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SINGULAR INTEGRAL INEQUALITIES AND NATURAL REGULARIZATIONS

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Abstract. We shall introduce natural regularizations for singular integrals from the viewpoint of numerical treatments and establish very good error estimates (fundamental inequalities for potentials) for the regularizations.

1. Introduction and motivation

In the paper [15], from its general ideas, we gave a concrete representation formula of some general inverse functions. When we consider some general methods and ideas for the inversion formulas for some general non-linear mappings, however, their formulations will be, in general, very involved. We shall recall the principle for our method for the representations of inverses of non-linear mappings based on [15]:

We shall consider some representation of the inversion ϕ^{-1} in terms of some integral form – at this moment, we need to postulate a natural assumption on the mapping ϕ . Then, we shall transform the integral representation by the mapping ϕ to the original space that is the defined domain of the mapping ϕ. Then, we will be able to obtain the representation of the inverse ϕ^{-1} in terms of the direct mapping ϕ. In [15], we considered the representation of the inverse ϕ^{-1} in some reproducing kernel Hilbert spaces, and in [22], we considered the representations of the inverse ϕ^{-1} for a very concrete situation and we gave a very fundamental representation of the inverse for some general functions on one dimensional spaces. When we consider its multi-dimensional versions, it seems that we can not find some simple representations by some concrete known reproducing kernels for some general domains, and indeed, we know the reproducing kernels only for special domains and for special reproducing kernel Hilbert spaces (cf. [16]). It seems that for the integral representations of some function spaces, the representations by regular integrals (not singular integrals) are very complicated numerically or as concrete representations. However, when we use singular integrals, we can consider the representations in some general ways.

Indeed, we shall recall the following fundamental facts:

We can represent a function f in terms of the delta function δ in the form

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$$f(q) = \int_D f(p)\delta(p-q)dp \tag{1.1}$$

in some domain, symbolically for some general function f. Moreover, a fundamental solution $G(p-q)$ for some linear differential operator L is given by the equation,

$$LG(p-q) = \delta(p-q). \tag{1.2}$$

So, from (1.1) we obtain the representation

$$f(q) = \int_D f(p)LG(p-q)dp. \tag{1.3}$$

Then, we can obtain the representation symbolically, by using the Green-Stokes formula, for some adjoint operator L^* for L,

$$f(q) = \int_D L^* f(p)G(p-q)dp + \text{some} \quad \text{boundary} \quad \text{integrals}. \tag{1.4}$$

We shall use this type of representation. In this approach, we will encounter a singular integral representation in the first term of (1.4). However, if $G(p-q)$ is integrable, then a simple regularization of $G(p-q)$ will enable us to realize the representation in numerical treatments. We are interested in some very concrete results that may be achieved by computers. For example, we shall consider very concrete cases in the two dimensional spaces.

Let $D \subset \mathbf{R}^2$ be a bounded domain with a finite number of piecewise C^1 class boundary components. Let f be a one-to-one C^1 class mapping from $\bar{D}$ into $\mathbf{R}^2$ and we assume that its Jacobian $J(x)$ is positive on D. We shall represent f as follows:

$$\begin{aligned} y_1 &= f_1(x) = f_1(x_1, x_2) \\ y_2 &= f_2(x) = f_2(x_1, x_2) \end{aligned} \tag{1.5}$$

and the inverse mapping f^{-1} of f as follows:

$$\begin{aligned} x_1 &= (f^{-1})_1(y) = (f^{-1})_1(y_1, y_2) \\ x_2 &= (f^{-1})_2(y) = (f^{-1})_2(y_1, y_2). \end{aligned} \tag{1.6}$$

Then, we shall represent

$$\begin{pmatrix} (f^{-1})_1(y^*) \\ (f^{-1})_2(y^*) \end{pmatrix} \tag{1.7}$$

in terms of the direct mapping (1.5).

Of course, we are interested in some numerical and practical solutions for the simultaneous non-linear equations (1.5).

For some representation of type (1.4) connected to (1.2), of functions containing the functions (1.6), we shall consider the representation by the fundamental solution for the Laplace equation. However, its representation will be complicated for the present situation (note that for the three dimensional case, the representation is very simple by using the fundamental solution of the Laplace equation) and furthermore, for the Laplace equation we shall consider the representation for the C^2 class functions. At this moment, we shall recall the representation by using the fundamental solution for the $\partial_{\bar{z}}$ equation for complex versions ([8, Theorem 1.2.1]). Then we can obtain the integral representations for the C^1 class functions by using the Green-Stokes formula. In the complex representation, we assume that the imaginary part is zero, then we can obtain the integral representation of the C^1 class functions. Then, following the general method and transforming by the mapping (1.5) of the integral representation, we can obtain the desired result.

