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A SHARP OSTROWSKI–GRÜSS TYPE INEQUALITY

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Abstract. The main purpose of this paper is to use a variant of Grüss inequality to obtain a sharp Ostrowski-Grüss type inequality for absolutely continuous functions whose derivatives are bounded both above and below almost everywhere. Thus we provide improvement and generalization of some previous results.

1. Introduction

In 1935, G. Grüss (see for example [13]), proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

THEOREM A. *Let* $h,g:[a,b]\to\mathbf{R}$ *be two integrable functions such that* $\phi\leqslant h(t)\leqslant\Phi$ *and* $\gamma\leqslant g(t)\leqslant\Gamma$ *for all* $t\in[a,b]$, *where* ϕ,Φ,γ,Γ *are real numbers. Then*

$$|T(h,g)|\leqslant\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),\tag{1}$$

where

$$T(h,g)=\frac{1}{b-a}\int_a^b h(t)g(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\tag{2}$$

and the inequality is sharp in the sense that the constant $\frac{1}{4}$ *cannot be replaced by a smaller one.*

From then on, (1) is well known in the literature as the Grüss inequality.

A premature Grüss inequality originated from the work of Grüss (see also [13]). It is embodied in the following theorem and was also considered and applied for the first time in the paper [12] by M. Matić, J. Pečarić and N. Ujević in 2000.

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THEOREM B. *Let $h,g:[a,b]\to\mathbf{R}$ be two integrable functions such that $\gamma\leqslant g(t)\leqslant\Gamma$ for all $t\in[a,b]$, where $\gamma,\Gamma\in\mathbf{R}$. Then*

$$|T(h,g)|\leqslant\frac{\Gamma-\gamma}{2}[T(h,h)]^{\frac{1}{2}}, \tag{3}$$

where $T(h,g)$ is as defined in (2).

In 2002, X. L. Cheng and J. Sun [5] have got the following variant of the Grüss inequality:

THEOREM C. *Let $h,g:[a,b]\to\mathbf{R}$ be two integrable functions such that $\gamma\leqslant g(t)\leqslant\Gamma$ for all $t\in[a,b]$, where $\gamma,\Gamma\in\mathbf{R}$. Then*

$$|T(h,g)|\leqslant\frac{\Gamma-\gamma}{2(b-a)}\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(u)\,du\right|dt, \tag{4}$$

where $T(h,g)$ is as defined in (2).

It is not difficult to find that the premature Grüss inequality (3) provides a sharper bound than the Grüss inequality (1) and the variant of Grüss inequality (4) provides a sharper bound than the premature Grüss inequality (3).

In [8], I. Fedotov and S. S. Dragomir have used Theorem A to show that if f has a first derivative on (a,b) and $\gamma\leqslant f'(t)\leqslant\Gamma$ for all $t\in(a,b)$, then

$$\begin{gathered}|(C-A)f(a)+(b-a-B+A)f(x)+(B-C)f(b)-\textstyle\int_a^b f(t)\,dt|\\ \leqslant\frac{1}{4}(\Gamma-\gamma)(M_x-m_x)(b-a),\end{gathered} \tag{5}$$

where $A,B\in\mathbf{R}$, $M_x=\sup\{p_x(t):t\in(a,b)\}$, $m_x=\inf\{p_x(t):t\in(a,b)\}$,

$$C=\frac{1}{2(b-a)}[(x-a)(x-a+2A)-(x-b)(x-b+2B)], \tag{6}$$

p_x is defined by

$$p_x(t)=\begin{cases}t-a+A, & t\in[a,x]\\ t-b+B, & t\in(x,b]\end{cases} \tag{7}$$

and M_x-m_x is expressed in terms of a,b and x and has the following complicated form:

