Mathematical
Inequalities
& Applications
Volume 13, Number 1 (2010), 217–225

ALMOST CONVEX FUNCTIONS ON LOCALLY COMPACT ABELIAN GROUPS

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(*Communicated by Zs. Páles*)

Abstract. Let G be a locally compact Abelian group divisible by 2. We prove that every almost convex function on G equals a convex function a.e.

The function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex*, if $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$ for every $x, y \in \mathbb{R}^n$. (Sometimes it is called *midconvex* or *Jensen convex*.) We shall say that the function $f : \mathbb{R}^n \to \mathbb{R}$ is *almost convex*, if $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$ for a.e. pair $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. M. Kuczma proved in [2] (see also [3, Chap. XVII, Sec. 8]) that every almost convex function equals a convex function a.e. In this paper we shall generalize Kuczma's theorem for some locally compact Abelian groups.

Let G be a locally compact Abelian group with Haar measure μ. We shall say that the function $f : G \to \mathbb{R}$ is *convex*, if $2f(x) \leqslant f(x+y) + f(x-y)$ holds for every $x, y \in G$. The function f is *almost convex*, if $2f(x) \leqslant f(x+y) + f(x-y)$ holds for μ^2-a.e. $(x,y) \in G^2$. Clearly, for $G = \mathbb{R}^n$ this is equivalent to Kuczma's condition. Our aim is to prove the following.

THEOREM 1. *Let G be a locally compact Abelian group divisible by 2. Then for every almost convex function $f : G \to \mathbb{R}$ there exists a unique convex function $g : G \to \mathbb{R}$ such that $f = g$ μ-a.e.*

Since every compact connected Abelian group is divisible, it follows that Kuczma's theorem is valid in these groups. We do not know whether or not Kuczma's theorem is valid in *every* locally compact Abelian group. The condition of divisibility by 2 is by no means necessary. Indeed, on a discrete group every almost convex function is convex, and thus Kuczma's theorem is valid on discrete groups. Also, it is easy to check that if G is a torsion group, then every almost convex function on G must be constant a.e., and thus Kuczma's theorem is valid on torsion groups as well. Thus the best candidates of groups on which Kuczma's theorem might be false are those non-discrete torsion free groups which are not divisible by 2. Such groups are, for example, the powers of the compact group of 2-adic integers. We do not know whether or not Kuczma's theorem is true on these groups.

Mathematics subject classification (2010): 22D99.
Keywords and phrases: Almost convex functions, loccaly compact Abelian group.
Research partially supported by the Hungarian National Foundation for Scientific Research, Grant No. T49786.

Paper MIA-13-17

Let $\frac{1}{n}E$ denote the set $\{x \in G : nx \in E\}$ for every $n = 1, 2, \dots$ and $E \subset G$. The outer measure of the set $E \subset G$ is defined by $\overline{\mu}(E) = \inf\{\mu(U) : E \subset U \subset G,\ U \text{ is open}\}$. Then $\overline{\mu}$ is an outer measure on G, and is an extension of μ. The family of Borel subsets of G will be denoted by $\mathscr{B}(G)$. As we shall check shortly, Kuczma's proof is based on the fact that the following properties are satisfied if $G = \mathbb{R}^n$.

(P_1) If $A \subset G$ is a set of positive outer measure, then the set $\frac{1}{2}A$ is also of positive outer measure.

(P_2) If $A, B \subset G$ are sets of positive outer measure and $N, M \subset G^2$ are sets of full measure, then

$$N \cap \phi^{-1}(M) \cap \left[\left(\frac{1}{2}A\right) \times \left(\frac{1}{2}B\right)\right] \neq \emptyset, \tag{1}$$

where ϕ is the transformation of $G \times G$ defined by $\phi(x, y) = (x + y, x - y)$.

Our proof of Theorem 1 consists of two parts. First we show that Kuczma's proof works in any group satisfying the properties (P_1) and (P_2). Then we prove that (P_1) and (P_2) are satisfied in every group which is divisible by 2.