PROPOSITION 1. ([24]) *For the mappings (1.5) and (1.6), we obtain the following representation of (1,7), for any* $y^* = (y_1^*, y_2^*) \in f(D)$*:*

$$\begin{pmatrix} (f^{-1})_1(y^*) \\ (f^{-1})_2(y^*) \end{pmatrix}$$

$$= \frac{1}{2\pi} \oint_{\partial D} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d\,\mathrm{Arctan}\, \frac{f_2(x) - y_2^*}{f_1(x) - y_1^*}$$

$$- \frac{1}{2\pi} \int\int_D \frac{1}{|f(x) - y^*|^2} \,\mathrm{adj}\, J(x) \begin{pmatrix} f_1(x) - y_1^* \\ f_2(x) - y_2^* \end{pmatrix} dx_1 dx_2.$$

Anyhow we can obtain many integral representations by using singular integrals for many function spaces. Now we are interested in the numerical calculations for the singular integrals. For example, in Proposition 1, we considered the very natural method for the singular integral: Given the singular integral

$$- \frac{1}{2\pi} \int\int_D \frac{1}{|f(x) - y^*|^2} \,\mathrm{adj}\, J(x) \begin{pmatrix} f_1(x) - y_1^* \\ f_2(x) - y_2^* \end{pmatrix} dx_1 dx_2,$$

we calculate the regularized integral, for a very small λ

$$- \frac{1}{2\pi} \int\int_D \frac{1}{|f(x) - y^*|^2 + \lambda} \,\mathrm{adj}\, J(x) \begin{pmatrix} f_1(x) - y_1^* \\ f_2(x) - y_2^* \end{pmatrix} dx_1 dx_2.$$

For this regularized integral, its computational calculation is very simple. At this moment, we are interested in the error estimate in these integrals. So, in this paper, we shall discuss this type of error estimates (fundamental inequalities for potentials).

There are huge amount of papers concerned with the integral operators of the type

$$Kf(x) = \int_{\mathbf{R}^n} k(x,y) f(y)\, dy,$$

where k is measurable function with certain integrability. From the viewpoint of harmonic analysis, [1, 2, 3, 4] are key papers when $k(x,y) = k(x-y)$ satisfies

$$|k(x)| \leqslant \frac{c}{|x|^n}, \; |\nabla k(x)| \leqslant \frac{c}{|x|^{n+1}}$$

and K itself is L^2-bounded. Also, when the kernel k is a bound $c|\cdot|^{\alpha-n}$, $\alpha > 0$ or K satisfies some estimates similar to this one, it is investigated from various viewpoints. A famous result, as is written in the textbook [21], says that K is bounded from L^p to L^q when $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $k(x) = |x|^{\alpha-n}$. If $k(x) = |x|^{\alpha-n}$, then K is called the potential operator. This boundedness has been investigated more subtly and we have huge amount of results. We can find a tip of iceberg of this vast field attempting to investigate the integrability of the fractional integral operators in more depth in [6, 7, 9, 10, 12, 13, 17, 18, 19, 20], where the kernel and the function spaces are very generalized.

In the present paper we are concerned with the potential operators with $k(x) = |x|^{\alpha-n}$ and $k(x) = \log|x|$. However, we are not oriented to further generalization of the results in [6, 7, 9, 10, 12, 13, 17, 18, 19, 20]. What we aim at is quite opposite: We shall mainly consider the operators of the form

$$Kf(x) = \int k(x-y)f(y)\,dy, \; If(x) = \int f(y)\log|x-y|\,dy.$$

In Section 2 we investigate some general inequalities for K. In Section 3 we consider the case If and in Section 4 we shall investigate the error estimates for the natural regularization. In the final section, Section 5, we shall give computational experiments for our natural regularizations.

2. Some general properties and inequalities for the operator K

LEMMA 1. *Denote by M the uncentered ball-maximal operator, that is, we define*

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(x)|\,dx$$

for measurable functions f, where B runs over all the balls containing x. Then

$$\|Mf\|_p \leqslant \left(\frac{p\,2^p\,3^n}{p-1}\right)^{\frac{1}{p}} \|f\|_p$$

for all $f \in L^p$ with $1 < p < \infty$.

This is a well-known theorem (see [21]). The number 3^n comes about by using a covering lemma called $5r$-covering lemma. The $5r$-coverling lemma, as the name suggests, involves the number 5. However, this covering lemma is refined and the number 3 comes about. We can find an example of this refined covering lemma in the celebrated paper [14].

To formulate our result we need to define two norms.

DEFINITION 1. *1. Given a locally integrable function f, we define*

$$\|f : BMO\| = \sup_B \left(\frac{1}{|B|}\int_B \left| f(x) - \frac{1}{|B|}\int_B f(y)\,dy\right| dx\right),$$

where B runs over all the balls in $\mathbf{R}^n$.

