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# INEQUALITIES FOR $L_p$–DUAL AFFINE SURFACE AREA

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*Abstract.* According to the notion of $L_p$-affine surface area which was defined by Lutwak, in this paper, we introduce the concept of $L_p$-dual affine surface area. Further, we establish the affine isoperimetric inequality and the Blaschke-Santaló inequality for $L_p$-dual affine surface area. Besides, the dual Brunn-Minkowski inequality for $L_p$-dual affine surface area is presented.

## 1. Introduction

During the past three decades, the investigations of the classical affine surface area have received great attention from many articles (see articles [3–6, 9, 10, 12, 13, 15, 16, 21, 25, 27] or books [7, 24]). Based on the classical affine surface area, Lutwak [18] introduced the notion of $L_p$-affine surface area and obtained some isoperimetric inequalities for it. Regarding the studies of $L_p$-affine surface area also see [20, 26, 28].

Let $\mathscr{K}_0^n$ be the set of convex bodies in $\mathbb{R}^n$ containing the origin in their interiors. For $K \in \mathscr{K}_0^n$, let $K^*$ denote the polar body of $K$ and $\mathscr{S}_0^n$ denote the set star bodies in $\mathbb{R}^n$ containing the origin in their interiors.

In [5], Leichtweiß defined the affine surface area $\Omega(K)$ by:

$$n^{-\frac{1}{n}}\Omega(K)^{\frac{n+1}{n}} = \inf\{nV_1(K,Q^*)V(Q)^{\frac{1}{n}} : Q \in \mathscr{S}_0^n\}. \tag{1.1}$$

In [18], Lutwak generalized to the $L_p$-affine surface area $\Omega_p(K)$ by using the Brunn-Minkowski-Fiery theory as follows:

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathscr{S}_0^n\}. \tag{1.2}$$

Obviously, if $p = 1$, $\Omega_1(K)$ is just the classical affine surface area $\Omega(K)$.

Moreover, Lutwak proved the following inequalities for the $L_p$-affine surface area.

THEOREM A. *Let $K \in \mathscr{K}_0^n$ and $p \geqslant 1$. Then*

$$\Omega_p(K)^{n+p} \leqslant n^{n+p}V(K)^nV(K^*)^p. \tag{1.3}$$

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THEOREM B. *Let $K \in \mathscr{K}_0^n$ and $p \geqslant 1$. Then*

$$\Omega_p(K)\Omega_p(K^*) \leqslant n^2 V(K)V(K^*). \tag{1.4}$$

It has slowly come to be recognized that along side the Brunn-Minkowski theory there is a dual theory. It is easily seen that the $L_p$-affine surface area belongs to the Brunn-Minkowski theory. A nature question is whether there is a dual analog of the $L_p$-affine surface area in the dual Brunn-Minkowski theory.

It is the aim of this paper to demonstrate the existence of this dual object, which is called $L_p$-dual affine surface area.

For star body $K$ and $0 < p < n$, we define the $L_p$-dual affine surface area, $\widetilde{\Omega}_p(K)$, of $K$ by

$$n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n\widetilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathscr{K}_c^n\}. \tag{1.5}$$

As applications, we establish the dual forms of inequalities (1.3) and (1.4). Our main results can be stated as follows:

THEOREM 1.1. *Let $K \in \mathscr{K}_c^n$ and $0 < p < n$. Then*

$$\widetilde{\Omega}_p(K)^{n+p} \geqslant n^{n+p} V(K)^n V(K^*)^p. \tag{1.6}$$

THEOREM 1.2. *Let $K \in \mathscr{K}_c^n$ and $0 < p < n$. Then*

$$\widetilde{\Omega}_p(K)\widetilde{\Omega}_p(K^*) \geqslant n^2 V(K)V(K^*). \tag{1.7}$$

The other aim of this paper is to establish the dual Brunn-Minkowski inequality for the $L_p$-dual affine surface area.

