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FUNCTIONAL CALCULUS WITH OPERATOR–MONOTONE FUNCTIONS

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Abstract. Let $f(t)$ be a non-negative operator-monotone function defined on $[0,\infty)$, and A,B positive definite operators on a Hilbert space. The inequalities $\langle Bx,x\rangle \leqslant f(\langle Ax,x\rangle)$ for every unit vector x do not imply the operator inequality $B \leqslant f(A)$. We prove, however, that when combined with the inequalities $\langle B^{-1}x,x\rangle^{-1} \geqslant f(\langle A^{-1}x,x\rangle^{-1})$, the relation $B = f(A)$ follows.

1. Introduction and main result

Let $\mathscr{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For bounded selfadjoint operators A,B on $\mathscr{H}$ the notation $A \geqslant B$ (resp. $A > B$) means that $A-B$ is positive semi-definite (resp. positive definite). In particular, $A \geqslant 0$ (resp. $A > 0$) means that A is positive semi-definite (resp. positive definite). In this paper capital letters $A,B,\cdots$ will denote bounded selfadjoint operators.

We consider only *non-negative strictly increasing continuous* functions $f(t)$ on $[0,\infty)$. Therefore the functional calculus for $A \geqslant 0$ produces again an operator $f(A) \geqslant 0$. Furthermore $A > 0$ implies $f(A) > 0$.

A function $f(t)$ is said to be *operator-monotone* if regardless of the dimension of $\mathscr{H}$

$$0 < A \leqslant B \quad \Longrightarrow \quad f(A) \leqslant f(B).$$

Let us recall some related classes of functions. A function $f(t)$ is said to be *operator-convex* if

$$f\Big(\frac{A+B}{2}\Big) \leqslant \frac{f(A)+f(B)}{2} \qquad (A,B > 0).$$

Correspondingly one may consider *operator-concave* functions.

It is shown in [4, Section 2] that a function $f(t)$ on $[0,\infty)$ is operator-concave if and only if it is operator-monotone and that it is operator-convex with $f(0) \leqslant 0$ if and only if the function $f(t)/t$ is operator-monotone. In particular, an operator-monotone function is concave.

Further if $f(t)$ with $f(0) = 0$ and $f(\infty) = \infty$ is operator-convex then its inverse function is operator-monotone. In fact, in [1, Lemma 5] it is shown that the inverse

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function of a function of the form $f(t) = tg(t)$ with operator-monotone $g(t)$ is operator-monotone. This fact on inverse functions is widely generalized in [6, Theorem 2.8]. Let us mention that also in [3, Corollary 3.4] the same fact is proved by using characterization of operator-monotonicity in terms of the Loewner matrices associated with the function.

A typical example of an operator-monotone function is $f(t) = t^p$ with $0 < p \leqslant 1$. The inverse function of a function $f(t) = t^p$ with $1 \leqslant p < \infty$ is operator-monotone while for $1 \leqslant p \leqslant 2$ the function $f(t) = t^p$ is operator-convex.

We refer the reader to the nice exposition in [2, Chapter V] for those classes of functions.

LEMMA 1. *If a function $f(t)$ is operator-monotone, so is the function $f(t^{-1})^{-1}$.*

Proof. In fact,

$$\begin{aligned} 0 < A \leqslant B \quad &\Longrightarrow \quad A^{-1} \geqslant B^{-1} \\ &\Longrightarrow \quad f(A^{-1}) \geqslant f(B^{-1}) \quad \Longrightarrow \quad f(A^{-1})^{-1} \leqslant f(B^{-1})^{-1}, \end{aligned}$$

showing the operator-monotonicity of the function $f(t^{-1})^{-1}$. □

Now since an operator-monotone function $f(t)$ and by Lemma 1 its associate $f(t^{-1})^{-1}$ are concave, the next lemma is an easy consequence of the Jensen inequality for the integral of a concave function with respect to a probability measure on an interval (cf. [4, Corollary 2.2]).