2. *Given a continuous function* f*, one defines*

$$\|f : Lip(\alpha)\| = \sup_{x,y\in\mathbf{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

for $0<\alpha<1$.

Our main result in this section reads:

THEOREM 1. *Suppose that* $k : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ *is a measurable function. Assume that there exist* $c_0, c_1, \delta, \alpha > 0$ *such that*

$$|k(x,y)| \leqslant c_0\,|x-y|^{-n+\alpha}$$
$$|\partial_x k(x,y)| \leqslant c_1\,|x-x'|^{\varepsilon}\,|x-y|^{-n+\alpha-\varepsilon}$$

for some $\varepsilon \in (0,1]$. *Define*

$$Kf(x) = \int k(x,y) f(y)\,dy$$

as long as the integral converges. Then we have the following.

1. *Let* $f \in L^{p_0}$ *for some* $1 \leqslant p_0 < \frac{n}{\alpha}$. *Then the integral defining* $Kf(x)$ *converges absolutely. Assume that*

$$\frac{1}{s_0} = \frac{1}{p_0} - \frac{\alpha}{n}.$$

Furthermore, it satisfies the following pointwise inequality

$$|Kf(x)| \leqslant \frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p_0\alpha)} Mf(x)^{\frac{p_0}{s_0}}\, \|f\|_{p_0}{}^{1-\frac{p_0}{s_0}},$$

where $B(1)$ *denotes the open unit ball in* $\mathbf{R}^n$.

2. *Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then the inequality

$$\|Kf\|_s \leqslant \frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p\alpha)} \left(\frac{p\,2^p\,3^n}{p-1}\right)^{\frac{1}{s}} \|f\|_p$$

holds for all $f \in L^p$.

3. *Assume that*

$$p = \frac{\alpha}{n},\ 1 < p_0 < \frac{n}{\alpha},\ \frac{1}{s_0} = \frac{1}{p_0} - \frac{\alpha}{n}.$$

Then the inequality

$$\|Kf : BMO\| \leqslant C(c_0, c_1, s, p, s_0, p_0, \alpha, n)\|f\|_p$$

holds for all $f \in L^p \cap L^1$. *Here*

$$C(c_0,c_1,s,p,s_0,p_0,\alpha,n) = 2^{1+\frac{n}{s_0}} \frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p_0\alpha)} \left(\frac{p_0\,2^{p_0}\,3^n}{p_0-1}\right)^{\frac{1}{s_0}} + c_1\,(n+\varepsilon-\alpha)\int_0^\infty \frac{(1+\ell)^{n-\frac{n}{p}}\,d\ell}{\ell^{1+n+\varepsilon-\alpha}}.$$

4. *Assume that*

$$\alpha < \frac{n}{p} < \alpha+1.$$

Then the inequality

$$\|f : Lip(\alpha - n/p)\| \leqslant D(c_0,c_1,s,p,s_0,p_0,\alpha,n)\|f\|_p$$

holds for all $f \in L^p \cap L^1$. *Here*

$$D(c_0,c_1,s,p,s_0,p_0,\alpha,n) = \left(\int_1^\infty \frac{(\ell+1)^{\frac{1}{p'}}}{\ell^{n+\varepsilon-\alpha+1}}\right)^{\frac{1}{p'}} + 4\left(\frac{|S^1|}{(n-p'(n-\alpha))}\right)^{\frac{1}{p'}} r^{n/p'-(n-\alpha)}.$$

Proof. By the Fubini theorem we have

$$\begin{aligned}|Kf(x)| &\leqslant c_0 \int_{\mathbf{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \\ &= c_0\,(n-\alpha)\int_0^\infty \frac{1}{\ell^{n-\alpha+1}}\left(\int_{B(x,\ell)}|f(y)|\,dy\right)d\ell \\ &= c_0\,(n-\alpha)\int_0^\infty \frac{1}{\ell^{n-\alpha+1}}\min(|B(1)|Mf(x)\ell^n, |B(1)|^{1-\frac{1}{p}}\|f\|_p\ell^{n-\frac{n}{p}})\,d\ell.\end{aligned}$$

Now let us calculate $\int_0^\infty \frac{1}{\ell^{n-\alpha+1}}\min(A\,\ell^n, B\,\ell^{n-\frac{n}{p}})\,d\ell$ with $A,B > 0$.