1. If $B-A\leqslant 0$, then $M_x-m_x=(b-a)-(B-A)$.

2. If $B-A>0$, There are three subcases.

(i) If $0\leqslant B-A\leqslant\frac{b-a}{2}$, then

$$M_x-m_x=\begin{cases}-x+b & \text{for }\ a\leqslant x\leqslant a+(B-A),\\ (b-a)-(B-A) & \text{for }\ a+(B-A)<x\leqslant b-(B-A),\\ x-a & \text{for }\ b-(B-A)<x\leqslant b.\end{cases}$$

(ii) If $\frac{b-a}{2}<B-A\leqslant b-a$, then

$$M_x-m_x=\begin{cases}-x+b & \text{for }\ a\leqslant x\leqslant b-(B-A),\\ B-A & \text{for }\ b-(B-A)<x\leqslant a+(B-A),\\ x-a & \text{for }\ a+(B-A)<x\leqslant b.\end{cases}$$

(iii) If $B-A>b-a$, then $M_x-m_x=B-a$.

In [11], C. E. M. Pearce et al have used Theorem B to prove that

$$\begin{aligned}&|(C-A)f(a)+(b-a-B+A)f(x)+(B-C)f(b)-\int_a^b f(t)\,dt|\\&\leqslant \tfrac{\Gamma-\gamma}{2}\Big(\tfrac{B^3-(x-b+B)^3+(x-a+A)^3-A^3}{3(b-a)}-\Big(\tfrac{B^2-(x-b+B)^2+(x-a+A)^2-A^2}{2(b-a)}\Big)^2\Big)^{\frac{1}{2}}(b-a)\end{aligned}$$

under the same conditions.

In this paper, we will use Theorem C to give a sharp Ostrowski-Grüss type inequality for absolutely continuous functions whose derivatives are bounded above and below almost everywhere. Some sharp integral inequalities of midpoint, trapezoid and Simpson type are obtained or recaptured as particular cases.

2. The results

THEOREM. *Let $f:[a,b]\to\mathbf{R}$ be a function which is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma,\Gamma\in\mathbf{R}$ such that $\gamma\leqslant f'(t)\leqslant\Gamma$ a.e. on $[a,b]$. Then for all $x\in[a,b]$ we have*

$$\begin{aligned}&|(C-A)f(a)+(b-a-B+A)f(x)+(B-C)f(b)-\int_a^b f(t)\,dt|\\&\leqslant \tfrac{\Gamma-\gamma}{2}I(a,b,A,B,x),\end{aligned}\tag{8}$$

where $A,B\in\mathbf{R}$, C is as defined in (6), and $I(a,b,A,B,x)$ has the following complicated form.

1. If $B-A\leqslant 0$, then

$$I(a,b,A,B,x)=\begin{cases}[\frac{b-a}{2}-\frac{B-A}{b-a}(x-a)]^2, & a\leqslant x\leqslant\xi,\\ (\frac{1}{2}-\frac{B-A}{b-a})[(b-a)^2-(x-a)^2-(b-x)^2], & \xi<x<\eta,\\ [\frac{b-a}{2}-\frac{B-A}{b-a}(b-x)]^2, & \eta\leqslant x\leqslant b,\end{cases}\tag{9}$$

with

$$\xi=a-\frac{(b-a)^2}{2[(B-A)-(b-a)]},\quad \eta=b+\frac{(b-a)^2}{2[(B-A)-(b-a)]},\tag{10}$$

and $a<\xi<\eta<b$.

2. If $B-A>0$, there are three subcases.

(i) If $0<B-A\leqslant\frac{b-a}{2}$, then

$$I(a,b,A,B,x)=\begin{cases}[\frac{b-a}{2}-\frac{B-A}{b-a}(x-a)]^2, & a\leqslant x\leqslant\eta,\\ [\frac{1}{4}+(\frac{1}{2}-\frac{B-A}{b-a})^2][(x-a)^2+(b-x)^2], & \eta<x<\xi,\\ [\frac{b-a}{2}-\frac{B-A}{b-a}(b-x)]^2, & \xi\leqslant x\leqslant b,\end{cases}\tag{11}$$

with ξ,η as defined in (10) and $a<\eta<\xi<b$.

