Mathematical
Inequalities
& Applications
Volume 13, Number 2 (2010), 271–288

STRIPS AND HYPERBOLAS FOR ZEROS OF POLYNOMIALS IN TERMS OF THEIR HERMITE EXPANSION

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(*Communicated by A. Laforgia*)

Abstract. We obtain estimates for strips and hyperbolas containing all the zeros of a polynomial given by its Hermite expansion, by combining some ideas of Turán and the classical methods of Fujiwara, Ballieu, Cowling and Thron.

1. Motivation and summary of results

As Turán [17] pointed out, in the study of the distribution of zeros of certain entire functions whose zeros lie in a horizontal strip, it might be advantageous to consider their expansion with respect to the set of Hermite polynomials, $H_n(z)$, $n = 0,1,2,\ldots$, where

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}(e^{-z^2}).$$

For polynomials having all the roots in a strip, Turán established in [18] and [19] the following results.

THEOREM A. *Suppose a complex polynomial $f(z)$ has the Hermite expansion $f(z) = \sum_{i=0}^{n} b_i H_i(z)$. Then all the roots of f lie in the strip*

$$|\mathrm{Im}\, z| \leqslant \frac{1}{2}\left(1 + \max_{0\leqslant i\leqslant n-1} \frac{|b_i|}{|b_n|}\right).$$

THEOREM B. *Suppose a complex polynomial $f(z)$ has the Hermite expansion $f(z) = \sum_{i=0}^{n} b_i H_i(z)$. Then all the roots of f lie in the strip*

$$|\mathrm{Im}\, z| \leqslant \frac{1}{2}\left(\sqrt[n]{\frac{|b_0|}{|b_n|}} + \sqrt[n-1]{\frac{|b_1|}{|b_n|}} + \ldots + \sqrt{\frac{|b_{n-2}|}{|b_n|}} + \frac{|b_{n-1}|}{|b_n|}\right).$$

Mathematics subject classification (2010): 12D10, 26C10, 11R09.
Keywords and phrases: Root location, Hermite polynomial, strip.
This work was partially supported by the CEEX Programme of the Romanian Ministry of Education and Research, contract 2-CEx06-11-20/2006.

Paper MIA-13-22

THEOREM C. *Suppose a complex polynomial $f(z)$ has the Hermite expansion $f(z)=\sum_{i=0}^{n} b_{2i}H_{2i}(z)$. Then all the roots of f lie in the strip*

$$|\operatorname{Im} z| \leqslant \frac{1}{2}\left(1+\frac{5}{\sqrt{2n-1}}\cdot \max_{0\leqslant i\leqslant n-1}\frac{|b_{2i}|}{|b_{2n}|}\right).$$

Here and henceforth it is supposed that the coefficient b_i of maximal index is nonzero. These results give no information on the location of the real parts of the roots. It is possible to obtain such an information by applying the previous theorems to the polynomial $g(z)=f(iz)$, but there is no simple uniform formula relating the coefficients of the Hermite expansions of f and g. However, Turán [19] has obtained an upper bound for the product of the real part and the imaginary part of the roots of an even polynomial.

THEOREM D. *Suppose a complex polynomial $f(z)$ has the Hermite expansion $f(z)=\sum_{i=0}^{n} b_{2i}H_{2i}(z)$. Then all the roots of f lie in the hyperbola*

$$|\operatorname{Im} z\cdot \operatorname{Re} z| \leqslant \frac{5}{4}\left(1+\max_{0\leqslant i\leqslant n-1}\frac{|b_{2i}|}{|b_{2n}|}\right).$$

For a study of the distribution of zeros of certain classes of entire functions in terms of their Hermite expansion, and for generalisations of Turán's results in [17], the reader is referred to the work of D. Bleecker and G. Csordas [2]. For estimates of the zeros of polynomials in terms of their expansion with respect to other families of orthogonal polynomials, we refer the reader to Specht [12]–[15] and Giroux [6].

The aim of this paper is to obtain further estimates for strips and hyperbolas containing all the zeros of a polynomial given by its Hermite expansion, by using the ideas of Turán and adapting the classical methods of Fujiwara, Ballieu, Cowling and Thron. Before stating our results, we remind the classical estimates that we will adapt by considering strips and hyperbolas instead of disks containing all the zeros of a complex polynomial. The first estimate we will adapt is a well-known theorem of M. Fujiwara [5] on the location of the roots of complex polynomials.

THEOREM E. *Let $P(z)=\sum_{i=0}^{n} a_i z^{d_i}\in\mathbb{C}[z]$, with $0=d_0<d_1<\cdots<d_n$ and $a_0a_1\ldots a_n\neq 0$. Let also $\mu_0,\ldots,\mu_{n-1}$ be positive real numbers such that $\frac{1}{\mu_0}+\ldots+\frac{1}{\mu_{n-1}}\leqslant 1$. Then all the roots of P are contained in the disk*

$$|z| \leqslant \max_{0\leqslant j\leqslant n-1}\left(\mu_j\frac{|a_j|}{|a_n|}\right)^{1/(d_n-d_j)}.$$

Another classical result that we will use, and which depends too on a set of parameters, is the following theorem of Ballieu (see [1], [10]).

THEOREM F. *Let* $P(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$ *with* $a_0 a_n \neq 0$ *and let* $\mu_0 = 0$ *and* $\mu_1, \ldots, \mu_n$ *be arbitrary positive constants. Then all the roots of* P *lie in the disc*

$$|z| \leqslant \max_{0 \leqslant j \leqslant n-1} \left\{ \frac{\mu_j}{\mu_{j+1}} + \frac{\mu_n}{\mu_{j+1}} \cdot \frac{|a_j|}{|a_n|} \right\}.$$

The third result that we will adapt in order to find regions containing all the zeros of a complex polynomial, is the following.

THEOREM G. *Let* $P(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$ *with* $a_0 a_n \neq 0$ *and let* $\mu_1, \ldots, \mu_n$ *be arbitrary positive constants. Then all the roots of* f *lie in the disk*

$$|z| \leqslant \max\left\{ \frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_2}, \ldots, \frac{\mu_n}{\mu_{n-1}}, \sum_{j=1}^{n} \frac{\mu_j}{\mu_n} \cdot \frac{|a_{j-1}|}{|a_n|} \right\}.$$

For the proof of this estimate one may apply to the companion matrix of the polynomial $\bar{P}(X) = \frac{1}{a_n} P(X)$ the following classical result [11].