$$\begin{aligned}\int_0^\infty \frac{1}{\ell^{n-\alpha+1}}\min(A\,\ell^n, B\,\ell^{n-\frac{n}{p}})\,d\ell &= \int_0^{\left(\frac{B}{A}\right)^{\frac{p}{n}}} A\,\ell^{\alpha-1}\,d\ell + \int_{\left(\frac{B}{A}\right)^{\frac{p}{n}}}^\infty B\,\ell^{\alpha-\frac{n}{p}-1}\,d\ell \\ &= \left(\frac{1}{\alpha}+\frac{1}{\frac{n}{p}-\alpha}\right)A^{1-\frac{p\alpha}{n}}B^{\frac{p\alpha}{n}} \\ &= \frac{n}{\alpha(n-p\alpha)}A^{1-\frac{p\alpha}{n}}B^{\frac{p\alpha}{n}}.\end{aligned}$$

As a consequence we obtain

$$|Kf(x)| \leqslant \frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p\alpha)} Mf(x)^{1-\frac{p\alpha}{n}}\|f\|_p{}^{\frac{p\alpha}{n}}.$$

1. is therefore established.

Now we prove the second assertion. If we invoke Lemma 1, then we obtain

$$\left(\int |Kf(x)|^s\,dx\right)^{\frac{1}{s}} \leqslant \frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p\alpha)}\left(\int Mf(x)^p\,dx\right)^{\frac{1}{s}}\|f\|_p{}^{\frac{p\alpha}{n}}$$
$$\leqslant \frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p\alpha)}\left(\frac{p\,2^p\,3^n}{p-1}\right)^{\frac{1}{s}}\|f\|_p.$$

Concerning the third assertion, we begin with fixing a cube B. Let us set

$$f_1 = \chi_{2B}f,\ f_2 = f - f_1.$$

By using this decomposition, we have

$$\frac{1}{|B|}\int_B |Kf(x) - m_B(Kf)|\,dx$$
$$\leqslant \frac{1}{|B|}\int_B |Kf_1(x) - m_B(Kf_1)|\,dx + \frac{1}{|B|}\int_B |Kf_2(x) - m_B(Kf_2)|\,dx.$$

Concerning the first term, we have chosen an auxiliary constant $1 < p_0 < \dfrac{n}{\alpha}$. Set

$$\frac{1}{s_0} = \frac{1}{p_0} - \frac{\alpha}{n}.$$

With this constant, we obtain

$$\frac{1}{|B|}\int_B |Kf_1(x) - m_B(Kf_1)|\,dx$$
$$\leqslant \frac{2}{|B|}\int_B |Kf_1(x)|\,dx$$
$$\leqslant 2\left(\frac{1}{|B|}\int_B |Kf_1(x)|^{s_0}\,dx\right)^{\frac{1}{s_0}}$$
$$\leqslant 2^{1+\frac{n}{p_0}}\frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p_0\alpha)}\left(\frac{p_0\,2^{p_0}\,3^n}{p_0-1}\right)^{\frac{1}{s_0}}|B|^{\frac{\alpha}{n}}\left(\frac{1}{|2B|}\int_{2B}|f(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}$$
$$\leqslant 2^{1+\frac{n}{s_0}}\frac{c_0\,(n-\alpha)n|B(1)|^{1-\frac{\alpha}{n}}}{\alpha(n-p_0\alpha)}\left(\frac{p_0\,2^{p_0}\,3^n}{p_0-1}\right)^{\frac{1}{s_0}}\left(\int |f(x)|^{\frac{n}{\alpha}}\,dx\right)^{\frac{\alpha}{n}}.$$

Now let us turn to the second term

$$\begin{aligned}
&\frac{1}{|B|}\int_B |Kf_2(x)-m_B(Kf_2)|\,dx\\
&=\frac{1}{|B|}\int_B \left|\int_{\mathbf{R}^n} K(x,y)f_2(y)\,dy-\frac{1}{|B|}\int_B\left(\int_{\mathbf{R}^n} K(z,y)f_2(y)\,dy\right)dz\right|dx\\
&=\frac{1}{|B|^2}\int_B \left|\int_{\mathbf{R}^n}\int_B K(x,y)f_2(y)-K(z,y)f_2(y)\,dy\,dz\right|dx\\
&=\frac{1}{|B|^2}\int_{B\times B\times \mathbf{R}^n\setminus 2B} |K(x,y)-K(z,y)|\cdot|f(y)|\,dz\,dx\,dy\\
&\leqslant c_1\,\frac{1}{|B|^2}\int_{B\times B\times \mathbf{R}^n\setminus 2B}\frac{r(B)^{\varepsilon}|f(y)|}{(|y-c(B)|-r(B))^{n+\varepsilon-\alpha}}\,dz\,dx\,dy\\
&\leqslant c_1\int_{\mathbf{R}^n\setminus 2B}\frac{r(B)^{\varepsilon}|f(y)|}{(|y-c(B)|-r(B))^{n+\varepsilon-\alpha}}\,dy.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{(|y-c(B)|-r(B))^{n+\varepsilon-\alpha}}&=(n+\varepsilon-\alpha)\int_{|y-c(B)|-r(B)}^{\infty}\frac{d\ell}{\ell^{1+n+\varepsilon-\alpha}}\\
&=(n+\varepsilon-\alpha)\int_0^{\infty}\chi_{B(c(B),r(B)+\ell)}(y)\,\frac{d\ell}{\ell^{1+n+\varepsilon-\alpha}}.
\end{aligned}$$