Before turning to the proof of Theorem 1 we clarify the notions of null sets and sets of full measure. We say that $E \subset G$ is *null*, if $\overline{\mu}(E) = 0$; that is, if E can be covered by a Borel set of measure zero. If G has an open subgroup G_0, then a null set cannot intersect uncountably many cosets of G_0. Indeed, the cosets of G_0 are open, and thus, if an open set U intersects uncountably many cosets of G_0, then necessarily $\mu(U) = \infty$. Thus any set that intersects uncountably many cosets of G_0 must have infinite outer measure.

We say that a set is *of full measure*, if its complement is null. Clearly, a set of full measure contains all but a countably many of the cosets of G_0.

We remark that in [3] Kuczma actually proves his theorem in a more general setting, in which the ideal of null sets is replaced by other ideals. The statement of Theorem 1 has similar generalizations as well. These generalizations will be presented by a forthcoming paper of the first author.

Proof of Theorem 1. First we prove that the theorem is true under the assumption that (P_1) and (P_2) are satisfied in G. Let $f : G \to \mathbb{R}$ be an almost convex function. For every $x \in G$ we define $g(x)$ as the infimum of those real numbers c for which the set

$$\{h \in G : f(x+h) + f(x-h) < 2c\}$$

is of positive outer measure. It is clear that $g(x) \in [-\infty, \infty)$ for every $x \in G$. Our aim is to prove that g is a finite valued convex function and is equal to f a.e. Let M denote the set of pairs $(x, h) \in G^2$ for which $2f(x) \leqslant f(x+h) + f(x-h)$. Then M is of full μ^2-measure in G^2. For every integer n, the map $(x, y) \mapsto (x + ny, y)$ is a topological isomorphism of G^2 onto itself and, consequently, if a subset of G^2 is null then so is its preimage. Therefore, the sets $M_n = \{(x, y) : (x + ny, y) \in M\}$ are of full measure in G^2. Then so is the set $P = \bigcap_{n=-\infty}^{\infty} M_n$. By Fubini's theorem, there is a set $Q \subset G$ of full measure such that for every $x \in Q$ we have $(x, y) \in P$ for μ-a.e. y. We shall prove that $g(x) = f(x)$ for every $x \in Q$.

Let $x \in Q$ be arbitrary. Then $(x,y) \in P \subset M$ for μ-a.e. y. Thus $2f(x) \leqslant f(x+h)+f(x-h)$ for μ-a.e. h, and hence $g(x) \geqslant f(x)$. Let $\varepsilon > 0$ be given; we prove that $g(x) \leqslant f(x)+\varepsilon$. For every positive integer k let

$$A_k = \{y \in G : \max(f(x+y), f(x-y)) \leqslant k\}.$$

Since $G = \bigcup_{k=1}^{\infty} A_k$, there is a k such that $\overline{\mu}(A_k) > 0$. Fix an integer n such that $2^n > (k-f(x))/\varepsilon$. By property (P_1), each of the sets $\frac{1}{2}A_k$, $\frac{1}{2}\left(\frac{1}{2}A_k\right) = \frac{1}{4}A_k$, $\frac{1}{2}\left(\frac{1}{4}A_k\right) = \frac{1}{8}A_k, \dots$ is of positive outer measure. Therefore, so is the set $S = \left(\frac{1}{2^n}A_k\right) \cap \{y \in G : (x,y) \in P\}$. Let $y \in S$ be arbitrary. Then for every i we have $(x,y) \in M_i$; that is, $(x+iy, y) \in M$ and

$$2f(x+iy) \leqslant f(x+(i+1)y) + f(x+(i-1)y).$$

Multiplying this inequality by $2^n - i$ and adding for $i = 1, \dots, 2^n - 1$ we obtain

$$2^n f(x+y) \leqslant (2^n-1)f(x) + f(x+2^n y) \leqslant (2^n-1)f(x) + k,$$

as $2^n y \in A_k$. Thus $f(x+y) \leqslant f(x) + (k-f(x))/2^n < f(x)+\varepsilon$. A similar argument gives $f(x-y) < f(x)+\varepsilon$. We have proved that

$$f(x+y)+f(x-y) \leqslant 2\max(f(x+y), f(x-y)) < 2(f(x)+\varepsilon)$$

for every $y \in S$, where S is a set of positive outer measure. This implies $g(x) \leqslant f(x)+\varepsilon$ and, as ε was an arbitrary positive number, we have $g(x) \leqslant f(x)$ for every $x \in Q$. Since Q is of full measure, this proves that $f = g$ a.e.