If $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ *is an arbitrary set of positive numbers, then all the characteristic roots of the* $n \times n$ *complex matrix* $\mathscr{M} = (a_{ij})$ *lie in the disk*

$$|z| \leqslant \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} \frac{\mu_j}{\mu_i} |a_{ij}|.$$

The last result we will use, depending too on a set of parameters, is due to Cowling and Thron ([3], [4]).

THEOREM H. *Let* $P(z) = a_0 z^{d_0} + a_1 z^{d_1} + \cdots + a_n z^{d_n} \in \mathbb{C}[z]$ *with all* $a_j \neq 0$ *and* $0 = d_0 < d_1 < \cdots < d_n$*. Let also* $\mu_0 = 0$*,* $\mu_n = 1$ *and* $\mu_1, \ldots, \mu_{n-1}$ *be arbitrary positive constants. Then all the roots of* P *lie in the disc*

$$|z| \leqslant \max_{1 \leqslant j \leqslant n} \left(\frac{(1 + \mu_{j-1})|a_{j-1}|}{\mu_j |a_j|} \right)^{1/(d_j - d_{j-1})}.$$

We note here that the estimate in the case when $\mu_1 = \ldots = \mu_n = 1$ was established earlier by Kojima (see [8], [9]).

In this paper we will prove the following analogous results for strips containing all the zeros of a complex polynomial.

THEOREM 1.1. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z) = \sum_{i=0}^{n} b_i H_i(z)$ *and let* $\mu_1, \ldots, \mu_n$ *be positive real numbers such that* $\mu_1 + \cdots + \mu_n \leqslant 1$*. Then all the roots of* f *lie in the strip*

$$|\operatorname{Im} z| \leqslant \frac{1}{2} \max_{1 \leqslant j \leqslant n} \left(\frac{|b_{n-j}|}{\mu_j |b_n|} \right)^{1/j}.$$

THEOREM 1.2. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z)=\sum_{i=0}^{n} b_i H_i(z)$ *and let* $\mu_0 = 0$ *and* $\mu_1,\dots,\mu_n$ *be arbitrary positive real numbers. Then all the roots of* f *lie in the strip*

$$|\operatorname{Im} z| \leqslant \frac{1}{2} \max_{0\leqslant j\leqslant n-1}\left(\frac{\mu_j}{\mu_{j+1}}+\frac{\mu_n}{\mu_{j+1}}\cdot\frac{|b_j|}{|b_n|}\right).$$

THEOREM 1.3. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z)=\sum_{i=0}^{n} b_i H_i(z)$ *and let* $\mu_1,\dots,\mu_n$ *be arbitrary positive real numbers. Then all the roots of* f *lie in the strip*

$$|\operatorname{Im} z| \leqslant \frac{1}{2}\max\left\{\frac{\mu_2}{\mu_1},\frac{\mu_3}{\mu_2},\dots,\frac{\mu_n}{\mu_{n-1}},\sum_{j=1}^{n}\frac{\mu_j}{\mu_n}\cdot\frac{|b_{j-1}|}{|b_n|}\right\}.$$

As one can see, the enclosing strips can be described only in terms of nonzero b_i, since a zero Hermite coefficient has no contribution to the expressions in the right hand sides of the inequalities defining the relevant sets. The next result explicitly refers to sparse Hermite expansions.

THEOREM 1.4. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z)=\sum_{i=0}^{n} b_i H_{d_i}(z)$ *with all* $b_i \neq 0$ *and* $0\leqslant d_0 < d_1 < \ldots < d_n$. *Let* $\mu_0 = 0$, $\mu_n = 1$ *and* $\mu_1,\dots,\mu_{n-1}$ *be arbitrary positive constants. Then all the roots of* f *lie in the strip*

$$|\operatorname{Im} z| \leqslant \frac{1}{2}\max_{1\leqslant j\leqslant n}\left(\frac{(1+\mu_{j-1})|b_{j-1}|}{\mu_j |b_j|}\right)^{1/(d_j-d_{j-1})}.$$

The forthcoming results describe regions delimited by hyperbolas containing all the roots of an even complex polynomial.

THEOREM 1.5. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z)=\sum_{i=0}^{n} b_{2i} H_{2i}(z)$ *and let* $\mu_1,\dots,\mu_n$ *be positive real numbers such that* $\mu_1+\cdots+\mu_n \leqslant 1$. *Then all the roots of* f *lie in the hyperbola*

$$|\operatorname{Re} z \operatorname{Im} z| \leqslant \frac{13}{20}\max_{1\leqslant j\leqslant n}\left(\frac{|b_{2n-2j}|}{\mu_j |b_{2n}|}\right)^{1/j}.$$

THEOREM 1.6. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z)=\sum_{i=0}^{n} b_{2i} H_{2i}(z)$ *and let* $\mu_0 = 0$ *and* $\mu_1,\dots,\mu_n$ *be arbitrary positive real numbers. Then all the roots of* f *lie in the hyperbola*

$$|\operatorname{Re} z \operatorname{Im} z| \leqslant \frac{13}{20}\max_{0\leqslant j\leqslant n-1}\left(\frac{\mu_j}{\mu_{j+1}}+\frac{\mu_n}{\mu_{j+1}}\cdot\frac{|b_{2j}|}{|b_{2n}|}\right).$$

THEOREM 1.7. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z)=\sum_{i=0}^{n} b_{2i} H_{2i}(z)$ *and let* $\mu_1,\dots,\mu_n$ *be arbitrary positive real numbers. Then all the roots of* f *lie in the hyperbola*

$$|\operatorname{Re} z \operatorname{Im} z| \leqslant \frac{13}{20}\max\left\{\frac{\mu_2}{\mu_1},\frac{\mu_3}{\mu_2},\dots,\frac{\mu_n}{\mu_{n-1}},\sum_{j=1}^{n}\frac{\mu_j}{\mu_n}\cdot\frac{|b_{2j-2}|}{|b_{2n}|}\right\}.$$

