

SCHATTEN p -NORM INEQUALITIES RELATED TO A CHARACTERIZATION OF INNER PRODUCT SPACES

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Abstract. Let A_1, \dots, A_n be operators acting on a separable complex Hilbert space such that $\sum_{i=1}^n A_i = 0$. It is shown that if A_1, \dots, A_n belong to a Schatten p -class, for some $p > 0$, then

$$2^{p/2} n^{p-1} \sum_{i=1}^n \|A_i\|_p^p \leq \sum_{i,j=1}^n \|A_i \pm A_j\|_p^p$$

for $0 < p \leq 2$, and the reverse inequality holds for $2 \leq p < \infty$. Moreover,

$$\sum_{i,j=1}^n \|A_i \pm A_j\|_p^2 \leq 2n^{2/p} \sum_{i=1}^n \|A_i\|_p^2$$

for $0 < p \leq 2$, and the reverse inequality holds for $2 \leq p < \infty$. These inequalities are related to a characterization of inner product spaces due to E.R. Lorch.

1. Introduction

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space \mathcal{H} . Let $A \in B(\mathcal{H})$ be compact, and let $0 < p < \infty$. The Schatten p -norm (quasi-norm) for $1 \leq p < \infty$ ($0 < p < 1$) is defined by $\|A\|_p = (\text{tr}|A|^p)^{1/p}$, where tr is the usual trace functional and $|A| = (A^*A)^{1/2}$. Clearly $\| |A|^2 \|_{p/2} = \|A\|_p^2$ for $p > 0$. For $p > 0$, the Schatten p -class, denoted by C_p , is defined to be the set of those compact operators A for which $\|A\|_p$ is finite. When $p = 2$, the Schatten p -norm $\|A\|_2 = (\text{tr}|A|^2)^{1/2}$ is called the Hilbert–Schmidt norm of A . For $p > 0$, C_p is a two-sided ideal in $B(\mathcal{H})$. For $1 \leq p < \infty$, C_p is a Banach space; in particular, if $A_1, \dots, A_n \in C_p$, then the triangle inequality for $\|\cdot\|_p$ asserts that

$$\left\| \sum_{i=1}^n A_i \right\|_p \leq \sum_{i=1}^n \|A_i\|_p. \quad (1.1)$$

However, for $0 < p < 1$, the quasi-norm $\|\cdot\|_p$ does not satisfy the triangle inequality. It has been shown in [2] (see, also, [3]) that if $A_1, \dots, A_n \in C_p$ are positive and $0 < p \leq 1$, then

$$\sum_{i=1}^n \|A_i\|_p \leq \left\| \sum_{i=1}^n A_i \right\|_p. \quad (1.2)$$

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For more information on the theory of the Schatten p -classes, the reader is referred to [1, 6].

It is well-known that a normed space X is an inner product space if and only if for every $x, y \in X$, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \quad (1.3)$$

The identity (1.3) is known as the parallelogram law. A version of this fundamental law holds for the Hilbert-Schmidt norm. Generalizations of the parallelogram law for the Schatten p -norms have been given in the form of the celebrated Clarkson inequalities (see [3] and references therein). These inequalities have proven to be very useful in analysis, operator theory and mathematical physics.

Another known characterization of inner product spaces is due to E.R. Lorch [5]. He proved that a normed space X is an inner product space if and only if for a fixed integer $n \geq 3$, and $x_1, \dots, x_n \in X$ with $\sum_{i=1}^n x_i = 0$, we have

$$\sum_{i,j=1}^n \|x_i - x_j\|^2 = 2n \sum_{i=1}^n \|x_i\|^2. \quad (1.4)$$

Since C_2 is a Hilbert space under the inner product $\langle A, B \rangle = \text{tr}(B^*A)$, it follows that if $A_1, \dots, A_n \in C_2$ with $\sum_{i=1}^n A_i = 0$, then

$$\sum_{i,j=1}^n \|A_i - A_j\|_2^2 = 2n \sum_{i=1}^n \|A_i\|_2^2. \quad (1.5)$$

In this paper, we establish operator inequalities for the Schatten p -norms that form natural generalizations of the identity (1.5). Our inequalities presented here seem natural enough and applicable to be widely useful.

2. Main results

To achieve our goal, we need the following lemma (see [2, 4]).

LEMMA 2.1. *Let $A_1, \dots, A_n \in C_p$ for some $p > 0$. If A_1, \dots, A_n are positive, then*

$$(a) \quad n^{p-1} \sum_{i=1}^n \|A_i\|_p^p \leq \left\| \sum_{i=1}^n A_i \right\|_p^p \leq \sum_{i=1}^n \|A_i\|_p^p$$

for $0 < p \leq 1$; and

$$(b) \quad \sum_{i=1}^n \|A_i\|_p^p \leq \left\| \sum_{i=1}^n A_i \right\|_p^p \leq n^{p-1} \sum_{i=1}^n \|A_i\|_p^p$$

for $1 \leq p < \infty$.