Hence it follows that

$$\frac{1}{|B|}\int_B |Kf_2(x)-m_B(Kf_2)|\,dx\leqslant c_1\,(n+\varepsilon-\alpha)\int_0^{\infty}\frac{(1+\ell)^{n-\frac{n}{p}}\,d\ell}{\ell^{1+n+\varepsilon-\alpha}}\,\|f\|_p.$$

In view of the definition of the BMO spaces, the proof follows.

Finally let us consider the last assertion. Let us fix two (distinct) points x,y and set

$$r=2|x-y|.$$

Then we have

$$Kf(x)-Kf(y)=\int_{B(x,r)}+\int_{B(y,r)\setminus B(x,r)}+\int_{\mathbf{R}^n\setminus B(x,r)\cup B(y,r)}(k(x,z)-k(y,z))f(z)\,dz.$$

As for the first term, we use the second assertion,

$$\begin{aligned}
\int_{B(x,r)}|(k(x,z)-k(y,z))f(z)|\,dz&\leqslant 2\int_{B(x,r)}\frac{|f(z)|}{|x-z|^{n-\alpha}}\,dz\\
&\leqslant 2\|f\|_p\left(\int_{B(x,r)}\frac{1}{|x-z|^{p'(n-\alpha)}}\,dz\right)^{\frac{1}{p'}}\\
&\leqslant 2\|f\|_p\left(\int_0^r\frac{|S^1|}{R^{p'(n-\alpha)-n+1}}\,dR\right)^{\frac{1}{p'}}\\
&\leqslant 2\|f\|_p\left(\frac{|S^1|}{(n-p'(n-\alpha))}\right)^{\frac{1}{p'}}r^{n/p'-(n-\alpha)}.
\end{aligned}$$

The second term can be estimated similarly.

As for the last term, we have

$$\int_{\mathbf{R}^n\setminus B(x,r)\cup B(y,r)} |k(x,z)-k(y,z)||f(z)|\,dz \leqslant \int_{\mathbf{R}^n\setminus B(x,r)} \frac{c_1\,|x-y|^{\varepsilon}|f(z)|\,dz}{(|x-z|-|x-y|)^{n+\varepsilon-\alpha}}.$$

A similar computation yields

$$\int_{\mathbf{R}^n\setminus B(x,r)} \frac{c_1\,|x-y|^{\varepsilon}|f(z)|\,dz}{(|x-z|-|x-y|)^{n+\varepsilon-\alpha}} \leqslant \|f\|_p|x-y|^{n-\alpha}\left(\int_1^{\infty}\frac{(\ell+1)^{\frac{1}{p'}}}{\ell^{n+\varepsilon-\alpha+1}}\right)^{\frac{1}{p'}}.$$

By combining the estimates above, we obtain the last inequality and the proof is complete.

We are in the position of applying Theorem 1 to the special case :

$$K(x,y)=\frac{1}{(|x-y|+\delta)^{n-\alpha}}-\frac{1}{(|x-y|+\delta')^{n-\alpha}}. \tag{2.8}$$

Then, we can obtain the desired error estimates

THEOREM 2. *1. Assume that*

$$1<\alpha<n+1,\,1<p<q<\infty,\,\frac{1}{q}=\frac{1}{p}-\frac{\alpha-1}{n}.$$

Then, the operator K defined by (2.8), satisfies $\|K\|_{L^p\to L^q}\leqslant c\,|\delta-\delta'|$, where c is independent of δ,δ'.

2. *Assume that*

$$1<\alpha<n+1,\,p=\frac{n}{\alpha-1}.$$

Then, the operator K defined by (2.8), satisfies $\|K\|_{L^p\to BMO}\leqslant c\,|\delta-\delta'|$, where c is independent of δ,δ'.

3. *Assume that*

$$1<\alpha<n+1,\,0<\beta=\alpha-1-\frac{n}{p}<1.$$

Then, the operator K defined by (2.8), satisfies $\|K\|_{L^p\to Lip^{\beta}}\leqslant c\,|\delta-\delta'|$, where c is independent of δ,δ'.