(ii) If $\frac{b-a}{2} < B-A \leqslant b-a$*, then*

$$I(a,b,A,B,x) = \begin{cases} [(\frac{a+b}{2}-x)+\frac{B-A}{b-a}(x-a)]^2, & a \leqslant x \leqslant \zeta, \\ [\frac{1}{4}+(\frac{B-A}{b-a}-\frac{1}{2})^2][(x-a)^2+(b-x)^2], & \zeta < x < \theta, \\ [(x-\frac{a+b}{2})+\frac{B-A}{b-a}(b-x)]^2, & \theta \leqslant x \leqslant b, \end{cases} \tag{12}$$

with

$$\zeta = b - \frac{(b-a)^2}{2(B-A)}, \quad \theta = a + \frac{(b-a)^2}{2(B-A)}, \tag{13}$$

and $a < \zeta < \theta < b$.

(iii) If $B-A > b-a$*, then*

$$I(a,b,A,B,x) = \begin{cases} [(\frac{a+b}{2}-x)+\frac{B-A}{b-a}(x-a)]^2, & a \leqslant x \leqslant \theta, \\ (\frac{B-A}{b-a}-\frac{1}{2})[(b-a)^2-(x-a)^2-(b-x)^2], & \theta < x < \zeta, \\ [(x-\frac{a+b}{2})+\frac{B-A}{b-a}(b-x)]^2, & \zeta \leqslant x \leqslant b, \end{cases} \tag{14}$$

with ζ, θ *as defined in (13) and* $a < \theta < \zeta < b$.

Proof. Integrating by parts produces the identity

$$\int_a^b p_x(t)f'(t)\,dt = Bf(b) - Af(a) - \int_a^b f(t)\,dt - [(B-A)-(b-a)]f(x) \tag{15}$$

where $p_x(t)$ is as defined in (7). Moreover,

$$\begin{aligned} \tfrac{1}{b-a}\textstyle\int_a^b p_x(t)\,dt &= \tfrac{(x-a)(x-a+2A)-(x-b)(x-b+2B)}{2(b-a)} \\ &= (1-\tfrac{B-A}{b-a})x + \tfrac{bB-aA}{b-a} - \tfrac{a+b}{2}. \end{aligned} \tag{16}$$

Applying the variant of Grüss inequality (4) by associating $g(t)$ with $f'(t)$ and $h(t)$ with $p_x(t)$ and multiply through by $(b-a)$ gives

$$\begin{aligned} &|\textstyle\int_a^b p_x(t)f'(t)\,dt - \tfrac{1}{b-a}\int_a^b p_x(t)\,dt \int_a^b f'(t)\,dt| \\ &\leqslant \tfrac{\Gamma-\gamma}{2}\textstyle\int_a^b |p_x(t) - \tfrac{1}{b-a}\int_a^b p_x(s)\,ds|\,dt. \end{aligned}$$

Then for any fixed $x \in [a,b]$ we can derive from (15), (16), (6) and (7) that

$$\begin{aligned} &|(C-A)f(a) + (b-a-B+A)f(x) + (B-C)f(b) - \textstyle\int_a^b f(t)\,dt| \\ &\leqslant \tfrac{\Gamma-\gamma}{2} I(a,b,A,B,x), \end{aligned} \tag{17}$$

where

$$\begin{aligned} I(a,b,A,B,x) &= \textstyle\int_a^x |t-a+A-(1-\frac{B-A}{b-a})x - \frac{bB-aA}{b-a} + \frac{a+b}{2}|\,dt \\ &\quad + \textstyle\int_x^b |t-b+B-(1-\frac{B-A}{b-a})x - \frac{bB-aA}{b-a} + \frac{a+b}{2}|\,dt \\ &= \textstyle\int_a^x |t-[\frac{B-A}{b-a}(b-x)+x-\frac{b-a}{2}]|\,dt + \int_x^b |t-[\frac{b-a}{2}+x-\frac{B-A}{b-a}(x-a)]|\,dt. \end{aligned} \tag{18}$$