LEMMA 2. *Let $f(t)$ be operator-monotone, and $A, B > 0$. Then*

$$B \leqslant f(A) \quad \Longrightarrow \quad \langle Bx, x\rangle \leqslant f(\langle Ax, x\rangle) \qquad (\|x\| = 1),$$

while

$$f(A) \leqslant B \quad \Longrightarrow \quad f(\langle A^{-1}x, x\rangle^{-1}) \leqslant \langle B^{-1}x, x\rangle^{-1} \qquad (\|x\| = 1).$$

For instance, let $A = \int_0^\infty \lambda dE(\lambda)$ be the spectral respresentation of the positive definite operator A. Then any unit vector x produces a probabiltiy measure $d\langle E(\lambda)x, x\rangle$ on $(0, \infty)$. Then the Jensen inequality shows

$$\langle f(A)x, x\rangle = \int_0^\infty f(\lambda) d\langle E(\lambda)x, x\rangle \leqslant f\Big(\int_0^\infty \lambda d\langle E(\lambda)x, x\rangle\Big) = f(\langle Ax, x\rangle).$$

Given an operator-monotone function $f(t)$ and $A, B > 0$, the converse implication in each of the statements in Lemma 2 does not hold. However if both the inequalities

$$\langle Bx, x\rangle \leqslant f(\langle Ax, x\rangle) \qquad (\|x\| = 1), \tag{$\sharp$}$$

and

$$f(\langle A^{-1}x, x\rangle^{-1}) \leqslant \langle B^{-1}x, x\rangle^{-1} \qquad (\|x\| = 1) \tag{$\flat$}$$

hold, a *sandwich technique* will show that $B = f(A)$. This is our main result.

THEOREM 3. *Let $f(t)$ be operator-monotone, and $A,B>0$. Then the relation $B=f(A)$ occurs if and only if the two groups of inequalities* ($\sharp$) *and* ($\flat$) *hold.*

Though a complete proof of Theorem 3 will be given in the next section, let us present a simple proof for the case of finite dimensional $\mathscr{H}$, that is, the matrix case. This is based on an idea of T. Hayashi [5].

By the finite dimensionality assumption, it suffices to prove that if x is a unit vector and $Ax=\alpha x$ for some $\alpha>0$ then $Bx=f(\alpha)x$. The assumption implies that $\langle Bx,x\rangle \leqslant f(\alpha)$ and $\langle B^{-1}x,x\rangle \leqslant f(\alpha)^{-1}$. Then since

$$\begin{aligned} 1 &= \langle x,x\rangle = \langle B^{1/2}x,B^{-1/2}x\rangle \\ &\leqslant \|B^{1/2}x\|\cdot\|B^{-1/2}x\| = \sqrt{\langle Bx,x\rangle\cdot\langle B^{-1}x,x\rangle} \leqslant \sqrt{f(\alpha)f(\alpha)^{-1}} = 1, \end{aligned}$$

we have $\langle B^{1/2}x,B^{-12/}x\rangle = \|B^{1/2}x\|\cdot\|B^{-1/2}x\|$, which is possible only when $B^{1/2}x = \beta B^{-1/2}x$, hence $Bx=\beta x$ for some $\beta>0$. Finally it follows from the assumption that $\beta \leqslant f(\alpha) \leqslant \beta$. This completes the proof.

The following is immediate from Theorem 3 by interchanging the places of A and B.

COROLLARY 4. *Suppose that the inverse function of a stricitly increasing function $f(t)$ on $[0,\infty)$ with $f(0)=0$ and $f(\infty)=\infty$ is operator-monotone, and let $A,B>0$. Then the relation $B=f(A)$ occurs if and only if the following two groups of inequalities hold:*

$$f(\langle A^{-1}x,x\rangle^{-1}) \geqslant \langle B^{-1}x,x\rangle^{-1} \quad \textit{and} \quad f(\langle Ax,x\rangle) \leqslant \langle Bx,x\rangle \qquad (\|x\|=1).$$

2. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. Since the only-if part is already in Lemma 2, let us suppose that both ($\sharp$) and ($\flat$) hold.

Notice first that both $f(t)$ and $g(t):=f(t^{-1})^{-1}$ are concave functins. Considering the tangent line to the curve $\{(t,f(t));t>0\}$ in $\mathbb{R}^2$ at the point $(\lambda,f(\lambda))$, we can conclude from the concavity of $f(t)$ that

$$f(t) \leqslant f'(\lambda)t + \{f(\lambda)-\lambda f'(\lambda)\} \qquad (t>0),$$

and further that

$$f(t) = \inf_{\lambda>0}\Big\{f'(\lambda)t + \{f(\lambda)-\lambda f'(\lambda)\}\Big\} \qquad (t>0). \tag{1}$$

Notice here that $f'(\lambda)>0$ and $f(\lambda)-\lambda f'(\lambda)>0$ ($\lambda>0$).