Now we prove, still assuming (P_1) and (P_2), that g is a finite valued convex function. Let $u, v \in G$ be arbitrary. It is enough to show that for every fixed $\varepsilon > 0$ we have $2g(u) \leqslant g(u+v) + g(u-v) + \varepsilon$. We shall prove this by showing that there are elements $h, k \in G$ such that the following relations hold:

$$\begin{aligned} 4g(u) &\leqslant 2g(u+h+k) + 2g(u-h-k) \\ &\leqslant [g(u-v+2k) + g(u+v+2h)] + [g(u-v-2k) + g(u+v-2h)] \\ &= [g(u+v+2h) + g(u+v-2h)] + [g(u-v+2k) + g(u-v-2k)] \\ &< [2g(u+v)+\varepsilon] + [2g(u-v)+\varepsilon]. \end{aligned} \tag{2}$$

It follows from the definition of g that the set

$$\{y \in G : 2g(u) \leqslant f(u+y) + f(u-y)\}$$

is of full measure. Since $f = g$ a.e., we obtain that the set $H = \{y \in G : 2g(u) \leqslant g(u+y) + g(u-y)\}$ is of full measure. Let $K = \{(h,k) : h+k \in H\}$. Then K is of full measure in G^2, and the first inequality of (2) holds for every $(h,k) \in K$.

Since f is almost convex and $f = g$ a.e., the function g is also almost convex. That is, the set $L = \{(x,y) \in G^2 : 2g(x) \leqslant g(x+y) + g(x-y)\}$ is of full measure. The second inequality of (2) holds for every (h,k) such that $(u+h+k, v+h-k) \in L$

and $(u-h-k, v-h+k) \in L$; that is, when

$$(h+k, h-k) \in (L-(u,v)) \cap (-L+(u,v)) \stackrel{\text{def}}{=} N.$$

This condition can be formulated as $(h,k) \in \phi^{-1}(N)$, where $\phi(x,y) = (x+y, x-y)$ $(x,y \in G)$. Note that the set N is of full measure in G^2.

Finally, the definition of g implies that the sets $\{y \in G : f(u+v+y)+f(u+v-y) < 2g(u+v)+\varepsilon\}$ and $\{y \in G : f(u-v+y)+f(u-v-y) < 2g(u-v)+\varepsilon\}$ are of positive outer measure. Since $f = g$ a.e., it follows that the sets $A = \{y \in G : g(u+v+y) + g(u+v-y) < 2g(u+v)+\varepsilon\}$ and $B = \{y \in G : g(u-v+y)+g(u-v-y) < 2g(u-v)+\varepsilon\}$ are of positive outer measure as well. Now the last inequality of (2) holds for every (h,k) such that $2h \in A$ and $2k \in B$. Summing up, (2) holds for every (h,k) such that

$$(h,k) \in K \cap \phi^{-1}(N) \cap \left[\left(\frac{1}{2}A\right) \times \left(\frac{1}{2}B\right)\right].$$

By property (P_2) this set is nonempty, and thus (2) holds for some $h,k \in G$. This proves that g is convex. We show that g is finite everywhere. Indeed, let $x \in G$ be such that $g(x) = f(x)$. Then we have

$$-\infty < 2f(x) = 2g(x) \leqslant g(x+y) + g(x-y)$$

for every $y \in G$. Thus $g(x+y)$ is finite for every y; that is, g is finite everywhere.

Now we prove the uniqueness of g. First we note that if f is convex, then the function g defined in the proof above equals f everywhere. We shall use the notation introduced in the course of the proof. If f is convex, then we have $M = M_n = G^2$ for every n, and thus $P = G^2$ and $Q = G$. As we have shown, $g(x) = f(x)$ for every $x \in Q$; that is, $g = f$.