THEOREM 1.8. *Suppose a complex polynomial* $f(z)$ *has the Hermite expansion* $f(z)=\sum_{i=0}^{n} b_{2i}H_{2d_i}(z)$ *with all* $b_i \neq 0$ *and* $0 \leqslant d_0 < d_1 < \ldots < d_n$. *Let* $\mu_0 = 0$, $\mu_n = 1$ *and* $\mu_1, \ldots, \mu_{n-1}$ *be arbitrary positive constants. Then all the roots of* f *lie in the hyperbola*

$$|\operatorname{Re} z \operatorname{Im} z| \leqslant \frac{13}{20} \max_{1 \leqslant j \leqslant n} \left(\frac{(1+\mu_{j-1})|b_{2j-2}|}{\mu_j |b_{2j}|} \right)^{1/(d_j - d_{j-1})}.$$

Our results are quite flexible and may be useful in various applications. The proofs of the main results are presented in the next section of the paper. They are based on several of the many properties that Hermite polynomials share with other classes of orthogonal polynomials, and invoke only a few results specific to Hermite polynomials. This makes it possible to prove similar statements referring to the coefficients of expansions with respect to other families of orthogonal polynomials. A series of corollaries and examples are given in Section 3 below, which also contains a discussion of possible extensions of our results and some considerations on practical aspects of the matter.

2. Proof of the main results.

Throughout this section, we denote by z_{im}, $i = 1, \ldots, m$, the zeros of the mth Hermite polynomial H_m, $m \geqslant 1$.

Proof of Theorem 1.1. By the well-known identity

$$H_m'(z) = 2mH_{m-1}(z)$$

we deduce the simple fraction development

$$\frac{H_{m-1}(z)}{H_m(z)} = \frac{1}{2m} \cdot \frac{H_m'(z)}{H_m(z)} = \frac{1}{2m} \sum_{i=1}^{m} \frac{1}{z - z_{im}},$$

which implies

$$\left| \frac{H_{m-1}(z)}{H_m(z)} \right| \leqslant \frac{1}{2m} \sum_{i=1}^{m} \frac{1}{|z - z_{im}|}. \tag{2.1}$$

Now, since all the zeros of $H_m(z)$ are real (see, e.g., Szegő [16]), we must have for nonreal z

$$\frac{1}{|z - z_{im}|} \leqslant \frac{1}{|\operatorname{Im} z|}, \tag{2.2}$$

so

$$\left| \frac{H_{m-1}(z)}{H_m(z)} \right| \leqslant \frac{1}{2\,|\operatorname{Im} z|}. \tag{2.3}$$

This shows that for any nonreal z and for all $j = 0, \ldots, n-1$ we have

$$\left| \frac{H_j(z)}{H_n(z)} \right| \leqslant \frac{1}{(2\,|\operatorname{Im} z|)^{n-j}}. \tag{2.4}$$

Let us assume by contradiction that f has at least one root z_0 satisfying

$$|\mathrm{Im}\, z_0| > \frac{1}{2} \max_{1 \leqslant j \leqslant n} \left(\frac{|b_{n-j}|}{\mu_j |b_n|} \right)^{1/j}. \tag{2.5}$$

Note in particular that such a z_0 can not be a root of $H_n(z)$. Assume now that $b_{n-j} \neq 0$ for some index $j \in \{1, \ldots, n\}$. In view of (2.4) and (2.5) we then obtain

$$\frac{\mu_j |b_n|}{|b_{n-j}|} > \frac{1}{(2\,|\mathrm{Im}\, z_0|)^j} \geqslant \left| \frac{H_{n-j}(z_0)}{H_n(z_0)} \right|,$$

and further

$$\mu_j |b_n| \cdot |H_n(z_0)| > |b_{n-j}| \cdot |H_{n-j}(z_0)|. \tag{2.6}$$

Since this obviously holds for $b_{n-j} = 0$, (2.6) must hold for all $j \in \{1, \ldots, n\}$. Adding term by term these inequalities, one obtains

$$|b_n| \cdot |H_n(z_0)| \geqslant \sum_{j=1}^{n} \mu_j |b_n| \cdot |H_n(z_0)| > \sum_{j=0}^{n-1} |b_j| \cdot |H_j(z_0)|.$$

On the other hand, since $f(z_0) = 0$, we must have

$$|b_n| \cdot |H_n(z_0)| \leqslant \sum_{j=0}^{n-1} |b_j| \cdot |H_j(z_0)|,$$

a contradiction. This completes the proof of the theorem. □

We notice that one obtains the same contradiction by using (2.6) only for those μ_j for which $b_{n-j} \neq 0$, so the μ_j for which $b_{n-j} = 0$ are irrelevant.

Proof of Theorem 1.2. Assume by contradiction that f has a root z_0 satisfying

$$|\mathrm{Im}\, z_0| > \frac{1}{2} \max_{0 \leqslant j \leqslant n-1} \frac{\mu_j |b_n| + \mu_n |b_j|}{\mu_{j+1} |b_n|}.$$

In view of (2.4) we then obtain

$$\left| \frac{H_{j+1}(z_0)}{H_j(z_0)} \right| \geqslant 2\,|\mathrm{Im}\, z_0| > \frac{\mu_j |b_n| + \mu_n |b_j|}{\mu_{j+1} |b_n|}$$

for $j = 0, \ldots, n-1$, so we have

$$\mu_{j+1} |b_n| \cdot |H_{j+1}(z_0)| > \mu_j |b_n| \cdot |H_j(z_0)| + \mu_n |b_j| \cdot |H_j(z_0)|$$

for $j = 0, \ldots, n-1$. Adding term by term these inequalities and canceling the equal terms on both sides, one obtains

$$|b_n| \cdot |H_n(z_0)| > \sum_{j=0}^{n-1} |b_j| \cdot |H_j(z_0)|.$$

On the other hand, since $f(z_0)=0$, we must have

$$|b_n|\cdot|H_n(z_0)| \leqslant \sum_{j=0}^{n-1} |b_j|\cdot|H_j(z_0)|,$$

a contradiction, and this completes the proof of the theorem. □

Proof of Theorem 1.3. Let z_0 be a root of f satisfying

$$|\operatorname{Im} z_0| > \frac{1}{2}\max\left\{\frac{\mu_2}{\mu_1},\frac{\mu_3}{\mu_2},\ldots,\frac{\mu_n}{\mu_{n-1}},\sum_{j=1}^{n}\frac{\mu_j}{\mu_n}\cdot\frac{|b_{j-1}|}{|b_n|}\right\}.$$