In a similar way, considering the tangent line to the curve $\{(t,g(t)):\ t>0\}$ in $\mathbb{R}^2$ at the point $(\lambda^{-1},g(\lambda^{-1}))$, we can conclude from the concavity of the function $g(t)$ that

$$g(t) \leqslant g'(\lambda^{-1})t + \{g(\lambda^{-1})-\lambda^{-1}g'(\lambda^{-1})\} \qquad (\lambda>0,\ t>0),$$

and further that

$$g(t) = \inf_{\lambda>0} \left\{ g'(\lambda^{-1})t + \{g(\lambda^{-1}) - \lambda^{-1}g'(\lambda^{-1})\} \right\} \qquad (t > 0).$$

Since

$$g'(\lambda^{-1}) = \frac{\lambda^2 f'(\lambda)}{f(\lambda)^2} \quad \text{and} \quad g(\lambda^{-1}) - \lambda^{-1}g'(\lambda^{-1}) = \frac{f(\lambda) - \lambda f'(\lambda)}{f(\lambda)^2} \qquad (\lambda > 0),$$

we can conclude

$$f(t^{-1})^{-1} \; = \; g(t) = \inf_{\lambda>0} \frac{\lambda^2 f'(\lambda)t + \{f(\lambda) - \lambda f'(\lambda)\}}{f(\lambda)^2} \qquad (t > 0). \tag{2}$$

Now it follows from (1) that the first inequalities ($\sharp$) are equivalent to the inequalities:

$$\langle Bx, x\rangle \leqslant f'(\lambda)\langle Ax, x\rangle + \{f(\lambda) - \lambda f'(\lambda)\} \qquad (\lambda > 0, \|x\| = 1). \tag{3}$$

Next let us rewrite these inequalities as operator-inequalities. For simplicity sake, let us represent a scalar multiple of the identity operator I simply by the scalar itself. Further let us write $(\alpha A + \beta I)(\gamma A + \delta I)^{-1}$ as $\frac{\alpha A+\beta}{\gamma A+\delta}$.

Now the inequalities (3), hence ($\sharp$), can be expressed as

$$B \leqslant f'(\lambda)A + \{f(\lambda) - \lambda f'(\lambda)\} \qquad (\lambda > 0), \tag{4}$$

or equivalently

$$B^{-1} \geqslant \frac{1}{f'(\lambda)A + \{f(\lambda) - \lambda f'(\lambda)\}} \qquad (\lambda > 0). \tag{5}$$

In a similar way it follows from (2) that the second inequalities ($\flat$) are equivalent to the following inequalities

$$\langle B^{-1}x, x\rangle \leqslant \frac{\lambda^2 f'(\lambda)\langle A^{-1}x, x\rangle + \{f(\lambda) - \lambda f'(\lambda)\}}{f(\lambda)^2} \qquad (\lambda > 0, \|x\| = 1).$$

As above, this can be converted to operator inequalities

$$B^{-1} \leqslant \frac{\lambda^2 f'(\lambda) + \{f(\lambda) - \lambda f'(\lambda)\}A}{f(\lambda)^2 A} \qquad (\lambda > 0), \tag{6}$$

or equivalently

$$B \geqslant \frac{f(\lambda)^2 A}{\lambda^2 f'(\lambda) + \{f(\lambda) - \lambda f'(\lambda)\}A} \qquad (\lambda > 0). \tag{7}$$

Notice here that since obviously $f(A) \leqslant f(A) \leqslant f(A)$, the operator inequalities (4) – (7) are also valid with $f(A)$ in place of B. Therefore it follows from (4) and (7) that

$$\pm\Big(B - f(A)\Big) \leqslant f'(\lambda)A + \{f(\lambda) - \lambda f'(\lambda)\} - \frac{f(\lambda)^2 A}{\lambda^2 f'(\lambda) + \{f(\lambda) - \lambda f'(\lambda)\}A} \qquad (\lambda > 0).$$

This becomes

$$\pm\Big(B - f(A)\Big) \leqslant \frac{f'(\lambda)\{f(\lambda) - \lambda f'(\lambda)\}}{\lambda^2 f'(\lambda) + \{f(\lambda) - \lambda f'(\lambda)\}A}(A-\lambda)^2 \qquad (\lambda > 0). \tag{8}$$

In a similar way it follows from (5) and (6) that

$$\pm\Big(B^{-1} - f(A)^{-1}\Big) \leqslant \frac{f'(\lambda)\{f(\lambda) - \lambda f'(\lambda)\}}{f(\lambda)^2 A\Big\{f'(\lambda)A + \{f(\lambda) - \lambda f'(\lambda)\}\Big\}}(A-\lambda)^2 \qquad (\lambda > 0). \tag{9}$$