Let f be an almost convex function on G. If g_1 is a convex function such that $f = g_1$ μ-a.e. on G, then

$$\begin{aligned} g(x) &= \inf\{c \in \mathbb{R} : \overline{\mu}(\{h \in G : f(x+h) + f(x-h) < 2c\}) > 0\} \\ &= \inf\{c \in \mathbb{R} : \overline{\mu}(\{h \in G : g_1(x+h) + g_1(x-h) < 2c\}) > 0\} \end{aligned}$$

for every x. Since g_1 is convex, it follows from the previous argument that $g_1 = g$ everywhere, which proves the uniqueness of g.

In order to complete the proof of Theorem 1 we have to show that the properties (P_1) and (P_2) are satisfied in every group divisible by 2. This will be proved by a series of lemmas.

LEMMA 2. *Let G be a locally compact Abelian group with Haar measure μ, and suppose that $G = \mathbb{R}^d \times H$, where H is compact. Suppose further that $\mu(nG) > 0$ for a positive integer n. Then there exists a number $c > 0$ such that*

$$\overline{\mu}\left(\frac{1}{n}E\right) = c \cdot \overline{\mu}(E) \tag{3}$$

for every $E \subset nG$.

Proof. First we shall consider the case when G is compact and E is Borel. We may assume that $\mu(G) = 1$. As nG is a compact subgroup of G of positive measure, it follows that nG is open and G/nG is finite. If $|G/nG| = N$, then we have $\mu(nG) = 1/N$. Let $\nu(E) = \mu(\frac{1}{n}E)$ for every $E \in \mathscr{B}(nG)$. Then ν is a finite and regular Borel measure on nG with $\nu(nG) = 1$. (Note that $\frac{1}{n}E$ is compact if E is compact, and is open if E is open.) If $a \in nG$ and $a = nb$, then we have $\frac{1}{n}(E+a) = (\frac{1}{n}E) + b$, and thus $\nu(E+a) = \nu(E)$ for every $E \in \mathscr{B}(nG)$. Thus ν is translation invariant and hence, by the uniqueness of Haar measure on nG we obtain $\nu(E) = N \cdot \mu(E)$ for every $E \in \mathscr{B}(nG)$. Therefore, (3) holds for Borel subsets of nG with $c = N$.

Now let $G = \mathbb{R}^d \times H$ with H compact, and suppose $\mu(nG) > 0$. Let ν denote the Haar measure on H. Then $\mu = \lambda_d \times \nu$, where λ_d is the Lebesgue measure on $\mathbb{R}^d$. Since $nG = \mathbb{R}^d \times (nH)$, the condition $\mu(nG) > 0$ gives $\nu(nH) > 0$. Then, as we proved above, there exists a number $N > 0$ such that $\nu\left(\frac{1}{n}E\right) = N \cdot \nu(E)$ for every Borel set $E \subset nH$. If $A \subset \mathbb{R}^d$ and $B \subset nH$ are Borel sets, then we have

$$\mu\left(\frac{1}{n}(A \times B)\right) = \mu\left(\left(\frac{1}{n}A\right) \times \left(\frac{1}{n}B\right)\right) = \lambda_d\left(\frac{1}{n}A\right) \cdot \nu\left(\frac{1}{n}B\right)$$
$$= \frac{1}{n^d} \cdot \lambda_d(A) \cdot N \cdot \nu(B) = \frac{N}{n^d} \cdot \mu(A \times B).$$

Then, by the definition of the product measure, we obtain that (3) holds for Borel subsets of nG with $c = N/n^d$.