Then we obtain on one hand

$$\mu_j(2|\operatorname{Im} z_0|)^{n-j} > \mu_n, \qquad j=1,\ldots,n-1, \tag{2.7}$$

and on the other hand

$$\mu_n|b_n|(2|\operatorname{Im} z_0|) > \sum_{j=1}^{n}\mu_j|b_{j-1}|. \tag{2.8}$$

Now, since $(2|\operatorname{Im} z_0|)^{n-j} \leqslant |H_{n-1}(z_0)|/|H_{j-1}(z_0)|$ for $j=1,\ldots,n-1$, by using (2.7) we obtain

$$\mu_j > \mu_n\frac{|H_{j-1}(z_0)|}{|H_{n-1}(z_0)|}, \qquad j=1,\ldots,n-1. \tag{2.9}$$

Finally, by combining (2.8) and (2.9) we have

$$\begin{aligned}\mu_n|b_n|\cdot|H_n(z_0)| &\geqslant \mu_n|b_n|(2|\operatorname{Im} z_0|)|H_{n-1}(z_0)|\\ &> \mu_n\sum_{j=1}^{n}|b_{j-1}|\cdot|H_{j-1}(z_0)|,\end{aligned}$$

and therefore $|b_n|\cdot|H_n(z_0)| > \sum_{j=1}^{n}|b_{j-1}|\cdot|H_{j-1}(z_0)|$. One concludes as before. □

Proof of Theorem 1.4. Assume that f has a root z_0 satisfying

$$|\operatorname{Im} z_0| > \frac{1}{2}\max_{1\leqslant j\leqslant n}\left(\frac{(1+\mu_{j-1})|b_{j-1}|}{\mu_j|b_j|}\right)^{1/(d_j-d_{j-1})}.$$

In view of (2.4) we obtain for each $j=1,\ldots,n$ the inequalities

$$\left|\frac{H_{d_{j-1}}(z_0)}{H_{d_j}(z_0)}\right| \leqslant \frac{1}{(2|\operatorname{Im} z_0|)^{d_j-d_{j-1}}} < \frac{\mu_j|b_j|}{(1+\mu_{j-1})|b_{j-1}|},$$

so we deduce that

$$\mu_j|b_j|\cdot|H_{d_j}(z_0)| > (1+\mu_{j-1})|b_{j-1}|\cdot|H_{d_{j-1}}(z_0)|$$

for each $j = 1,\dots,n$. After summation and cancellation of equal terms on each side, we obtain

$$|b_n| \cdot |H_{d_n}(z_0)| > \sum_{j=0}^{n-1} |b_j| \cdot |H_{d_j}(z_0)|,$$

which leads us to the desired contradiction. $\square$

The starting point of all the previous proofs was the simple fraction development of the quotient of two consecutive Hermite polynomials, wherefrom the upper bound (2.1) for the module of this quotient resulted by triangle inequality. When the quantity of interest is the product of the real part and the imaginary part of a complex number z, it will be useful to bound from above the module of the quotient of two consecutive *even* Hermite polynomials evaluated at z. This will allow us to deduce the following inequality concerning the Hermite polynomials, valid for any complex z which is neither real nor purely imaginary.

$$\left|\frac{H_{2k}(z)}{H_{2n}(z)}\right| \leqslant \left(\frac{13}{20} \cdot \frac{1}{|\mathrm{Re}\, z\, \mathrm{Im}\, z|}\right)^{n-k}, \qquad k = 0,1,\dots,n-1. \tag{2.10}$$

This relation improves upon that used by Turán in his proof of Theorem D.

Inequation (2.10) is a consequence of the following result.

LEMMA 2.1. *Let z be a complex number with $\mathrm{Re}\, z\, \mathrm{Im}\, z \neq 0$. Then for every positive integer k one has*

$$\left|\frac{H_{2k-2}(z)}{H_{2k}(z)}\right| \leqslant \frac{13}{20} \cdot \frac{1}{|\mathrm{Re}\, z\, \mathrm{Im}\, z|}.$$

Proof. Let us consider for $m \geqslant 2$ the quotient $H_{m-2}(z)/H_m(z)$. Since the zeros x_{jm} of $H_m(z)$ are simple, one obtains

$$\frac{H_{m-2}(z)}{H_m(z)} = \sum_{j=1}^{m} \frac{H_{m-2}(x_{jm})}{H'_m(x_{jm})} \cdot \frac{1}{z - x_{jm}}.$$

Using now the fact that $H'_m(z) = 2mH_{m-1}(z)$, we deduce that

$$\frac{H_{m-2}(z)}{H_m(z)} = \frac{1}{2m} \sum_{j=1}^{m} \frac{H_{m-2}(x_{jm})}{H_{m-1}(x_{jm})} \cdot \frac{1}{z - x_{jm}}.$$

By the recursion formula $H_m(z) = 2zH_{m-1}(z) - 2(m-1)H_{m-2}(z)$ we get

$$\frac{H_{m-2}(x_{jm})}{H_{m-1}(x_{jm})} = \frac{x_{jm}}{m-1},$$

and hence

$$\frac{H_{m-2}(z)}{H_m(z)} = \frac{1}{2m(m-1)} \sum_{j=1}^{m} \frac{x_{jm}}{z - x_{jm}}. \tag{2.11}$$

The zeros of the even function $H_{2k}(z)$ are symmetric with respect to the origin. Let us denote by $x_{j,2k}$, $j=1,2,\ldots,k$, the positive roots of the polynomial H_{2k}. Using (2.11) and taking into account the symmetry of the zeros $x_{j,2k}$, we obtain

$$\frac{H_{2k-2}(z)}{H_{2k}(z)} = \frac{1}{2k(2k-1)} \sum_{j=1}^{k} \frac{x_{j,2k}^2}{z^2 - x_{j,2k}^2}.$$