Proof. Note that

$$|K(x,y)|\leqslant\frac{|\delta-\delta'|}{|x-y|^{n-\alpha+1}}$$

for all $x,y\in\mathbf{R}^n$ with $x\neq y$ and

$$|\partial_x K(x,y)|=|\partial_y K(x,y)|\leqslant\frac{(n-\alpha)|\delta-\delta'|}{|x-y|^{n-\alpha+2}}$$

for all $x,y\in\mathbf{R}^n$ with $x\neq y$. As a consequence we can use Theorem 1 with $c_0=|\delta-\delta'|$ and $c_1=(n-\alpha)|\delta-\delta'|$ and the proof follows.

3. The logarithmic potential case

In this section, we shall consider the integral operator

$$If(x) := \int_{\Omega} f(y) \log|x-y|\, dy,$$

where Ω is a bounded domain in $\mathbf{R}^n$. Here, we assume that $n \geqslant 2$.

Let us set

$$\Omega^* := \{x-y : x, y \in \Omega\},$$

which is still bounded. Assume that $\Omega \subset B(R)$, where $B(R)$ denotes the open ball centered at the origin of radius $R > 0$. We also denote by σ_n the volume of the unit surface: $\sigma_n = 2\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)^{-1}$.

Our main result in this section reads:

THEOREM 3. *Let $n \geqslant 2$ and $1 \leqslant p \leqslant \infty$. Suppose Ω is a bounded domain in $\mathbf{R}^n$.*

1. The operator I given by

$$If(x) := \int_{\Omega} f(y) \log|x-y|\, dy$$

converges for almost every $x \in \Omega$, whenever $f \in L^1(\Omega)$.

2. The operator I is $L^p(\Omega)$-bounded. More precisely, we have the following inequality

$$\|I\|_{L^p(\Omega) \to L^p(\Omega)} \leqslant M(I) := \sigma_n \int_0^{2R} r^{n-1} |\log r|\, dr.$$

3. Let $\delta > 0$ and define

$$I_\delta f(x) := \int f(y) \log(|x-y| + \delta)\, dy.$$

Then we have the following inequality

$$\| I_\delta - I \|_{L^p(\Omega) \to L^p(\Omega)} \leqslant \frac{2^{n-1}\delta}{n-1} R^{n-1}.$$

We remark that $M^*(I)$ is finite because we are assuming that $n \geqslant 2$.

Proof. Let us begin with the proof of *2.* because *1.* is tacitly contained in *2.*. Let $f \in L^p(\Omega)$ and $x \in \Omega$. Then we have

$$If(x) = \int_{\Omega} f(y) \cdot (\chi_{\Omega^*}(x-y) \log|x-y|)\, dy.$$

Below it will be understood that any function f defined in Ω or Ω^* is extended by setting $f \equiv 0$ outside the domain where f is defined. By the Young inequality we have

$$\|If\|_{L^p(\Omega)} = \|If\|_{L^p} \leqslant \|\chi_{\Omega^*} \log|\cdot|\,\|_1 \|f\|_{L^p} = \|\chi_{\Omega^*} \log|\cdot|\,\|_1 \|f\|_{L^p(\Omega)}.$$

If we write $\|\chi_{\Omega^*}\log|\cdot|\|_1$ in the polar coordinate, then we have

$$\|\chi_{\Omega^*}\log|\cdot|\|_1 \leqslant \|\chi_{B(2R)}\log|\cdot|\|_1 = \sigma_n\int_0^{2R} r^{n-1}|\log r|\,dr$$

and the proof of *2.* follows. Also the proof of *3.* is similar. In fact, note that the operator norm of $I-I_\delta$ is bounded by

$$\int_{\Omega^*}|\log|y|-\log(|y|+\delta)|\,dy.$$

If we use the trivial estimate $\log(1+t)\leqslant 1+t$, then we obtain

$$\begin{aligned}\int_{\Omega^*}|\log|y|-\log(|y|+\delta)|\,dy &= \int_{\Omega^*}\log\left(1+\frac{\delta}{|y|}\right)dy\\ &\leqslant \int_{B(2R)}\log\left(1+\frac{\delta}{|y|}\right)dy \leqslant \int_{B(2R)}\frac{\delta}{|y|}\,dy.\end{aligned}$$

Again by writing the integral in the polar coordinate we obtain

$$\int_{B(2R)}\frac{\delta}{|y|}\,dy = \delta\,\sigma_n\int_0^{2R} r^{n-2}\,dr = \frac{2^{n-1}\delta}{n-1}R^{n-1}.$$

Therefore, the proof is complete.

4. Various inequalities connected to the operator I

THEOREM 4. *Let $1\leqslant p\leqslant q\leqslant\infty$. Define P by*

$$1+\frac{1}{q}=\frac{1}{p}+\frac{1}{P}.$$

Then we have

$$\|If\|_{L^q(\Omega)} \leqslant {v_n}^{\frac{1}{P}}\left(\int_0^{2R} r^{n-1}|\log r|^P\,dr\right)^{\frac{1}{P}}\|f\|_{L^p(\Omega)},$$

where R is a constant such that $\Omega\subset B(R)$.