A commutative version of Lemma 2.1 can be formulated for scalars as follows: If a_1, \dots, a_n are nonnegative real numbers, then

$$n^{p-1} \sum_{i=1}^n a_i^p \leq \left(\sum_{i=1}^n a_i \right)^p \leq \sum_{i=1}^n a_i^p \tag{2.1}$$

for $0 < p \leq 1$; and

$$\sum_{i=1}^n a_i^p \leq \left(\sum_{i=1}^n a_i \right)^p \leq n^{p-1} \sum_{i=1}^n a_i^p \tag{2.2}$$

for $1 \leq p < \infty$. These inequalities follow, respectively, from the concavity of the function $f(t) = t^p, t \in [0, \infty)$ for $0 < p \leq 1$, and the convexity of the function $f(t) = t^p, t \in [0, \infty)$ for $1 \leq p < \infty$.

Our first main result can be stated as follows. It furnishes a generalization of (1.5).

THEOREM 2.2. *Let $A_1, \dots, A_n, B_1, \dots, B_n \in C_p$, for some $p > 0$, such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then*

$$2^{p/2-1} n^{p-1} \left(\sum_{i=1}^n \|A_i\|_p^p + \sum_{i=1}^n \|B_i\|_p^p \right) \leq \sum_{i,j=1}^n \|A_i \pm B_j\|_p^p$$

for $0 < p \leq 2$; and

$$\sum_{i,j=1}^n \|A_i \pm B_j\|_p^p \leq 2^{p/2-1} n^{p-1} \left(\sum_{i=1}^n \|A_i\|_p^p + \sum_{i=1}^n \|B_i\|_p^p \right)$$

for $2 \leq p < \infty$.

Proof. Let $0 < p \leq 2$. Then

$$\begin{aligned} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^p &= \sum_{i,j=1}^n \left\| |A_i \pm B_j|^2 \right\|_{p/2}^{p/2} \\ &\geq \left\| \sum_{i,j=1}^n |A_i \pm B_j|^2 \right\|_{p/2}^{p/2} \\ &\quad \text{(by the second inequality of Lemma 2.1(a))} \\ &= \left\| \sum_{i,j=1}^n (|A_i|^2 + |B_j|^2 \pm A_i^* B_j \pm B_j^* A_i) \right\|_{p/2}^{p/2} \\ &= \left\| \sum_{i,j=1}^n |A_i|^2 + \sum_{i,j=1}^n |B_j|^2 \right\|_{p/2}^{p/2} \end{aligned}$$

$$\begin{aligned}
 &= n^{p/2} \left\| \sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2 \right\|_{p/2}^{p/2} \\
 &\geq (2n)^{p/2-1} n^{p/2} \left(\sum_{i=1}^n \|A_i\|_p^p + \sum_{i=1}^n \|B_i\|_p^p \right) \\
 &\quad \text{(by the first inequality of Lemma 2.1(a))} \\
 &= 2^{p/2-1} n^{p-1} \left(\sum_{i=1}^n \|A_i\|_p^p + \sum_{i=1}^n \|B_i\|_p^p \right).
 \end{aligned}$$

This proves the first part of the theorem.

Based on Lemma 2.1(b), one can employ an argument similar to that used in the proof of the first part of the theorem to prove the second part.

An application of Theorem 2.2 can be seen in the following result, which is a natural generalization of (1.5).

COROLLARY 2.3. *Let $A_1, \dots, A_n \in C_p$, for some $p > 0$, such that $\sum_{i=1}^n A_i = 0$. Then*

$$2^{p/2} n^{p-1} \sum_{i=1}^n \|A_i\|_p^p \leq \sum_{i,j=1}^n \|A_i \pm A_j\|_p^p$$

for $0 < p \leq 2$; and

$$\sum_{i,j=1}^n \|A_i \pm A_j\|_p^p \leq 2^{p/2} n^{p-1} \sum_{i=1}^n \|A_i\|_p^p$$

for $2 \leq p < \infty$. In particular, (1.5) holds in the case where $p = 2$.

Proof. Since $\sum_{i=1}^n A_i = 0$, we have $(\sum_{i=1}^n A_i)^* (\sum_{i=1}^n A_i) = 0$. Consequently, $\sum_{i,j=1}^n A_i^* A_j = 0$. Utilizing Theorem 2.2 with $B_j = A_j$ ($1 \leq j \leq n$), we obtain the result.