Since $-\sqrt{4k+1} \leqslant x_{j,2k} \leqslant \sqrt{4k+1}$ for $j=1,2,\ldots,2k$ (see [16]), one has

$$\left|\frac{H_{2k-2}(z)}{H_{2k}(z)}\right| \leqslant \frac{4k+1}{2k(2k-1)} \sum_{j=1}^{k} \frac{1}{|z^2 - x_{j,2k}^2|}.$$

Now, since

$$\begin{aligned} |z^2 - x_{j,2k}^2| &= |(\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 - x_{j,2k}^2 + (2\operatorname{Re} z \operatorname{Im} z)\mathrm{i}| \\ &\geqslant 2|\operatorname{Re} z \operatorname{Im} z|, \end{aligned}$$

we get for $k \geqslant 3$

$$\left|\frac{H_{2k-2}(z)}{H_{2k}(z)}\right| \leqslant \frac{4k+1}{4(2k-1)} \cdot \frac{1}{|\operatorname{Re} z \operatorname{Im} z|} \leqslant \frac{13}{20} \cdot \frac{1}{|\operatorname{Re} z \operatorname{Im} z|}.$$

Using the explicit form of the first even Hermite polynomials, one obtains after some computations

$$\left|\frac{H_0(z)}{H_2(z)}\right| = \frac{1}{|4z^2-2|} \leqslant \frac{1}{8|\operatorname{Re} z \operatorname{Im} z|} < \frac{13}{20|\operatorname{Re} z \operatorname{Im} z|},$$

$$\left|\frac{H_2(z)}{H_4(z)}\right|^2 = \frac{u^2+16v}{(2u^2-8u-4-32v)^2+256v(u-2)^2} < \frac{1}{64v},$$

where $z = x+\mathrm{i}y$, $u = 2x^2-2y^2-1$, $v = x^2y^2$ $(x,y \in \mathbb{R})$. $\square$

Proof of Theorem 1.5. We argue by reduction to absurd. Let us assume that f has a root z_0 satisfying

$$|\operatorname{Re} z_0 \operatorname{Im} z_0| > \frac{13}{20} \max_{1 \leqslant j \leqslant n} \left(\frac{|b_{2n-2j}|}{\mu_j |b_{2n}|}\right)^{1/j}. \tag{2.12}$$

Note in particular that such a z_0 can not be a root of $H_{2n}(z)$. Assume now that $b_{2n-2j} \neq 0$ for some index $j \in \{1,\ldots,n\}$. In view of (2.10) and (2.12) we then obtain

$$\frac{\mu_j |b_{2n}|}{|b_{2n-2j}|} > \left(\frac{13}{20} \cdot \frac{1}{|\operatorname{Re} z_0 \operatorname{Im} z_0|}\right)^j \geqslant \left|\frac{H_{2n-2j}(z_0)}{H_{2n}(z_0)}\right|,$$

and further

$$\mu_j |b_{2n}| \cdot |H_{2n}(z_0)| > |b_{2n-2j}| \cdot |H_{2n-2j}(z_0)|. \tag{2.13}$$

Since this obviously holds for $b_{2n-2j}=0$, it follows that (2.13) must hold for all $j\in\{1,\dots,n\}$. Adding term by term these inequalities, we get

$$|b_{2n}|\cdot|H_{2n}(z_0)|\geqslant\sum_{j=1}^{n}\mu_j|b_{2n}|\cdot|H_{2n}(z_0)|>\sum_{j=0}^{n-1}|b_{2j}|\cdot|H_{2j}(z_0)|.$$

On the other hand, since $f(z_0)=0$, we must have

$$|b_{2n}|\cdot|H_{2n}(z_0)|\leqslant\sum_{j=0}^{n-1}|b_{2j}|\cdot|H_{2j}(z_0)|,$$

which is a contradiction. This completes the proof of the theorem. □

Proof of Theorem 1.6. Assume by contradiction that f has at least one root z_0 satisfying

$$|\operatorname{Re} z_0\operatorname{Im} z_0|>\frac{13}{20}\max_{0\leqslant j\leqslant n-1}\frac{\mu_j|b_{2n}|+\mu_n|b_{2j}|}{\mu_{j+1}|b_{2n}|}.$$

In view of (2.10) we then obtain

$$\left|\frac{H_{2(j+1)}(z_0)}{H_{2j}(z_0)}\right|\geqslant\frac{20}{13}|\operatorname{Re} z_0\operatorname{Im} z_0|>\frac{\mu_j|b_{2n}|+\mu_n|b_{2j}|}{\mu_{j+1}|b_{2n}|}$$

for $j=0,\dots,n-1$, so we have

$$\mu_{j+1}|b_{2n}|\cdot|H_{2(j+1)}(z_0)|>\mu_j|b_{2n}|\cdot|H_{2j}(z_0)|+\mu_n|b_{2j}|\cdot|H_{2j}(z_0)|$$

for $j=0,\dots,n-1$. Adding term by term these inequalities and canceling the equal terms on both sides results in

$$|b_{2n}|\cdot|H_{2n}(z_0)|>\sum_{j=0}^{n-1}|b_{2j}|\cdot|H_{2j}(z_0)|.$$

On the other hand, since $f(z_0)=0$, we must have

$$|b_{2n}|\cdot|H_{2n}(z_0)|\leqslant\sum_{j=0}^{n-1}|b_{2j}|\cdot|H_{2j}(z_0)|,$$

a contradiction, and this completes the proof of the theorem. □

Proof of Theorem 1.7. Let z_0 be a root of f satisfying

$$|\operatorname{Re} z_0\operatorname{Im} z_0|>\frac{13}{20}\max\left\{\frac{\mu_2}{\mu_1},\frac{\mu_3}{\mu_2},\dots,\frac{\mu_n}{\mu_{n-1}},\sum_{j=1}^{n}\frac{\mu_j}{\mu_n}\cdot\frac{|b_{2j-2}|}{|b_{2n}|}\right\}.$$

Then we obtain on one hand

$$\mu_j \left(\frac{20}{13} |\operatorname{Re} z_0 \operatorname{Im} z_0| \right)^{n-j} > \mu_n, \qquad j = 1, \dots, n-1, \tag{2.14}$$

and on the other hand

$$\mu_n |b_{2n}| \cdot \left(\frac{20}{13} |\operatorname{Re} z_0 \operatorname{Im} z_0| \right) > \sum_{j=1}^{n} \mu_j |b_{2j-2}|. \tag{2.15}$$

