) \in (SM). \end{aligned} \tag{7}$$

We show that if $x \in X \setminus X_{UM}$, $y \in Y$, $0 \leqslant u_X \neq 0$ with $\mu(\operatorname{supp} x \div \operatorname{supp} y) = 0$, $\operatorname{supp} u_X \subset \operatorname{supp} x$ and $\|x+u_X\|_X = \|x\|_X$ then $\|z\|_\rho < \|(x,y)\|_\infty$, where $z = \rho(x,y)$. Suppose that the previous assumption holds. Then $z = \rho(x+u_X, y-u_Y)$ for some $0 \leqslant u_Y \subset \operatorname{supp} x$, $u_Y \neq 0$. Then

$$\|z\|_\rho \leqslant \|(x+u_X, y-u_Y)\|_\infty \leqslant \|(x,y)\|_\infty .$$

If $\|z\|_\rho = \|(x,y)\|_\infty$, then $\|z\|_\rho = \|(x+u_X, y-u_Y)\|_\infty$. But this is a contradiction with $\rho(X,Y) \in (nm)$ because $\|y-u_Y\|_Y < \|y\|_Y$. To prove that case (*b*) of Definition 35 (*ii*) is not satisfied we go similarly. Obviously, the case (*c*) can't be fulfilled.

(*ii*) The proof is the same as above because having the same notation we would have $\|(x+u_X, y-u_Y)\|_p < \|(x,y)\|_p$. □

EXAMPLE 37. (*i*) Let $X_1 = \left(L^1[0,1] \oplus L^\infty[1,2]\right)_1$ and $Y_1 = \left(L^\infty[0,1] \oplus L^1[1,2]\right)_1$. Clearly $(X_1,Y_1) \notin (\text{non} - SM)_{\text{jointly}}$. If additionally $p = \infty$ and $\rho(X_1,Y_1) \in (nm)$ then $\rho(X_1,Y_1) \notin (A)$.

(ii) Set $X_2 = \left(L^1[0,1] \oplus L^\infty[1,2]\right)_1$ and $Y_2 = \left(L^1[0,3/2] \oplus L^\infty[3/2,2]\right)_1$. Hence $(X_2, Y_2) \in (\text{non} - SM)_{\text{jointly}}$. Furthermore, taking $p = \infty$, $x = \chi_{\left[\frac{3}{2},\frac{7}{4}\right]} + \frac{1}{2}\chi_{\left[\frac{7}{4},2\right]}$, $y = \chi_{\left[\frac{3}{2},2\right]}$ and $\rho(u,v) = \sqrt{uv}$ we see that $\rho(X_2, Y_2) \in (A)$, because $\|\rho(x,y)\|_\rho = \|(x,y)\|_\infty = 1$ and $\|x + u_X\|_{X_2} = 1$ with $u_X = \frac{1}{2}\chi_{\left[\frac{7}{4},2\right]}$.

QUESTION. Is the converse implication of Lemma 36 true? If yes, the following characterization would become more clear, specified and easier to verify in particular cases of X and Y.

THEOREM 38. (i) *Assume that* $p = \infty$. *The space* $\rho(X,Y)$ *is strictly monotone if and only if* $\rho(X,Y) \in (nm_R)$, $\rho(X,Y) \in (nm_L)$ *and* $\rho(X,Y) \notin (A)$.

(ii) *Suppose that* $1 \leqslant p < \infty$. *The space* $\rho(X,Y)$ *is strictly monotone if and only if* $\rho(X,Y) \notin (A)$.

Proof. (i) *Necessity*. Assume that $\rho(X,Y) \notin (nm_R)$. Then, by Lemma 27, there are $w \in \rho(X,Y)$, $x \in X_+$ and $y \in Y_+$ with $|w| = \rho(x,y)$ and $1 = \|w\|_\rho = \|(x,y)\|_\infty = \|x\|_X > \|y\|_Y$.