The last two integrals can be calculated as follows:

For brevity, we put

$$q_1(t) := t - [\tfrac{B-A}{b-a}(b-x) + x - \tfrac{b-a}{2}],\ t \in [a,x],$$
$$q_2(t) := t - [\tfrac{b-a}{2} + x - \tfrac{B-A}{b-a}(x-a)],\ t \in [x,b]$$

and denote $t_1 = \frac{B-A}{b-a}(b-x) + x - \frac{b-a}{2}$, $t_2 = \frac{b-a}{2} + x - \frac{B-A}{b-a}(x-a)$.

It is clear that both $q_1(t)$ and $q_2(t)$ are strictly increasing on $[a,x]$ and $[x,b]$ respectively. Moreover, we have

$$q_1(a) = \tfrac{a+b}{2} - x - \tfrac{B-A}{b-a}(b-x), \qquad q_1(x) = \tfrac{b-a}{2} - \tfrac{B-A}{b-a}(b-x);$$
$$q_2(x) = \tfrac{B-A}{b-a}(x-a) - \tfrac{b-a}{2}, \qquad q_2(b) = \tfrac{a+b}{2} - x + \tfrac{B-A}{b-a}(x-a).$$

We further denote

$$\xi = a - \tfrac{(b-a)^2}{2[(B-A)-(b-a)]}, \qquad \eta = b + \tfrac{(b-a)^2}{2[(B-A)-(b-a)]};$$
$$\zeta = b - \tfrac{(b-a)^2}{2(B-A)}, \qquad \theta = a + \tfrac{(b-a)^2}{2(B-A)}.$$

For $B-A \leqslant 0$, it is clear that $q_1(x) > 0$, $q_2(x) < 0$ and $a < \xi \leqslant \eta < b$. In case $x \in [a,\xi]$, we see $q_1(a) \geqslant 0$ which implies that $q_1(t) \geqslant 0$ for $t \in [a,x]$, and $q_2(b) \geqslant 0$ with $t_2 \in (x,b)$ such that $q_2(t_2) = 0$. So, we have

$$\begin{aligned}\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt &= \int_a^x (t-t_1)\,dt + \int_x^{t_2} (t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\ &= [\tfrac{b-a}{2} - \tfrac{B-A}{b-a}(x-a)]^2.\end{aligned} \tag{19}$$

In case $x \in (\xi,\eta)$, we see $q_1(a) > 0$ which implies that $q_1(t) > 0$ for $t \in [a,x]$, and $q_2(b) < 0$ which implies that $q_2(t) < 0$ for $t \in [x,b]$. So, we have

$$\begin{aligned}\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt &= \int_a^x (t-t_1)\,dt + \int_x^b (t_2-t)\,dt \\ &= \tfrac{(b-a)^2}{4} - \tfrac{(B-A)(b-a)}{2} - [1 - \tfrac{2(B-A)}{b-a}](x - \tfrac{a+b}{2})^2 \\ &= (\tfrac{1}{2} - \tfrac{B-A}{b-a})[(b-a)^2 - (x-a)^2 - (b-x)^2].\end{aligned} \tag{20}$$

In case $x \in [\eta,b]$, we see $q_1(a) \leqslant 0$ with $t_1 \in (a,x]$ such that $q_1(t_1) = 0$, and $q_2(b) \leqslant 0$ which implies that $q_2(t) \leqslant 0$ for $t \in [x,b]$. So, we have

$$\begin{aligned}\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt &= \int_a^{t_1} (t_1-t)\,dt + \int_{t_1}^x (t-t_1)\,dt + \int_x^b (t_2-t)\,dt \\ &= [\tfrac{b-a}{2} - \tfrac{B-A}{b-a}(b-x)]^2.\end{aligned} \tag{21}$$