In order to prove (3) for arbitrary subsets of nG first we show that if $F \subset G$ is closed, then so is nF. It is enough to show that if $x_0 \in \mathbb{R}^d$, $y_0 \in H$ and $(x_0, y_0) \notin nF$, then (x_0, y_0) is not in the closure of nF. Let $r > |x_0|/n$ be a positive number, and put $B = \{x \in \mathbb{R}^d : |x| \leqslant r\}$ and $C = \{x \in \mathbb{R}^d : |x| \geqslant r\}$. Then $\mathbb{R}^d = B \cup C$ and $F \subset [C \times H] \cup [(B \times H) \cap F]$. Now the set $D_1 = n[C \times H] = (nC) \times (nH)$ is closed, and $|x_0| < n \cdot r$ gives $(x_0, y_0) \notin D_1$. Also, the set $(B \times H) \cap F$ is compact, and thus so is $D_2 = n \cdot [(B \times H) \cap F]$. Since $D_2 \subset nF$, we have $(x_0, y_0) \notin D_2$. Then, by $nF \subset D_1 \cup D_2$ and $(x_0, y_0) \notin D_1 \cup D_2$ it follows that (x_0, y_0) is not in the closure of nF.

Let $K = \{x \in G : nx = 0\}$. Then $K = \{0\} \times \{x \in H : nx = 0\}$, and thus K is a compact subgroup of G. We show that for every open set V containing zero there is an open set W also containing zero such that $\frac{1}{n}W \subset K + V$. Indeed, $K + V$ is open, $G \setminus (K + V)$ is closed, and thus $D = n \cdot [G \setminus (K + V)]$ is also closed. Now $0 \notin D$, since $nx = 0$ implies $x \in K$, and then $x \notin G \setminus (K + V)$. Therefore, we may take $W = G \setminus D$.

Now let E be an arbitrary subset of nG. If B is a Borel set such that $E \subset B \subset nG$ and $\mu(B) = \overline{\mu}(E)$, then $\frac{1}{n}E \subset \frac{1}{n}B$, and thus

$$\overline{\mu}\left(\frac{1}{n}E\right) \leqslant \mu\left(\frac{1}{n}B\right) = c \cdot \mu(B) = c \cdot \overline{\mu}(E).$$

To prove the inequality $\overline{\mu}\left(\frac{1}{n}E\right) \geqslant c \cdot \overline{\mu}(E)$, let $\varepsilon > 0$ be fixed, and let U be an open set containing $\frac{1}{n}E$ such that $\mu(U) < \overline{\mu}\left(\frac{1}{n}E\right) + \varepsilon$. It is clear that if $y \in \frac{1}{n}E$, then $K + y \subset \frac{1}{n}E$. Thus $K + y \subset U$ and hence, as $K + y$ is compact, we may find an open set V_y containing zero such that $K + y + V_y \subset U$ [1, (4.10) Theorem, p. 20]. Choose an open set W_y containing zero such that $\frac{1}{n}W_y \subset K + V_y$. Then $W = \bigcup\{W_y + ny : y \in \frac{1}{n}E\}$ is an open set containing E. We prove that $\frac{1}{n}W \subset U$. Indeed, $nx \in W$ implies that $nx \in W_y + ny$ for some $y \in \frac{1}{n}E$. Then $n(x-y) \in W_y$, $x - y \in \frac{1}{n}W_y \subset K + V_y$, and $x \in K + V_y + y \subset U$. Thus $D = W \cap nG$ is a Borel set containing E such that $\frac{1}{n}D \subset U$. Therefore,

$$\overline{\mu}\left(\frac{1}{n}E\right) + \varepsilon > \mu(U) \geqslant \mu\left(\frac{1}{n}D\right) = c \cdot \mu(D) \geqslant c \cdot \overline{\mu}(E),$$

which completes the proof. □

LEMMA 3. *Let G be a locally compact Abelian group with Haar measure μ, and let n be a positive integer. If $E \subset nG$ is a set of positive outer measure, then $\frac{1}{n}E$ is also of positive outer measure.*