Take $0 \leqslant y_0 \neq 0, y_0 \in Y$ with $\operatorname{supp} y_0 \subset \operatorname{supp} y$ such that $\|y_0\|_Y < 1 - \|y\|_Y$ and define $w_0 = \rho(x, y + y_0)$. Hence $0 \leqslant |w| \leqslant w_0, |w| \neq w_0$. Furthermore

$$1 = \|w\|_\rho \leqslant \|w_0\|_\rho \leqslant \|(x, y+y_0)\|_\infty = \|x\|_X = 1.$$

Then $\rho(X,Y) \notin (SM)$. The necessity of the condition $\rho(X,Y) \in (nm_L)$ can be proved similarly. Suppose that $\rho(X,Y) \in (A)$. Let x, y, u_X be as in the Definition $(35)\,(ii)\,(a)$. Define $z = \rho(x,y)$ and $z_1 = \rho(x + u_X, y)$. Then $0 \leqslant z \leqslant z_1, z \neq z_1$. Moreover,

$$\|z\|_\rho \leqslant \|z_1\|_\rho \leqslant \|(x + u_x, y)\|_\infty = \|(x,y)\|_\infty = \|z\|_\rho.$$

Hence $\rho(X,Y) \notin (SM)$. If the case (b) or (c) of Definition $(35)\,(ii)$ holds we proceed analogously.

Sufficiency. Let $0 \leqslant z \leqslant w \in \rho(X,Y)$ and $z \neq w$. Take $(x,y) \in (X \oplus Y)_+$ with $w = \rho(x,y)$ and $\|w\|_\rho = \|(x,y)\|_\infty$. We find $u \in L^0$ such that $z = \rho(u,y)$. Clearly $u \leqslant x$, so $u \in X$. Furthermore

$$u \leqslant \frac{u+x}{2} \leqslant x \text{ and } u \neq \frac{u+x}{2} \neq x.$$

Similarly, one can find $v \in L^0$ with $z = \rho\left(\frac{u+x}{2}, v\right)$. Hence $v \leqslant y, v \neq y$, so $v \in Y$. If $\|z\|_\rho < \left\|\left(\frac{u+x}{2}, v\right)\right\|_\infty$, then

$$\|z\|_\rho < \left\|\left(\frac{u+x}{2}, v\right)\right\|_\infty \leqslant \|(x,y)\|_\infty = \|w\|_\rho,$$

whence $\rho(X,Y) \in (SM)$. Assume now that $\|z\|_\rho = \left\|\left(\frac{u+x}{2}, v\right)\right\|_\infty$. Consequently $\left\|\frac{u+x}{2}\right\|_X = \|v\|_Y$, by $\rho(X,Y) \in (nm)$ and Corollary 28. If $\frac{u+x}{2} \in X_{UM}$ or $v \in Y_{UM}$, then

$$\|z\|_\rho = \left\|\left(\frac{u+x}{2}, v\right)\right\|_\infty < \|(x,y)\|_\infty = \|w\|_\rho$$

as desired. Finally, suppose that $\frac{u+x}{2} \in X \backslash X_{UM}$ and $v \in Y \backslash Y_{UM}$. Since $\rho(X,Y) \notin (A)$ we conclude that $\left\| \frac{u+x}{2} \right\|_X < \|x\|_X$ or $\|v\|_Y < \|y\|_Y$. Really, if $\mu(\operatorname{supp} w \setminus \operatorname{supp} z) = 0$ we apply the condition (a) or (b) from Definition (35) (ii). If $\mu(\operatorname{supp} w \setminus \operatorname{supp} z) > 0$ then case (c) of Definition (35) (ii) is applied.

(ii) We follow as in the case (i). □

REMARK 39. If $p = \infty$, the space $\rho(X,Y)$ cannot be strictly monotone when $X = L^\infty$ and $Y \notin (SM)$ (see Corollary 42 below).

EXAMPLE 40. Let $p = \infty$, $X = \left(L^1[0,1] \oplus L^\infty[1,2]\right)_1$ and $Y = \left(L^\infty[0,1] \oplus L^1[1,2]\right)_1$. Clearly $X, Y \notin (SM)$ but $(X,Y) \notin (\text{non} - SM)_{\text{jointly}}$. If $\rho(u,v) = \sqrt{uv}$, then $\rho \in \Delta_2(L,R,\mathbb{R}_+)$. Consequently, $\rho(X,Y) \in (SM)$ by Lemma 31, Lemma 36 and Theorem 38.

From Lemma 31, Lemma 36 and Theorem 38 we get the following sufficient conditions more clear, specified and easier to verify.

COROLLARY 41. *Suppose that* $p = \infty$.