For $0 < B-A \leqslant \frac{b-a}{2}$, it is clear that $q_1(x) > 0$, $q_2(x) < 0$ and $a \leqslant \eta < \xi \leqslant b$. In case $x \in [a,\eta]$, we see $q_1(a) \geqslant 0$ which implies that $q_1(t) \geqslant 0$ for $t \in [a,x]$, and $q_2(b) > 0$ with $t_2 \in (x,b)$ such that $q_2(t_2) = 0$. So, we have

$$\begin{aligned}\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt &= \int_a^x (t-t_1)\,dt + \int_x^{t_2} (t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\ &= [\tfrac{b-a}{2} - \tfrac{B-A}{b-a}(x-a)]^2.\end{aligned} \tag{22}$$

In case $x \in (\eta, \xi)$, we see $q_1(a) < 0$ with $t_1 \in (a,x)$ such that $q_1(t_1) = 0$, and $q_2(b) > 0$ with $t_2 \in (x,b)$ such that $q_2(t_2) = 0$. So, we have

$$\begin{aligned}
&\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt \\
&\quad = \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^x (t-t_1)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\
&\quad = \tfrac{(b-a)^2}{4} - \tfrac{(B-A)(b-a)}{2} + [1-\tfrac{2(B-A)}{b-a}](x-\tfrac{a+b}{2})^2 + (\tfrac{B-A}{b-a})^2[(x-a)^2+(b-x)^2] \\
&\quad = [\tfrac14 + (\tfrac12 - \tfrac{B-A}{b-a})^2][(x-a)^2+(b-x)^2].
\end{aligned} \tag{23}$$

In case $x \in [\xi, b]$, we see $q_1(a) < 0$ with $t_1 \in (a,x)$ such that $q_1(t_1) = 0$, and $q_2(b) \leqslant 0$ which implies that $q_2(t) \leqslant 0$ for $t \in [x,b]$. So, we have

$$\begin{aligned}
\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt &= \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^x (t-t_1)\,dt + \int_x^b (t_2-t)\,dt \\
&= [\tfrac{b-a}{2} - \tfrac{B-A}{b-a}(b-x)]^2.
\end{aligned} \tag{24}$$

For $\frac{b-a}{2} < B-A \leqslant b-a$, it is clear that $q_1(a) < 0$, $q_2(b) > 0$ and $a < \zeta \leqslant \theta < b$. In case $x \in [a, \zeta]$, we see $q_1(x) \leqslant 0$ which implies that $q_1(t) \leqslant 0$ for $t \in [a,x]$, and $q_2(x) < 0$ with $t_2 \in (x,b)$ such that $q_2(t_2) = 0$. So, we have

$$\begin{aligned}
&\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt \\
&\quad = \int_a^x (t_1-t)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\
&\quad = (B-A)(\tfrac{a+b}{2} - x) - [\tfrac{2(B-A)}{b-a} - 1](x-\tfrac{a+b}{2})^2 + (\tfrac{B-A}{b-a})^2(x-a)^2 \\
&\quad = [(\tfrac{a+b}{2} - x) + \tfrac{B-A}{b-a}(x-a)]^2.
\end{aligned} \tag{25}$$

In case $x \in (\zeta, \theta)$, we see $q_1(x) > 0$ with $t_1 \in (a,x)$ such that $q_1(t_1) = 0$, and $q_2(x) < 0$ with $t_2 \in (x,b)$ such that $q_2(t_2) = 0$. So, we have

$$\begin{aligned}
&\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt \\
&\quad = \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^x (t-t_1)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\
&\quad = \tfrac{(b-a)^2}{4} - \tfrac{(B-A)(b-a)}{2} - [\tfrac{2(B-A)}{b-a} - 1](x-\tfrac{a+b}{2})^2 + (\tfrac{B-A}{b-a})^2[(x-a)^2+(b-x)^2] \\
&\quad = [\tfrac14 + (\tfrac{B-A}{b-a} - \tfrac12)^2][(x-a)^2+(b-x)^2].
\end{aligned} \tag{26}$$