Proof. By the structure theorem [4, 2.4.1 Theorem, p. 40], G has an open subgroup G_0 such that G_0 is topologically isomorphic to $\mathbb{R}^d \times H$, where H is a compact. If $\frac{1}{n}E$ intersects more than countable of the cosets of G_0, then $\overline{\mu}\left(\frac{1}{n}E\right) = \infty$, and in that case the statement is true. Therefore, we may assume that there are group elements g_i $(i = 1, 2, \ldots)$ such that

$$\frac{1}{n}E \subset \bigcup_{i=1}^{\infty}(G_0 + g_i).$$

Then

$$E = n \cdot \left(\frac{1}{n}E\right) = \bigcup_{i=1}^{\infty} n \cdot \left[\left(\frac{1}{n}E\right) \cap (G_0 + g_i)\right],$$

and thus there exists an i such that the set $E' = n \cdot \left[\left(\frac{1}{n}E\right) \cap (G_0 + g_i)\right]$ has positive outer measure. Since the set $E'' = n \cdot \left[\left(\left(\frac{1}{n}E\right) - g_i\right) \cap G_0\right]$ is a translate of E', it has positive outer measure as well. Note that $E'' \subset nG_0$, and thus $\mu(nG_0) > 0$. Therefore, $\frac{1}{n}E''$ has positive outer measure by Lemma 2. If $nx \in E''$, then $nx = ny$ for some $y \in \frac{1}{n}E - g_i$. Then $n(x-y) = 0$, and $x \in \left(\frac{1}{n}E - g_i\right) + (x - y) = \frac{1}{n}E - g_i$, since $\frac{1}{n}E + (x-y) = \frac{1}{n}E$. Therefore, $\frac{1}{n}E'' \subset \frac{1}{n}E - g_i$, which proves that $\frac{1}{n}E$ has positive outer measure. □

LEMMA 4. *If a locally compact Abelian group is divisible by 2, then it satisfies property* (P_1).

Proof. If G is divisible by 2, then $2G = G$, and then we may apply Lemma 3 with $n = 2$. □

LEMMA 5. *Let G be a locally compact Abelian group with Haar measure μ, and suppose that $G=\mathbb{R}^d\times H$, where H is compact. If $\mu(2G)>0$, then there exists a number $c>0$ such that*

$$\mu^2\left(\phi^{-1}(E)\right)\leqslant c\cdot\mu^2(E) \tag{4}$$

for every Borel set $E\subset G^2$, where $\phi(x,y)=(x+y,x-y)$ $(x,y\in G)$.

Proof. By Lemma 2, there exists a number $c>0$ such that (3) holds for every $E\subset 2G$ with $n=2$. Our first aim is to show that

$$\int_G f(2x)\,dx\leqslant c\cdot\int_G f(x)\,dx \tag{5}$$

for every non-negative Borel measurable $f:G\to\mathbb{R}$. Clearly, it is enough to check (5) in the case when f is the characteristic function χ_E of a Borel set E. Then we have $\int_G f(x)\,dx=\mu(E)$, and thus

$$\int_G f(2x)\,dx=\int_G\chi_{\frac{1}{2}E}\,dx=\mu\left(\frac{1}{2}E\right)=\mu\left(\frac{1}{2}(E\cap 2G)\right)$$
$$=c\cdot\mu(E\cap 2G)\leqslant c\cdot\mu(E)=c\cdot\int_G f(x)\,dx.$$

It is enough to prove (4) in the case when $E=A\times B$, where A and B are Borel subsets of G. Then we have $(x,y)\in\phi^{-1}(A\times B)$ if and only if $x\in(A-y)\cap(B+y)$, and thus the measure of the $y-$ section $\{x\in G\colon(x,y)\in\phi^{-1}(A\times B)\}$ equals $\mu((A-y)\cap(B+y))=\mu(A\cap(B+2y))$. Thus, by Fubini's theorem, we obtain

$$\mu^2\left(\phi^{-1}(A\times B)\right)=\int_G\mu(A\cap(B+2y))\,dy\leqslant c\cdot\int_G\mu(A\cap(B+y))\,dy$$

by (5) with $f(x)=\mu(A\cap(B+x))$. Since

$$\int_G\mu(A\cap(B+y))\,dy=\int_{G^2}\chi_A(t)\cdot\chi_B(t-y)\,dt\,dy=\mu(A)\cdot\mu(B)=\mu^2(A\times B),$$

the proof is complete. □

LEMMA 6. *Let G be a locally compact Abelian group with Haar measure μ, and suppose that $G=\mathbb{R}^d\times H$, where H is compact. If $\mu(2G)>0$ and $E\subset G\times G$ is a set of full measure, then $\phi^{-1}(E)$ is of full measure, where $\phi(x,y)=(x+y,x-y)$ $(x,y\in G)$.*