THEOREM 1.3. *Let $K, L \in \mathscr{S}_0^n$ and $1 \leqslant p \leqslant n-1$. Then*

$$\widetilde{\Omega}_p(K \tilde{+}_{n-p} L)^{\frac{n+p}{n}} \leqslant \widetilde{\Omega}_p(K)^{\frac{n+p}{n}} + \widetilde{\Omega}_p(L)^{\frac{n+p}{n}}, \tag{1.8}$$

*with equality if and only if $K$ and $L$ are dilates.*

Please see the next section for above interrelated notations, definitions and their background materials.

## 2. Notation and preliminaries

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) and $\mathscr{K}_0^n$ denote the subset of $\mathscr{K}^n$ that contains the origin in their interiors in $\mathbb{R}^n$. Let $\mathscr{K}_c^n$ denote the set of convex bodies whose centroids lie at the origin. As usual, $S^{n-1}$ denotes the unit sphere, $B_n$ the unit ball, $\omega_n$ the volume of $B_n$.

For $\phi \in GL(n)$, let $\phi^t, \phi^{-1}$, and $\phi^{-t}$, denote the transpose, inverse, and inverse of the transpose of $\phi$, respectively. If $K \in \mathscr{K}^n$, then the support function of $K$, $h_K = h(K,\cdot) : \mathbb{R}^n \longrightarrow (0,\infty)$, is defined by

$$h(K,u) = \max\{u \cdot x : x \in K\}, \qquad u \in S^{n-1},$$

where $u \cdot x$ denotes the standard inner product of $u$ and $x$.

For a compact subset $L$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L,\cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$,

$$\rho(L,u) = \rho_L(u) = \max\{\lambda > 0 : \lambda u \in L\}. \tag{2.1}$$

If $\rho(L,\cdot)$ is continuous and positive, $L$ will be called a star body, and $\mathscr{S}_0^n$ will be used to denote the class of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. Two star bodies $K$ and $L$ are said to be dilates (of one another) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$.

Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows: if $K,L \in \mathscr{S}_0^n$, then

$$\tilde{\delta}(K,L) = | \rho_K - \rho_L |_\infty . \tag{2.2}$$

If $L \in \mathscr{S}_0^n$, then obviously

$$\rho(\phi L,u) = \rho(L,\phi^{-1}u), \ \ for \ \ \phi \in GL(n). \tag{2.3}$$

For $K \in \mathscr{K}_0^n$, let $K^*$ denote the polar of $K$; i.e.,

$$K^* = \{x \in \mathbb{R}^n : \ x \cdot y \leqslant 1, \text{ for all } y \in K\}. \tag{2.4}$$

It is easy to get that

$$\rho(K^*,\cdot) = 1/h(K,\cdot) \quad \text{ and } \quad h(K^*,\cdot) = 1/\rho(K,\cdot).$$

For $0 < p < n$, the $L_p$-dual mixed volume, $\widetilde{V}_p(K,L)$, of $K,L \in \mathscr{S}_0^n$, is defined by

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L,u)^p dS(u). \tag{2.5}$$

Obviously,

$$\widetilde{V}_p(K,K) = V(K). \tag{2.6}$$

## 3. $L_p$-dual affine surface area

LEMMA 3.1. *If $0 < p < n$, and $K,L \in \mathscr{S}_0^n$, then for $\phi \in SL(n)$,*

$$\widetilde{V}_p(\phi K,\phi L) = \widetilde{V}_p(K,L). \tag{3.1}$$

*Proof.* From (2.3) and (2.5), we have

$$\begin{aligned}
\widetilde{V}_p(\phi K,\phi L) &= \frac{1}{n}\int_{S^{n-1}} \rho(K,\phi^{-1}u)^{n-p}\rho(L,\phi^{-1}u)^p du \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K,z)^{n-p}\rho(L,z)^p d(\phi z) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K,z)^{n-p}\rho(L,z)^p dz \\
&= \widetilde{V}_p(K,L). \qquad \square
\end{aligned}$$

LEMMA 3.2. *If* $0 < p < n$, *and* $K, L \in \mathscr{S}_0^n$, *then*

$$\widetilde{V}_p(K,L)^n \leqslant V(K)^{n-p}V(L)^p, \tag{3.2}$$

*with equality if and only if* $K$ *and* $L$ *are dilates.*

*Proof.* Using (2.5) and the Hölder inequality, we get

$$\begin{aligned}\widetilde{V}_p(K,L) &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L,u)^p dS(u)\\ &\leqslant \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n dS(u)\right)^{\frac{n-p}{n}} \left(\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^n dS(u)\right)^{\frac{p}{n}}\\ &= V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}.\end{aligned}$$