In case $x \in [\theta, b]$, we see $q_1(x) > 0$ with $t_1 \in (a,x)$ such that $q_1(t_1) = 0$, and $q_2(x) \geqslant 0$ which implies that $q_2(t) \geqslant 0$ for $t \in [x,b]$. So, we have

$$\begin{aligned}
&\int_a^x |t-t_1|\,dt + \int_x^b |t-t_2|\,dt \\
&\quad = \int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^x (t-t_1)\,dt + \int_x^b (t-t_2)\,dt \\
&\quad = (B-A)(x-\tfrac{a+b}{2}) - [\tfrac{2(B-A)}{b-a} - 1](x-\tfrac{a+b}{2})^2 + (\tfrac{B-A}{b-a})^2(b-x)^2 \\
&\quad = [(x-\tfrac{a+b}{2}) + \tfrac{B-A}{b-a}(b-x)]^2.
\end{aligned} \tag{27}$$

For $B-A>b-a$, it is clear that $q_1(a)<0$, $q_2(b)>0$ and $a<\theta<\zeta<b$. In case $x\in[a,\theta]$, we see $q_1(x)<0$ which implies that $q_1(t)<0$ for $t\in[a,x]$, and $q_2(x)\leqslant 0$ with $t_2\in(x,b)$ such that $q_2(t_2)=0$. So, we have

$$\begin{aligned}&\textstyle\int_a^x|t-t_1|\,dt+\int_x^b|t-t_2|\,dt\\&\quad=\textstyle\int_a^x(t_1-t)\,dt+\int_x^{t_2}(t_2-t)\,dt+\int_{t_2}^b(t-t_2)\,dt\\&\quad=(B-A)(\tfrac{a+b}{2}-x)-[\tfrac{2(B-A)}{b-a}-1](x-\tfrac{a+b}{2})^2+(\tfrac{B-A}{b-a})^2(x-a)^2\\&\quad=[(\tfrac{a+b}{2}-x)+\tfrac{B-A}{b-a}(x-a)]^2.\end{aligned}\tag{28}$$

In case $x\in(\theta,\zeta)$, we see $q_1(x)<0$ which implies that $q_1(t)<0$ for $t\in[a,x]$, and $q_2(x)>0$ which implies that $q_2(t)>0$ for $t\in[x,b]$. So, we have

$$\begin{aligned}&\textstyle\int_a^x|t-t_1|\,dt+\int_x^b|t-t_2|\,dt\\&\quad=\textstyle\int_a^x(t_1-t)\,dt+\int_x^b(t-t_2)\,dt\\&\quad=-\tfrac{(b-a)^2}{4}+\tfrac{(B-A)(b-a)}{2}-[\tfrac{2(B-A)}{b-a}-1](x-\tfrac{a+b}{2})^2\\&\quad=(\tfrac{B-A}{b-a}-\tfrac{1}{2})[(b-a)^2-(x-a)^2-(b-x)^2].\end{aligned}\tag{29}$$

In case $x\in[\zeta,b]$, we see $q_1(x)\geqslant 0$ with $t_1\in(a,x)$ such that $q_1(t_1)=0$, and $q_2(x)>0$ which implies that $q_2(t)>0$ for $t\in[x,b]$. So, we have

$$\begin{aligned}&\textstyle\int_a^x|t-t_1|\,dt+\int_x^b|t-t_2|\,dt\\&\quad=\textstyle\int_a^{t_1}(t_1-t)\,dt+\int_{t_1}^x(t-t_1)\,dt+\int_x^b(t-t_2)\,dt\\&\quad=(B-A)(x-\tfrac{a+b}{2})-[\tfrac{2(B-A)}{b-a}-1](x-\tfrac{a+b}{2})^2+(\tfrac{B-A}{b-a})^2(b-x)^2\\&\quad=[(x-\tfrac{a+b}{2})+\tfrac{B-A}{b-a}(b-x)]^2.\end{aligned}\tag{30}$$

Consequently, the inequality (8) with (9)–(14) follows from (17)–(30).