Proof. Let us keep to the same notation and the same convention as the proof of Theorem 3. By the Hölder inequality we have

$$|If(x)| \leqslant \int_{\mathbf{R}^n}|f(y)|\chi_{\Omega^*}|\log(x-y)|\,dy.$$

Therefore, by the Young inequality, we obtain

$$\|If\|_{L^q} \leqslant \|f\|_{L^p}\left(\int_{\Omega^*}|\log x|^P\,dx\right)^{\frac{1}{P}} \leqslant \|f\|_{L^p}\left(\int_{B(2R)}|\log x|^P\,dx\right)^{\frac{1}{P}}$$

If we write $\left(\int_{B(2R)} |\log x|^P \, dx\right)^{\frac{1}{P}}$ in the polar coordinate, then we obtain the desired result.

Now we consider the condition for If to be continuous.

THEOREM 5. *Let $1 \leqslant p < \infty$. Then If belongs to BUC, the set of all continuous functions, with the norm estimate*

$$\|If\|_{BUC} \leqslant v_n{}^{\frac{1}{p'}} \left(\int_0^{2R} r^{n-1} |\log r|^{p'} \, dr\right)^{\frac{1}{p'}} \|f\|_{L^p(\Omega)}.$$

Proof. Let $p < \infty$. If f is a continuous function with compact support and contained in Ω, then $If = (\chi_{\Omega^*} \log|\cdot|) * f$ is continuous and supported compactly in $\mathbf{R}^n$. As a result If is a uniformly continuous function for such f. Now, since the set of all continuous functions with support contained in Ω forms a dense subset in $L^p(\Omega)$, we obtain $f \in BUC$ for all $f \in L^p(\Omega)$.

If $p = \infty$, then inclusion $f \in L^\infty(\Omega) \subset L^1(\Omega)$ gives us that $If \in BUC$ if $f \in L^\infty(\Omega)$. The proof is therefore complete.

Truncation procedure (Subcritical case)

THEOREM 6. *Assume that $1 \leqslant p \leqslant \infty, 0 < \theta \leqslant 1$ satisfies $p'\theta < n$. Then we have*

$$\| I - I_\delta \|_{L^p(\Omega) \to BUC} \leqslant c_{p,\theta} \delta \, v_n{}^{\frac{1}{p'}} (n - p')^{-\frac{1}{p'}} R^{\frac{n}{p'}-1}.$$

Proof. As before, the crux of the proof is to show

$$\| I - I_\delta \|_{L^p(\Omega) \to L^\infty(\Omega)} \leqslant c_\theta \delta^\theta \left(\int_{B(R)} \frac{dx}{|x|^{p'\theta}}\right).$$

The difference lies in the point that we use

$$\log(1+t) \leqslant c_\theta t^\theta, t > 0.$$

Indeed, if we write $I - I_\delta$ out in full, we obtain

$$If(x) - I_\delta f(x) = \int_\Omega f(y) \log\left(1 + \frac{\delta}{|x-y|}\right) dy.$$

Hence it follows that

$$|If(x) - I_\delta f(x)| \leqslant \int_\Omega |f(y)| \chi_{\Omega^*}(x-y) \log\left(1 + \frac{\delta}{|x-y|}\right) dy,$$

where $\Omega^* = \{x - y : x, y \in \Omega\}$. As a consequence, it follows that

$$|If(x) - I_\delta f(x)| \leqslant c_\theta \int_\Omega |f(y)| \chi_{\Omega^*}(x-y) \left(\frac{\delta}{|x-y|}\right)^\theta dy.$$

If we use the Young inequality, then we obtain

$$\|If - I_\delta f\|_{L^\infty(\Omega)} \leqslant \|f\|_{L^p(\Omega)} \cdot \left\{ \int_{B(2R)} \left(\frac{\delta}{|x-y|} \right)^{p'\theta} dy \right\}^{\frac{1}{p'}}.$$

If we calculate $\left\{ \int_{B(2R)} \left(\frac{\delta}{|x-y|} \right)^{p'\theta} dy \right\}^{\frac{1}{p'}}$, we have the desired result.

Truncation procedure (Limit case)

THEOREM 7. *Let* $0 < \delta < 1$. *Then we have*

$$\|I - I_\delta\|_{L^{\frac{n}{n-1}}(\Omega) \to Lip(\delta)} \leqslant c\delta,$$

where c *is independent of* δ.

Proof. This is just a corollary of Theorem 1.