Proof. Since $F=(G\times G)\setminus E$ is null, there exists a Borel set B such that $F\subset B\subset G\times G$ and $\mu^2(B)=0$. Then $\phi^{-1}(B)$ is null by Lemma 5. Since

$$(G\times G)\setminus\phi^{-1}(E)\subset\phi^{-1}(F)\subset\phi^{-1}(B),$$

it follows that $\phi^{-1}(E)$ is of full measure. □

The next lemma will complete the proof of Theorem 1.

LEMMA 7. *If a locally compact Abelian group is divisible by* 2, *then it satisfies property* (P_2).

Proof. Let $A, B \subset G$ be sets of positive outer measure and $N, M \subset G^2$ are sets of full measure. We prove (1). By the structure theorem, G has an open subgroup G_0 such that G_0 is topologically isomorphic to $\mathbb{R}^d \times H$, where H is a compact. We shall distinguish between two cases.

I. First we assume that $\frac{1}{2}A$ intersects uncountably many cosets of G_0. Since the set $N^y = \{(x,y) : x \in N\}$ is of full measure for μ-a.e. y and the set $\frac{1}{2}B$ is of positive outer measure by Lemma 4, we can fix a $y \in \frac{1}{2}B$ such that N^y is of full measure. Then N^y contains all but countably many cosets of G_0. Therefore, $N^y \cap \frac{1}{2}A$ intersects uncountably many cosets of G_0. If $x \in N^y \cap \frac{1}{2}A$, then

$$(x,y) \in N \cap \left[\left(\frac{1}{2}A\right) \times \left(\frac{1}{2}B\right)\right] \stackrel{\text{def}}{=} T.$$

Therefore, $(x+y, x-y) \in \phi(T)$ for every $x \in N^y \cap \frac{1}{2}A$. Since y is fixed and $N^y \cap \frac{1}{2}A$ intersects uncountably many cosets of G_0, it follows that $\phi(T)$ intersects uncountably many cosets of $G_0 \times G_0$. Therefore, $\phi(T)$ is not null, and $M \cap \phi(T) \neq \emptyset$, whence $\phi^{-1}(M) \cap T \neq \emptyset$, which proves (1). A similar argument works if $\frac{1}{2}B$ intersects uncountably many cosets of G_0.

II. Next we assume that $\frac{1}{2}A$ and $\frac{1}{2}B$ intersect only countably many cosets of G_0. If $\frac{1}{2}A \subset \bigcup_{a\in I}(G_0+a)$, where I is a countable subset of G, then $A \subset \bigcup_{a\in I}(2G_0+2a)$. Thus $\mu(2G_0) > 0$, and there is an a such that $A \cap (2G_0 + 2a)$ is of positive outer measure. We may assume that $A \subset 2G_0 + 2a$. Similarly, we may assume that $B \subset G_0 + 2b$ for a suitable $b \in G$. Then the sets $A' = A - 2a$, $B' = B - 2b$ are of positive outer measure in G_0, and $\left(\frac{1}{2}A'\right) \times \left(\frac{1}{2}B'\right)$ is of positive outer measure in $G_0 \times G_0$.

The sets $N' = N - (a,b)$ and $M' = M - (a+b, a-b)$ are of full measure in $G_0 \times G_0$. By Lemma 6, $\phi^{-1}(M')$ is of full measure in $G_0 \times G_0$, and thus the set

$$N' \cap \phi^{-1}(M') \cap \left[\left(\frac{1}{2}A'\right) \times \left(\frac{1}{2}B'\right)\right] \stackrel{\text{def}}{=} X$$

is nonempty. If $(h,k) \in X$, then $(h+a, k+b)$ is an element of the left hand side of (1), proving that the set in question is nonempty. This completes the proof of Lemma 6 and of Theorem 1. □

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(Received February 20, 2008)

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