According to the condition of equality holds in Hölder inequality, we see with equality in inequality (3.2) if and only if $\rho(K,u) = c\rho(L,u)$, for any $u \in S^{n-1}$, i.e., $K$ and $L$ are dilates. □

LEMMA 3.3. *If* $0 < p < n$, *and* $L \in \mathscr{K}_0^n$, *then* $\widetilde{V}_p(\cdot,L): \ \mathscr{S}_0^n \to (0,\infty)$ *is continuous.*

*Proof.* If $K_1, K_2 \in \mathscr{S}_0^n$, suppose $K_1 \to K_2$, by definition, $\rho_{K_1} \to \rho_{K_2}$, uniformly on $S^{n-1}$. Since both $\rho_{K_1}$ and $\rho_{K_2}$ are continuous and uniformly bounded away from 0. It follows that $\rho_{K_1}^{n-p} \to \rho_{K_2}^{n-p}$, uniformly on $S^{n-1}$, and thus that

$$\rho_{K_1}^{n-p}\rho_L^p \to \rho_{K_2}^{n-p}\rho_L^p,$$

uniformly on $S^{n-1}$. Then the desired result is obtained. □

An immediate consequence of the definition of $\widetilde{\Omega}_p$ and Lemma 3.1 is:

LEMMA 3.4. *If* $0 < p < n$ *and* $K \in \mathscr{S}_0^n$, *then*

$$\widetilde{\Omega}_p(\phi K) = \widetilde{\Omega}_p(K), \tag{3.3}$$

*for all* $\phi \in SL(n)$.

An immediate consequence of the definition of $\widetilde{\Omega}_p$ and Lemma 3.3 is:

LEMMA 3.5. *For* $0 < p < n$, *the functional,* $\widetilde{\Omega}_p : \mathscr{S}_0^n \to (0,\infty)$, *is lower semi-continuous.*

THEOREM 3.6. *If* $0 < p < n$ *and* $K \in \mathscr{S}_0^n$, *then*

$$\widetilde{\Omega}_p(K)^{n+p} \leqslant n^{n+p}\omega_n^{2p}V(K)^{n-p}, \tag{3.4}$$

*with equality if and only if* $K$ *is an ellipsoid.*

*Proof.* By Lemma 3.2 and the Blaschke-Santaló inequality, we get

$$\widetilde{V}_p(K,Q^*)^n V(Q)^p \leqslant V(K)^{n-p}V(Q^*)^p V(Q)^p \leqslant \omega_n^{2p}V(K)^{n-p}.$$

According to the definition $\widetilde{\Omega}_p(K)$, the desired inequality is obtained. By the equality condition of Blaschke-Santaló inequality and (3.2), the equality of (3.4) holds if and only if $K$ is an ellipsoid. □

COROLLARY 3.7. *If* $0 < p < n$ *and* $K \in \mathscr{S}_0^n$, *then*

$$\widetilde{\Omega}_p(K)\widetilde{\Omega}_p(K^*) \leqslant (n\omega_n)^2, \tag{3.5}$$

*with equality if and only if* $K$ *is an ellipsoid.*

*Proof.* Theorem 3.6 for $K$ and $K^*$, immediately yields:

$$\widetilde{\Omega}_p(K)^{n+p} \leqslant n^{n+p}\omega_n^{2p}V(K)^{n-p}, \tag{3.6}$$

$$\widetilde{\Omega}_p(K^*)^{n+p} \leqslant n^{n+p}\omega_n^{2p}V(K^*)^{n-p}, \tag{3.7}$$

Combining with (3.6) and (3.7), we obtain

$$\widetilde{\Omega}_p(K)^{n+p}\widetilde{\Omega}_p(K^*)^{n+p} \leqslant n^{2(n+p)}\omega_n^{4p}(V(K)V(K^*))^{n-p}.$$