The proof is completed. □

REMARK. It is not difficult to prove that the inequality (8) with (9)–(14) is sharp in the sense that we can construct the function f to attain the equality in (8) with (9)–(14). Indeed, if $B-A\leqslant 0$ then we may choose f such that

$$f(t)=\begin{cases}\Gamma(t-a), & a\leqslant t<x,\\ \gamma(t-x)+(x-a)\Gamma, & x\leqslant t<t_2,\\ \Gamma(t-t_2+x-a)+(t_2-x)\gamma, & t_2\leqslant t\leqslant b\end{cases}$$

for any $x\in[a,\xi]$, and

$$f(t)=\begin{cases}\Gamma(t-a), & a\leqslant t<x,\\ \gamma(t-x)+(x-a)\Gamma, & x\leqslant t\leqslant b\end{cases}$$

for any $x \in (\xi, \eta)$, and

$$f(t) = \begin{cases} \gamma(t-a), & a \leqslant t < t_1, \\ \Gamma(t-t_1) + (t_1-a)\gamma, & t_1 \leqslant t < x, \\ \gamma(t-x+t_1-a) + (x-t_1)\Gamma, & x \leqslant t \leqslant b \end{cases}$$

for any $x \in [\eta, b]$. If $0 < B - A \leqslant \frac{b-a}{2}$ then we may choose f such that

$$f(t) = \begin{cases} \Gamma(t-a), & a \leqslant t < x, \\ \gamma(t-x) + (x-a)\Gamma, & x \leqslant t < t_2, \\ \Gamma(t-t_2+x-a) + (t_2-x)\gamma, & t_2 \leqslant t \leqslant b \end{cases}$$

for any $x \in [a, \eta]$, and

$$f(t) = \begin{cases} \gamma(t-a), & a \leqslant t < t_1, \\ \Gamma(t-t_1) + (t_1-a)\gamma, & t_1 \leqslant t < x, \\ \gamma(t-x+t_1-a) + (x-t_1)\Gamma, & x \leqslant t < t_2, \\ \Gamma(t-t_2+x-t_1) + (t_2-x+t_1-a)\gamma, & t_2 \leqslant t \leqslant b \end{cases}$$

for any $x \in (\eta, \xi)$, and

$$f(t) = \begin{cases} \gamma(t-a), & a \leqslant t < t_1, \\ \Gamma(t-t_1) + (t_1-a)\gamma, & t_1 \leqslant t < x, \\ \gamma(t-x+t_1-a) + (x-t_1)\Gamma, & x \leqslant t \leqslant b \end{cases}$$

for any $x \in [\xi, b]$. If $\frac{b-a}{2} < B - A \leqslant b - a$ then we may choose f such that

$$f(t) = \begin{cases} \gamma(t-a), & a \leqslant t < t_2, \\ \Gamma(t-t_2) + (t_2-a)\gamma, & t_2 \leqslant t \leqslant b \end{cases}$$

for any $x \in [a, \zeta]$, and

$$f(t) = \begin{cases} \gamma(t-a), & a \leqslant t < t_1, \\ \Gamma(t-t_1) + (t_1-a)\gamma, & t_1 \leqslant t < x, \\ \gamma(t-x+t_1-a) + (x-t_1)\Gamma, & x \leqslant t < t_2, \\ \Gamma(t-t_2+x-t_1) + (t_2-x+t_1-a)\gamma, & t_2 \leqslant t \leqslant b \end{cases}$$

for any $x \in (\zeta, \theta)$, and

$$f(t) = \begin{cases} \gamma(t-a), & a \leqslant t < t_1, \\ \Gamma(t-t_1) + (t_1-a)\gamma, & t_1 \leqslant t \leqslant b \end{cases}$$

for any $x \in [\theta, b]$. If $B - A > b - a$ then we may choose f such that

$$f(t) = \begin{cases} \gamma(t-a), & a \leqslant t < t_2, \\ \Gamma(t-t_2) + (t_2-a)\gamma, & t_2 \leqslant t \leqslant b \end{cases}$$

for any $x \in [a,\theta]$, and

$$f(t)=\begin{cases}\gamma(t-a), & a\leqslant t<x,\\ \Gamma(t-x)+(x-a)\gamma, & x\leqslant t\leqslant b\end{cases}$$

for any $x \in (\theta,\zeta)$, and

$$f(t)=\begin{cases}\gamma(t-a), & a\leqslant t<t_1,\\ \Gamma(t-t_1)+(t_1-a)\gamma, & t_1\leqslant t\leqslant b\end{cases}$$

for any $x \in [\zeta,b]$.