Now, since we have $\left(\frac{20}{13} |\operatorname{Re} z_0 \operatorname{Im} z_0|\right)^{n-j} \leqslant |H_{2n-2}(z_0)/H_{2j-2}(z_0)|$ for $j = 1, \dots, n-1$, by using (2.14) we obtain

$$\mu_j > \mu_n \frac{|H_{2j-2}(z_0)|}{|H_{2n-2}(z_0)|}, \qquad j = 1, \dots, n-1. \tag{2.16}$$

Finally, by combining (2.15) and (2.16) we have

$$\begin{aligned} \mu_n |b_{2n}| \cdot |H_{2n}(z_0)| &\geqslant \mu_n |b_{2n}| \cdot \left(\frac{20}{13} |\operatorname{Re} z_0 \operatorname{Im} z_0| \right) |H_{2n-2}(z_0)| \\ &> \mu_n \sum_{j=1}^{n} |b_{2j-2}| \cdot |H_{2j-2}(z_0)|, \end{aligned}$$

and therefore $|b_{2n}| \cdot |H_{2n}(z_0)| > \sum_{j=1}^{n} |b_{2j-2}| \cdot |H_{2j-2}(z_0)|$. One concludes as before. □

Proof of Theorem 1.8. Assume that f has a root z_0 satisfying

$$|\operatorname{Re} z_0 \operatorname{Im} z_0| > \frac{13}{20} \max_{1 \leqslant j \leqslant n} \left(\frac{(1+\mu_{j-1})|b_{2j-2}|}{\mu_j |b_{2j}|} \right)^{1/(d_j - d_{j-1})}.$$

In view of (2.10) we then obtain for each $j = 1, \dots, n$ the inequalities

$$\left| \frac{H_{2d_{j-1}}(z_0)}{H_{2d_j}(z_0)} \right| \leqslant \left(\frac{13}{20} \cdot \frac{1}{|\operatorname{Re} z_0 \operatorname{Im} z_0|} \right)^{d_j - d_{j-1}} < \frac{\mu_j |b_{2j}|}{(1+\mu_{j-1})|b_{2j-2}|},$$

so we deduce that

$$\mu_j |b_{2j}| \cdot |H_{2d_j}(z_0)| > (1+\mu_{j-1}) |b_{2j-2}| \cdot |H_{2d_{j-1}}(z_0)|$$

for each $j = 1, \dots, n$. After summation and cancellation of equal terms on each side, we obtain

$$|b_{2n}| \cdot |H_{2d_n}(z_0)| > \sum_{j=0}^{n-1} |b_{2j}| \cdot |H_{2d_j}(z_0)|,$$

which leads us to the desired contradiction.

REMARK. From the proof of Lemma 2.1 it is obvious that the constant $13/20$ is obtained as the largest value taken by the function

$$f(k)=\frac{4k+1}{4(2k-1)}$$

when its argument is at least 3. For polynomials having Hermite expansion of the form $f(z)=\sum_{i=m}^{n}b_{2i}H_{2i}(z)$ with $m>3$, the constant $13/20$ in the statement of Lemma 2.1, in inequality (2.10) and in the statements of Theorems 1.5–1.8 can be improved to $f(m)<13/20$. □

3. Applications and examples.

In this section we present several possibilities to extend the results proved so far, along with a few consequences of the statements or of the proofs of the main results.

3.1. Particular cases

One may obtain various estimates for the roots of a polynomial by choosing different sequences of positive real numbers $\mu_1,\dots,\mu_n$ satisfying the requirements of each of Theorems 1.1–1.8. For instance, the hypothesis $\mu_1+\dots+\mu_n\leqslant 1$ from Theorem 1.1 is satisfied by $\mu_j=1/n$, or $\mu_j=2^{-n}\binom{n}{j}$, or more generally $\mu_j=\lambda^j(1-\lambda)^{n-j}\binom{n}{j}$ (with $0<\lambda<1$) for all j. As we saw in the proof of Theorem 1.1, we may consider only the relevant μ_j, namely those μ_j for which $b_{n-j}\neq 0$. For an example when the μ_j depend on the coefficients b_j, we take $\mu_j=|b_{n-j}|/\sum_i|b_i|$ for all the indices j for which $b_{n-j}\neq 0$, which results in the next statement.

COROLLARY 3.1. *If a complex polynomial $f(z)$ has the Hermite expansion $f(z)=\sum_{i=0}^{n}b_iH_i(z)$, with $|b_n|\geqslant|b_0|+\dots+|b_{n-1}|$, then all the roots of f lie in the strip $|\operatorname{Im}z|\leqslant 1/2$.*

Applying Theorem 1.2 with $\mu_1=\mu_2=\dots=\mu_n>0$ we obtain:

COROLLARY 3.2. *If a complex polynomial $f(z)$ has the Hermite expansion $f(z)=\sum_{i=0}^{n}b_iH_i(z)$, then all the roots of f lie in the strip*

$$|\operatorname{Im}z|\leqslant\frac{1}{2}\max\left(\frac{|b_0|}{|b_n|},1+\frac{|b_1|}{|b_n|},1+\frac{|b_2|}{|b_n|},\dots,1+\frac{|b_{n-1}|}{|b_n|}\right).$$

Note that Corollary 3.2 is stronger than Theorem A whenever $|b_0|$ is sufficiently large, precisely for $|b_0|>|b_n|+\max\limits_{1\leqslant k\leqslant n-1}|b_k|$.

By taking $\mu_j=1$ for $j=1,\dots,n$ we obtain a particularly simple instance of Theorem 1.3.

COROLLARY 3.3. *If a complex polynomial $f(z)$ has the Hermite expansion $f(z) = \sum_{i=0}^{n} b_i H_i(z)$, then all the roots of f lie in the strip*

$$|\operatorname{Im} z| \leqslant \frac{1}{2} \max\left(1,\ \frac{|b_0|+|b_1|+\ldots+|b_{n-1}|}{|b_n|}\right).$$

Note that Corollary 3.1 is implied by Corollary 3.3.

Similarly, from Theorem 1.4 one obtains the following.