5. Some numerical experiments

For the case of $n = 2$ in Theorem 6, Figure 1 shows the graphs of the difference $If(x) - I_\delta f(x)$ on $\Omega = [0,1]^2$ for $f(y) = 1$ (left), $y_1 + y_2$ (middle), $y_1 y_2$ (right), $\delta = 10^{-2}$ (bottom), 10^{-3} (middle), 10^{-4} (top). For the case of $n = 3, \alpha = 2$ in Theorem 1, Figure 2 shows the graphs of the differences $Kf|_{[0,1]^2 \times \{0.5\}}$ on $[0,1]^2 \times \{0.5\}$ for $f(y) = 1$ (left), $y_1 + y_2$ (middle), $y_1 y_2$ (right), $\delta = 10^{-2}$ (bottom), 10^{-3} (middle), 10^{-4} (top), $\delta' = 10^{-10}$. From these figures we see that the differences $If(x) - I_\delta f(x)$ and $Kf(x) - K_\delta f(x)$ seem to converge to zero as parameters δ and δ' tend to zero.

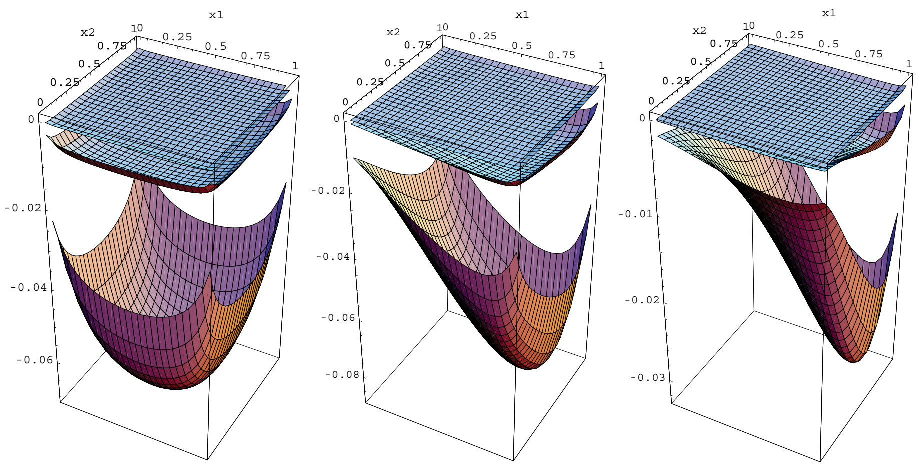

Figure 1. (case of $n = 2$ in Theorem 6): For $f(y) = 1$ (left), $y_1 + y_2$ (middle), $y_1 y_2$ (right), $\delta = 10^{-2}$ (bottom), 10^{-3} (middle), 10^{-4} (top), the graphs of $If(x) - I_\delta f(x)$ on $\Omega = [0,1]^2$.

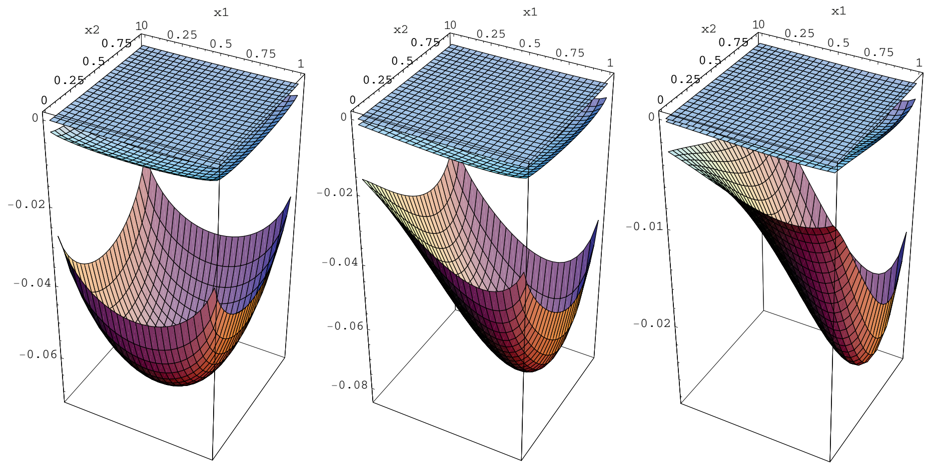

Figure 2. (case of $n = 3$, $\alpha = 2$ in Theorem 1): For $f(y) = 1$ (left), $y_1 + y_2$ (middle), $y_1 y_2$ (right), $\delta = 10^{-2}$ (bottom), 10^{-3} (middle), 10^{-4} (top), $\delta' = 10^{-10}$, the graphs of $Kf|_{[0,1]^2 \times \{0.5\}}$ on $[0,1]^2 \times \{0.5\}$.

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