It is clear that the above all $f(t)$ are absolutely continuous on $[a,b]$.

COROLLARY 1. *Let the assumptions of Theorem hold. Then for all $x \in [a,b]$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\leqslant\frac{(\Gamma-\gamma)(b-a)}{8}. \tag{31}$$

Proof. Letting $A=B=0$ in (8) readily produces the result (31) from (9) on noting that $I(a,b,0,0,x)=\frac{(b-a)^2}{4}$.

It should be noted that (31) is a sharp perturbed Ostrowski inequality with a uniform bound independent of x which provides an improvement of the main result in [7], and in particular, if we choose in (31), $x=\frac{a+b}{2}$, we get a sharp midpoint inequality

$$\left|\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right|\leqslant\frac{(\Gamma-\gamma)(b-a)^2}{8},$$

which has been given in [9] and improves the results in [3] and [11]. □

COROLLARY 2. *Let the assumptions of Theorem hold. Then for all $x \in [a,b]$, we have*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-\tfrac{1}{2}[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)]\right|\\ &\qquad\leqslant\tfrac{\Gamma-\gamma}{8}[(x-a)^2+(b-x)^2].\end{aligned} \tag{32}$$

Proof. Letting $B-A=\frac{b-a}{2}$ in (8) readily produces the result (32) from (11) on noting that $C-A=\frac{x-a}{2}$, $B-C=\frac{b-x}{2}$, $\eta=a$, $\xi=b$, and then for all $x\in[a,b]$ follows $I(a,b,A,B,x)=\frac{1}{4}[(x-a)^2+(b-x)^2]$.

It should be noted that we can find the inequality (32) in [4] and [15] with different proofs. However, we here have pointed out that the inequality (32) is sharp in the sense that we can find f such that the equality in (32) holds. Taking $x=\frac{a+b}{2}$ in (32) produces a sharp simple three point inequality as

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\leqslant\frac{(\Gamma-\gamma)(b-a)^2}{16}. \tag{33}$$

□

COROLLARY 3. *Let the assumptions of Theorem hold. Then we have*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \leqslant \frac{5(\Gamma-\gamma)(b-a)^2}{72}. \tag{34}$$

Proof. Letting $B-A=\frac{b-a}{3}$ and $x=\frac{a+b}{2}$ in (8) readily produces the result (34) from (11) on noting that $I(a,b,A,B,\frac{a+b}{2})=\frac{5(b-a)^2}{36}$.

It is interesting to note that from (33) and (34) we can conclude that an average of the midpoint quadrature rule and trapezoidal quadrature rule has a better estimation of error than the well-known Simpson quadrature rule when we estimate the error in terms of the first derivative f' of integrand f. The same conclusion can also be found in the previous papers [1], [6] and [14]. However, we here provide a generalization of the result in [6], and since both (33) and (34) are sharp, our assertion is more convincing than that stated in [1] and [14]. □

COROLLARY 4. *Let the assumptions of Theorem hold. Then we have*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)]\right| \leqslant \frac{(\Gamma-\gamma)(b-a)^2}{8}. \tag{35}$$

Proof. Letting $B-A=b-a$ and $x=\frac{a+b}{2}$ in (8) readily produces the result (35) from (12) on noting that $I(a,b,A,B,\frac{a+b}{2})=\frac{(b-a)^2}{4}$.

Thus we recapture the sharp trapezoid inequality which has been given in [10] and improves the result in [2]. □

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