Using the same reasoning as in the proof of Theorem 2.2, one can obtain the following result concerning operators having orthogonal ranges. Recall that ranges of two operators $A, B \in B(\mathcal{H})$ are orthogonal (written $\text{ran}A \perp \text{ran}B$) if and only if $A^*B = 0$.

THEOREM 2.4. *Let $A_1, \dots, A_n \in C_p$, for some $p > 0$, such that $\text{ran}A_i \perp \text{ran}A_j$ for $i \neq j$. Then*

$$(2n \pm 2)^{p/2} n^{p/2-1} \sum_{i=1}^n \|A_i\|_p^p \leq \sum_{i,j=1}^n \|A_i \pm A_j\|_p^p$$

for $0 < p \leq 2$; and

$$\sum_{i,j=1}^n \|A_i \pm A_j\|_p^p \leq (2n \pm 2)^{p/2} n^{p/2-1} \sum_{i=1}^n \|A_i\|_p^p$$

for $2 \leq p < \infty$.

Our second main result, which also leads to a generalization of (1.5), can be stated as follows.

THEOREM 2.5. *Let $A_1, \dots, A_n, B_1, \dots, B_n \in C_p$, for some $p > 0$, such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then*

$$\sum_{i,j=1}^n \|A_i \pm B_j\|_p^2 \leq n^{2/p} \sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^2$$

for $0 < p \leq 2$; and

$$n^{2/p} \sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^2 \leq \sum_{i,j=1}^n \|A_i \pm B_j\|_p^2$$

for $2 \leq p < \infty$.

Proof. Let $0 < p \leq 2$. Then

$$\begin{aligned} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^2 &= \sum_{i,j=1}^n \left\| |A_i \pm B_j|^2 \right\|_{p/2} \\ &\leq \left\| \sum_{i,j=1}^n |A_i \pm B_j|^2 \right\|_{p/2} \quad (\text{by (1.2)}) \\ &= \left\| \sum_{i,j=1}^n (|A_i|^2 + |B_j|^2 \pm A_i^* B_j \pm B_j^* A_i) \right\|_{p/2} \\ &= \left\| \sum_{i,j=1}^n |A_i|^2 + \sum_{i,j=1}^n |B_j|^2 \right\|_{p/2} \\ &= n \left\| \sum_{i=1}^n (|A_i|^2 + |B_i|^2) \right\|_{p/2} \\ &\leq n \left(\sum_{i=1}^n \left\| |A_i|^2 + |B_i|^2 \right\|_{p/2}^{p/2} \right)^{2/p} \\ &\quad (\text{by the second inequality of Lemma 2.1(a)}) \\ &= n \left(\sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^p \right)^{2/p} \\ &\leq n^{2/p} \sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^2. \\ &\quad (\text{by the second inequality of (2.2)}). \end{aligned}$$

This proves the first part of the theorem.

Based on (1.1), the first inequality of Lemma 2.1(b), and the first inequality of (2.1), one can employ an argument similar to that used in the proof of the first part of the theorem to prove the second part.

An application of Theorem 2.5, which is another natural generalization of (1.5), can be seen as follows.

COROLLARY 2.6. *Let $A_1, \dots, A_n \in C_p$, for some $p > 0$, such that $\sum_{i=1}^n A_i = 0$. Then*

$$\sum_{i,j=1}^n \|A_i \pm A_j\|_p^2 \leq 2n^{2/p} \sum_{i=1}^n \|A_i\|_p^2$$

for $0 < p \leq 2$; and

$$2n^{2/p} \sum_{i=1}^n \|A_i\|_p^2 \leq \sum_{i,j=1}^n \|A_i \pm A_j\|_p^2$$

for $2 \leq p < \infty$. In particular, (1.5) holds in the case where $p = 2$.

Using the same reasoning as in the proof of Theorem 2.5, one can obtain the following result concerning operators having orthogonal ranges.

THEOREM 2.7. *Let $A_1, \dots, A_n \in C_p$, for some $p > 0$, such that $\text{ran}A_i \perp \text{ran}A_j$ for $i \neq j$. Then*

$$\sum_{i,j=1}^n \|A_i \pm A_j\|_p^2 \leq 2n^{2/p-1}(n \pm 1) \sum_{i=1}^n \|A_i\|_p^2$$

for $0 < p \leq 2$; and

$$2n^{2/p-1}(n \pm 1) \sum_{i=1}^n \|A_i\|_p^2 \leq \sum_{i,j=1}^n \|A_i \pm A_j\|_p^2$$

for $2 \leq p < \infty$.

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