Let λ_{min} (resp. λ_{max}) be the minimum (resp. maximum) spectrum of A and consider the interval $\Delta := [\lambda_{min}, \lambda_{max}]$. Take positive constants γ_1 and γ_2 such that

$$\gamma_1 \geqslant \frac{f'(\lambda)\{f(\lambda) - \lambda f'(\lambda)\}}{\lambda^2 f'(\lambda) + \{f(\lambda) - \lambda f'(\lambda)\}A} \qquad (\lambda \in \Delta)$$

and

$$\gamma_2 \geqslant \frac{f'(\lambda)\{f(\lambda) - \lambda f'(\lambda)\}}{f(\lambda)^2 A\Big\{f'(\lambda)A + \{f(\lambda) - \lambda f'(\lambda)\}\Big\}} \qquad (\lambda \in \Delta).$$

For an orthoprojection P and a postive definite operator X, let us denote the inverse of the operator PXP restricted to the range space of P simply by $(PXP)^{-1}$. Then it is easy to see the formula

$$(PX^{-1}P)^{-1} = PXP - (PXP^{\perp}) \cdot (P^{\perp}XP^{\perp})^{-1} \cdot P^{\perp}XP, \tag{10}$$

where $P^{\perp} := I - P$.

Denote by $\mathscr{O}\mathscr{P}$ the set of orthoprojections and by $\mathscr{O}\mathscr{P}_A$ the subset of those *commuting* with A. Then it follows from (8) and (9) that

$$\|PBP - Pf(A)P\| \leqslant \gamma_1 \|(A-\lambda)P\|^2 \qquad (\lambda \in \Delta,\ P \in \mathscr{O}\mathscr{P}), \tag{11}$$

and

$$\|PB^{-1}P - Pf(A)^{-1}P\| \leqslant \gamma_2 \|(A-\lambda)P\|^2 \qquad (\lambda \in \Delta,\ P \in \mathscr{O}\mathscr{P}). \tag{12}$$

Since

$$(PB^{-1}P)^{-1} - (Pf(A)^{-1}P)^{-1} = (PB^{-1}P)^{-1}\Big(Pf(A)^{-1}P - PB^{-1}P\Big)(Pf(A)^{-1}P)^{-1},$$

and

$$\|(PB^{-1}P)^{-1}\| \leqslant \|B\| \quad \text{and} \quad \|(Pf(A)^{-1}P)^{-1}\| \leqslant \|f(A)\|,$$

with $\gamma_3 := \gamma_2\|B\| \cdot \|f(A)\|$ it follows from (12) that

$$\|(PB^{-1}P)^{-1} - (Pf(A)^{-1}P)^{-1}\| \leqslant \gamma_3\|(A-\lambda)P\|^2 \qquad (\lambda \in \Delta, P \in \mathscr{O}\mathscr{P}). \tag{13}$$

Now since $(Pf(A)^{-1}P)^{-1} = Pf(A)P$ for $P \in \mathscr{O}P_A$, we can conclude from (11) and (13) that

$$\|PBP - (PB^{-1}P)^{-1}\| \leqslant (\gamma_1+\gamma_3)\|(A-\lambda)P\|^2 \qquad (\lambda \in \Delta, P \in \mathscr{O}\mathscr{P}_A). \tag{14}$$

Using the general formula (10) and the inequality $(P^{\perp}BP^{\perp})^{-1} \geqslant \|B\|^{-1}P^{\perp}$, we can see

$$\|P^{\perp}BP\|^2 \leqslant \|B\| \cdot \|PBP - (PB^{-1}P)^{-1}\|.$$

Then it follows from (14) that

$$\|P^{\perp}BP\|^2 \leqslant (\gamma_1+\gamma_3)\|B\| \cdot \|(A-\lambda)P\|^2 \qquad (\lambda \in \Delta,\ P \in \mathscr{O}\mathscr{P}_A). \tag{15}$$

Now divide the interval $\Delta = [\lambda_{min}, \lambda_{max}]$ into n disjoint (half-open) subintervals of equal length, and let P_j the spectral projection of A corresponding to the j-th subinterval. Take λ_j in the j-th subinterval. Then we have, with $l := \lambda_{max} - \lambda_{min}$,

$$\|(A-\lambda_j)P_j\|^2 \leqslant \left(\frac{l}{n}\right)^2 \qquad (j = 1,2,\ldots,n). \tag{16}$$