(i) *If* $X, Y \in (SM)$, *then* $\rho(X,Y)$ *is strictly monotone.*

(ii) *If* $X \in (SM)$ *and* $\rho(X,Y) \in (nm_L)$, *then* $\rho(X,Y)$ *is strictly monotone.*

(iii) *If* $Y \in (SM)$ *and* $\rho(X,Y) \in (nm_R)$, *then* $\rho(X,Y)$ *is strictly monotone.*

(iv) *If* $X \in (SM)$ *and* $\rho \in \Delta_2(L,\mathbb{R}_+)$, *then* $\rho(X,Y)$ *is strictly monotone.*

(v) *If* $Y \in (SM)$ *and* $\rho \in \Delta_2(R,\mathbb{R}_+)$, *then* $\rho(X,Y)$ *is strictly monotone.*

(vi) *If* $(X,Y) \notin (non - SM)_{\text{jointly}}$ *and* $\rho \in \Delta_2(L,R,\mathbb{R}_+)$ *then* $\rho(X,Y)$ *is strictly monotone.*

Suppose that $p \in [1,\infty)$.

(vii) *If* $X \in (SM)$ *or* $Y \in (SM)$, *then* $\rho(X,Y)$ *is strictly monotone.*

COROLLARY 42. (Lemma 2.5 in [12]) *If* $X = L^\infty$ *and* $p = \infty$, *then* $\rho(L^\infty, Y) \in (SM)$ *iff* $Y \in (SM)$ *and* $\rho(L^\infty, Y) \in (nm_R)$.

Proof. Necessity. We claim that if $X = L^\infty$ and $Y \notin (SM)$, then $\rho(X,Y) \in (A)$. Really, taking $0 \leqslant y, y_0 \in Y$ with $\operatorname{supp} y_0 \subset \operatorname{supp} y$, $1 = \|y\|_Y = \|y + y_0\|_Y$ and $z = \rho\left(\chi_{\operatorname{supp} y}, y\right)$, we get $\|z\|_\rho \leqslant 1$. If $\|z\|_\rho < 1$, then $\|(x_1,y_1)\|_\infty < 1$, where $z = \rho(x_1,y_1)$. Then $\|x_1\|_{L^\infty} < 1$, so $y_1 \geqslant y$ and $\|y_1\|_Y \geqslant 1$, a contradiction. Thus $\|z\|_\rho = 1$, which proves the claim. Then $Y \in (SM)$ by Theorem 38.

Sufficiency. If $Y \in (SM)$, then $\rho(L^\infty, Y) \in (nm_L)$, by Lemma 31 (iv). The thesis is obvious by Theorem 38. □

LEMMA 43. *Let* $p = \infty$, $X \notin (SM)$, $Y \notin (SM)$, $(X,Y) \notin (non - SM)_{\text{jointly}}$, $z = \rho(x,y) \in \rho(X,Y)$ *and* $\|z\|_\rho = \|(x,y)\|_\infty$. *Let* T_1, T_2 *be as in (6) and (7). Then:*

(i) *if* $\operatorname{supp} z \subset T_1$, *then* $\|z\|_\rho = \|y\|$.

(ii) *if* $\operatorname{supp} z \subset T_2$, *then* $\|z\|_\rho = \|x\|$.

(iii) *if* $\mu(\operatorname{supp} z \cap T_1) > 0$ *and* $\mu(\operatorname{supp} z \cap T_2) > 0$, *then* $\|z\|_\rho = \|x\| = \|y\|$.

Proof. It can be done similarly as the proof of Lemma 31 (iii). □

4. Discussion

We have seen that the problem of a full characterization of order continuous spaces $\rho(X,Y)$ remains open. The crucial point here is to prove the necessity of Δ_2 condition. Clearly, such criteria for $X = L^{\infty}$ are known, but L^{∞} is the worst space in the class $\mathscr{X}$ of spaces lacking OC. Moreover, $\mathscr{X}$ is quite rich and we have a lot of possibilities in general case of $\rho(X,Y)$. Considering the property SM, the respective criterion is not easy to verify even in the case $X = L^{\infty}$, because of the nm_R condition (some clear characterization of nm_R condition by properties of φ and E is given in Lemma 2.3 from [12] only in one particular case). Of course, the case $X = L^{\infty}$ is much simpler than the general one because of the fact that L^{∞} has the strong unit. Moreover, the main difficulty in looking for a clear characterization of the condition nm_R, nm_L or $\rho(X,Y) \in A$ consists in the fact that we know almost nothing about the set $\left\{(x,y) : \|(x,y)\|_p =_p \|z\|_\rho \text{ and } z = \rho(x,y)\right\}$.

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