COROLLARY 3.4. *If a complex polynomial $f(z)$ has the Hermite expansion $f(z) = \sum_{i=0}^{n} b_i H_i(z)$ with $b_0 b_1 \cdots b_n \neq 0$, then all the roots of f lie in the strip*

$$|\operatorname{Im} z| \leqslant \frac{1}{2} \max\left(\frac{|b_0|}{|b_1|}, \frac{2|b_1|}{|b_2|}, \frac{2|b_2|}{|b_3|}, \ldots, \frac{2|b_{n-1}|}{|b_n|}\right).$$

In a similar way one may obtain analogous corollaries for hyperbolas instead of strips containing all the roots. We illustrate the idea with a single example.

COROLLARY 3.5. *If ρ is the unique positive root of the polynomial $X^n - X^{n-1} - \cdots - X - 1$, then each root of a complex polynomial $f(z)$ given by its Hermite expansion $f(z) = \sum_{i=0}^{n} b_i H_i(z)$, with $b_0 b_1 \cdots b_n \neq 0$, satisfies*

$$|\operatorname{Re} z \operatorname{Im} z| \leqslant \frac{13\rho}{20} \max_{1 \leqslant j \leqslant n} \left(\frac{|b_{n-j}|}{|b_n|}\right)^{1/j}.$$

Proof. Take $\mu_j = \rho^{-j}$, $j = 1, 2, \ldots, n$ in Theorem 1.5. □

A close look at the proofs reveals that we did not use the full strength of the hypothesis z_0 is a root of f, all what is actually needed in the proofs of Theorems 1.1, 1.2 and 1.3 is the fact that z_0 belongs to the set

$$A := \left\{ z : |b_n H_n(z)| \leqslant \sum_{j=0}^{n-1} |b_j H_j(z)| \right\}.$$

Therefore, we can state the following.

THEOREM 3.6. *Suppose a complex polynomial $f(z)$ has the Hermite expansion $f(z) = \sum_{i=0}^{n} b_i H_i(z)$ and let $\mu_0 = 0$ and $\mu_1, \ldots, \mu_n$ be arbitrary positive real numbers. Then*

$$A \subseteq \left\{ z : |\operatorname{Im} z| \leqslant \frac{1}{2} \max_{0 \leqslant j \leqslant n-1} \left(\frac{\mu_j}{\mu_{j+1}} + \frac{\mu_n}{\mu_{j+1}} \cdot \frac{|b_j|}{|b_n|}\right) \right\},$$

$$A \subseteq \left\{ z : |\operatorname{Im} z| \leqslant \frac{1}{2} \max\left(\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_2}, \ldots, \frac{\mu_n}{\mu_{n-1}}, \sum_{j=1}^{n} \frac{\mu_j}{\mu_n} \cdot \frac{|b_{j-1}|}{|b_n|}\right) \right\}.$$

If moreover $\mu_1 + \ldots + \mu_n \leqslant 1$, then

$$A \subseteq \left\{ z : |\operatorname{Im} z| \leqslant \frac{1}{2} \max_{1 \leqslant j \leqslant n} \left(\frac{|b_{n-j}|}{\mu_j |b_n|}\right)^{1/j} \right\}.$$

Define now

$$A := \left\{ z : |b_{2n}H_{2n}(z)| \leqslant \sum_{j=0}^{n-1} |b_{2j}H_{2j}(z)| \right\}. \tag{3.1}$$

Following the proofs of Theorems 1.5–1.7, one can prove the result below.

THEOREM 3.7. *Suppose a complex polynomial $f(z)$ has the Hermite expansion $f(z) = \sum_{i=0}^{n} b_{2i}H_{2i}(z)$ and let $\mu_0 = 0$ and $\mu_1, \ldots, \mu_n$ be arbitrary positive real numbers. Then the set A defined by (3.1) is contained in $B \cap C$, where*

$$B := \left\{ z : |\mathrm{Re}\, z \mathrm{Im}\, z| \leqslant \frac{13}{20} \max_{0 \leqslant j \leqslant n-1} \left(\frac{\mu_j}{\mu_{j+1}} + \frac{\mu_n}{\mu_{j+1}} \cdot \frac{|b_{2j}|}{|b_{2n}|} \right) \right\}$$

and

$$C := \left\{ z : |\mathrm{Re}\, z \mathrm{Im}\, z| \leqslant \frac{13}{20} \max \left(\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_2}, \ldots, \frac{\mu_n}{\mu_{n-1}}, \sum_{j=1}^{n} \frac{\mu_j}{\mu_n} \cdot \frac{|b_{2j-2}|}{|b_{2n}|} \right) \right\}.$$

If moreover $\mu_1 + \ldots + \mu_n \leqslant 1$, then

$$A \subseteq \left\{ z : |\mathrm{Re}\, z \mathrm{Im}\, z| \leqslant \frac{13}{20} \max_{1 \leqslant j \leqslant n} \left(\frac{|b_{2n-2j}|}{\mu_j |b_{2n}|} \right)^{1/j} \right\}.$$

3.2. Modified Hermite expansion

As already mentioned, it is possible to bound the imaginary part of the roots of a polynomial given by its expansion with respect to other orthogonal polynomials. We state a set of results invoking the modified Hermite polynomials introduced by Jörgensen [7] and defined by

$$He_n(z) := 2^{-n/2} H_n\left(\frac{z}{\sqrt{2}}\right).$$

The coefficients of Hermite polynomials grow very fast with the degree. Thus, the leading coefficient of H_n is 2^n, and many other coefficients of H_n are even bigger. This fact has as consequence the appearance in the Hermite expansion of a “nicely” looking polynomial of coefficients b_i differing by several orders of magnitude. This in turn implies a large upper bound in each of the Theorems 1.1–1.8. The coefficients of He_n are much smaller than the coefficients of the corresponding H_n. Unfortunately, this feature does not necessarily imply better bounds for the roots of f.

It is not difficult to obtain similar results for polynomials given by their expansion with respect to the modified Hermite polynomials. In the forthcoming results, the coefficient c_i of maximal index is supposed to be nonzero.