Then from (11), (15) and (16) we can conclude that there is a constant $\gamma > 0$, not depending on n, such that

$$\|P_jBP_j - P_jf(A)P_j\|,\ \|P_j^{\perp}BP_j\|^2 \leqslant \left(\frac{\gamma}{n}\right)^2 \qquad (j = 1,2,\ldots,n). \tag{17}$$

Since $P_i^{\perp}f(A)P_j = 0$ $(i \neq j)$ and

$$B - f(A) = \sum_{j=1}^{n}\{P_jBP_j - P_jf(A)P_j\} + \sum_{j=1}^{n} P_j^{\perp}BP_j,$$

we have

$$\|B - f(A)\| \leqslant \sum_{j=1}^{n}\|P_jBP_j - P_jf(A)P_j\| + \|\sum_{j=1}^{n} P_j^{\perp}BP_j\|.$$

Since by (17) for any unit vector x

$$\begin{aligned}\|(\sum_{j=1}^{n} P_j^{\perp}BP_j)x\| &\leqslant \sum_{j=1}^{n}\|P_j^{\perp}BP_j\| \cdot \|P_jx\| \\ &\leqslant \sqrt{\sum_{j=1}^{n}\|P_j^{\perp}BP_j\|^2}\sqrt{\sum_{j=1}^{n}\|P_jx\|^2} \leqslant \frac{\gamma}{\sqrt{n}},\end{aligned}$$

we have

$$\|\sum_{j=1}^{n} P_j^{\perp} B P_j\| \leqslant \frac{\gamma}{\sqrt{n}}, \quad \text{hence} \quad \|B - f(A)\| \leqslant \frac{\gamma^2}{n} + \frac{\gamma}{\sqrt{n}} \qquad (n = 1, 2, \ldots).$$

Letting $n \to \infty$, we can conclude $B = f(A)$. This completes the proof of Theorem 3.

We have proved really the following.

THEOREM 5. *Let $f(t)$ be operator-monotone, and $A, B > 0$. Then the relation $B = f(A)$ occurs if and only if the following two groups of operator inequalities hold:*

$$B \leqslant f'(\lambda)A + \{f(\lambda) - \lambda f'(\lambda)\}I \qquad (\lambda > 0),$$

and

$$B \geqslant f(\lambda)^2 A \Big\{\lambda^2 f'(\lambda) I + \{f(\lambda) - \lambda f'(\lambda)\}A\Big\}^{-1} \qquad (\lambda > 0).$$

3. Case of functions $f(t) = t^p$

The following is immediate from Theorem 5.

THEOREM 6. *Let $0 < p \leqslant 1$ and $A, B > 0$. Then the relation $B = A^p$ occurs if and only if*

$$\lambda^p A\{p\lambda I + (1-p)A\}^{-1} \leqslant B \leqslant \lambda^{p-1}\{pA + (1-p)\lambda I\} \qquad (\lambda > 0).$$

COROLLARY 7. *Let $1 < p < \infty$ and $A, B > 0$. Then the relation $B = A^p$ occurs if and only if*

$$p\lambda^{p-1}A - (p-1)\lambda^p I \leqslant B \quad \textit{and} \quad \lambda^{-p}\{p\lambda A^{-1} - (p-1)I\} \leqslant B^{-1} \qquad (\lambda > 0)$$

Proof. The relation $B = A^p$ is equivalent to $A = f(B)$ with an operator-monotone function $f(t) = t^{1/p}$. Therefore by Theorem 6 this occurs if and only if

$$\lambda^{1/p} B\{(1/p)\lambda I + (1 - 1/p)B\}^{-1} \leqslant A \leqslant \lambda^{1/p-1}\{(1/p)B + (1-1/p)\lambda I\} \qquad (\lambda > 0).$$

By replacing λ by λ^p, it is easy to see that these inequalities are written by the two inequalities in the assertion. □

COROLLARY 8. *Let $1 < p \leqslant 2$ and $A, B > 0$. Then the relation $B = A^p$ occurs if and only if*

$$\lambda^{p-1}A^2\{(p-1)\lambda I + (2-p)A\}^{-1} \leqslant B \leqslant \lambda^{p-2}\{(p-1)A^2 + (2-p)\lambda A\} \qquad (\lambda > 0).$$

In fact, $B = A^p$ is equivaent to $A^{-1/2}BA^{-1/2} = f(A)$ with an operator-monotone function $f(t) = t^{p-1}$. Now the assertion follows from Theorem 6.

The results of this section are found also in [5].

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