THEOREM 3.8. *Suppose a complex polynomial $f(z)$ has the modified Hermite expansion $f(z) = \sum_{i=0}^{n} c_i He_i(z)$ and let $\mu_1, \ldots, \mu_n$ be positive real numbers such that $\mu_1 + \cdots + \mu_n \leqslant 1$. Then all the roots of f lie in the strip*

$$|\mathrm{Im}\, z| \leqslant \max_{1 \leqslant j \leqslant n} \left(\frac{|c_{n-j}|}{\mu_j |c_n|} \right)^{1/j}.$$

THEOREM 3.9. *Suppose a complex polynomial $f(z)$ has the modified Hermite expansion $f(z) = \sum_{i=0}^{n} c_{2i} He_{2i}(z)$ and let $\mu_0 = 0$ and $\mu_1, \ldots, \mu_n$ be arbitrary positive real numbers. Then all the roots of f lie in the hyperbola*

$$|\operatorname{Re} z \operatorname{Im} z| \leqslant \frac{13}{5} \max_{0 \leqslant j \leqslant n-1} \left(\frac{\mu_j}{\mu_{j+1}} + \frac{\mu_n}{\mu_{j+1}} \cdot \frac{|c_{2j}|}{|c_{2n}|} \right).$$

3.3. Numerical examples

The explicit examples below illustrate the ideas discussed so far in this section.

EXAMPLE 3.10. Let us consider the polynomial

$$f_1 := (z^2 + 2z + 37)(z^2 + 2z + 25) = z^4 + 4z^3 + 66z^2 + 124z + 925.$$

Then the Hermite expansion for the polynomial $24 f_1$ is

$$3H_4 + 32H_3 + 654H_2 + 1680H_1 + 23472H_0.$$

According to Theorem 1.1, all the roots of f_1 satisfy

$$|\operatorname{Im} z| \leqslant \frac{1}{2} \max \left(\frac{32}{3\mu_1}, \sqrt{\frac{218}{\mu_2}}, \sqrt[3]{\frac{560}{\mu_3}}, \sqrt[4]{\frac{7824}{\mu_4}} \right),$$

for any positive real numbers μ_1, μ_2, μ_3, μ_4 with sum at most 1.

The optimal choice for the parameters μ_j is when the four numbers whose maximum defines the enclosing strip for the roots are as close as possible to each other. Thus, by choosing

$$\mu_1 = 0.48,\ \mu_2 = 0.44,\ \mu_3 = 0.04, \mu_4 = 0.04$$

one obtains

$$|\operatorname{Im} z| \leqslant 12.051,$$

while the slightly different choice

$$\mu_1 = 0.48,\ \mu_2 = 0.44,\ \mu_3 = 0.05, \mu_4 = 0.03$$

yields the comparatively better estimate

$$|\operatorname{Im} z| \leqslant 11.300.$$

If one applies Theorem 1.2 with μ_j forming a geometric progression of ratio 22.3, one gets an even better estimate

$$|\operatorname{Im} z| \leqslant 11.150.$$

This is only slightly worse than the result

$$|\operatorname{Im} z| \leqslant 11.148$$

provided by Theorem 1.4 with

$$\mu_1 = 0.627,\ \mu_2 = 0.188,\ \mu_3 = 1.089.$$

EXAMPLE 3.11. Put

$$f_2 = z^4 + 81,$$

so that

$$16f_2 = H_4 + 12H_2 + 1308H_0$$

and

$$f_2 = He_4 + 6He_2 + 84.$$

Choosing $\mu_2 = 0.282$ and $\mu_4 = 0.718$ in Theorem 1.1 results in

$$|\operatorname{Im} z| \leqslant 3.267,$$

while Theorem 1.4 applied with $\mu_1 = 2.555$ yields

$$|\operatorname{Im} z| \leqslant 3.266.$$

Since f_2 is an even polynomial, we may also use Theorems 1.5–1.8. Thus, from Theorem 1.5 for $\mu_1 = 0.282$, $\mu_2 = 0.718$ one obtains that

$$|\operatorname{Re} z \operatorname{Im} z| < 27.747,$$

and Theorem 1.8 gives for $\mu_1 = 2.555$

$$|\operatorname{Re} z \operatorname{Im} z| < 27.730.$$

Using the expansion of f_2 with respect to the modified Hermite polynomials, one gets worse estimations:

$$|\operatorname{Im} z| \leqslant 3.557,$$

according to Theorem 3.8 with $\mu_2 = 0.475$, $\mu_4 = 0.525$, and

$$|\operatorname{Re} z \operatorname{Im} z| < 32.875,$$

by choosing $\mu_1 = 6.644\mu_2$ in Theorem 3.9.

As the next example shows, the discrepancies between the estimates obtained from the Hermite expansion and those corresponding to modified Hermite expansion are less important when the imaginary parts of the roots have greater moduli.

EXAMPLE 3.12. The roots of the polynomial

$$f_3 = z^4 + 1296 = \frac{1}{16}H_4 + \frac{3}{4}He_2 + \frac{5187}{4} = He_4 + 6He_2 + 1299$$

are twice the roots of f_2. Theorem 1.1 with $\mu_2 = 0.08$ and $\mu_4 = 0.92$ yields the estimate

$$|\operatorname{Im} z| \leqslant 6.128,$$

which is 44.44% bigger than the true value of $|\operatorname{Im} z|$. Note that in the previous Example the overestimation is around 54%. Similarly, for the roots of f_3 one obtains from Theorem 3.8 a strip whose width is 47.63% larger than the width of the optimal strip, while the corresponding ratio for f_2 is 67.68%. The same phenomenon can be easily detected when using the modified Hermite expansion.

3.4. Future work

Besides properties specific to orthogonal polynomials (e.g., recurrence relation, estimates for the roots, relationship with the derivative), two essential ingredients of the proofs are the triangle inequality (see Eq. (2.1)) and the inequality (2.2) relating the hypotenuse to the legs of a rectangle triangle. The quality of the final estimates for the roots seems to be downgraded because of these two inequalities, which are too general and do not take advantage on any information specific to orthogonal polynomials. Apparently, obtaining sharper estimates for strips and hyperbolas containing all the roots of a polynomial will be possible by replacing Eqs. (2.1)–(2.2) above by more adequate inequalities, adapted to the context we are working in.

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(Received September 11, 2006)

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