Using the Blaschke-Santaló inequality, we have

$$\widetilde{\Omega}_p(K)\widetilde{\Omega}_p(K^*) \leqslant (n\omega_n)^2.$$

By the equality condition of Blaschke-Santaló inequality and (3.4), the equality of (3.5) holds if and only if $K$ is an ellipsoid □

*Proof of Theorem 1.1.* From definition (1.5), it follows that for $Q \in \mathscr{K}_c^n$,

$$n^{-p}\widetilde{\Omega}_p(K)^{n+p} \geqslant n^n\widetilde{V}_p(K,Q^*)^nV(Q)^p.$$

Take $K^*$ for $Q$, and notice that $K^{**} = K$, we get

$$\widetilde{\Omega}_p(K)^{n+p} \geqslant n^{n+p}\widetilde{V}_p(K,K)^nV(K^*)^p = n^{n+p}V(K)^nV(K^*)^p.$$

Then the desired inequality is obtained. □

*Proof of Theorem 1.2.* Theorem 1.1 for $K$ and $K^*$, immediately yields:

$$\widetilde{\Omega}_p(K)^{n+p} \geqslant n^{n+p}V(K)^nV(K^*)^p, \tag{3.8}$$

$$\widetilde{\Omega}_p(K^*)^{n+p} \geqslant n^{n+p}V(K^*)^nV(K)^p. \tag{3.9}$$

Combining with (3.8) and (3.9), we obtain that

$$\widetilde{\Omega}_p(K)\widetilde{\Omega}_p(K^*) \geqslant n^2V(K)V(K^*).$$

We complete the proof of the Theorem 1.2. □

Using the reverse Blaschke-Santaló inequality (see [8], [23]),

$$V(K)V(K^*) \geqslant \frac{4^n}{n!},$$

with equality if and only if $K$ is a zonoids, we can get the following corollary.

COROLLARY 3.8. *Let* $K$ *be a zonoids in* $\mathbb{R}^n$ *and* $0 < p < n$. *Then*

$$\widetilde{\Omega}_p(K)\widetilde{\Omega}_p(K^*) \geqslant \frac{n^2 4^n}{n!}.$$

## 4. Dual Brunn-Minkowski inequality for the $L_p$-dual affine surface area

Let $\mathscr{F}_s^n$ denote the set of all origin-symmetric convex bodies which have a positive continuous function, and $K\breve{+}_pL$ denote the Blaschke $L_p$-combination of convex bodies $K$ and $L$. In [28], Wang established the Brunn-Minkowski inequality for the $L_p$-affine surface area as follows:

THEOREM C. [28] *Let $K,L\in\mathscr{F}_s^n$ and $n\neq p\geqslant 1$. Then*

$$\Omega_p(K\breve{+}_pL)^{\frac{n+p}{p}}\geqslant\Omega_p(K)^{\frac{n+p}{p}}+\Omega_p(L)^{\frac{n+p}{p}}, \tag{4.1}$$

*with equality for $p=1$ if and only if $K$ and $L$ are homothetic, for $p>1$ if and only if $K$ and $L$ are dilates.*

Let $K$ and $L$ be star bodies in $\mathbb{R}^n$ and $p\geqslant 1$, the $L_p$-radial linear combination $K\tilde{+}_pL$ is a star body whose radial function is given by

$$\rho(K\tilde{+}_pL,u)^p=\rho(K,u)^p+\rho(L,u)^p. \tag{4.2}$$

Under the definition of the $L_p$-radial linear combination, we can give the dual Brunn-Minkowski inequality for the $L_p$-dual affine surface area, the Theorem 1.3, which is just the dual of Theorem C.

*Proof of Theorem 1.3.* From the definition, we have:

$$\begin{aligned}
&n^{-\frac{p}{n}}\widetilde{\Omega}_p(K\tilde{+}_{n-p}L)^{\frac{n+p}{n}}\\
&\quad=\sup\{n\widetilde{V}_p(K\tilde{+}_{n-p}L,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{K}_c^n\}\\
&\quad=\sup\{n[\widetilde{V}_p(K,Q^*)+V_p(L,Q^*)]V(Q)^{\frac{p}{n}}:Q\in\mathscr{K}_c^n\}\\
&\quad\leqslant\sup\{n\widetilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{K}_c^n\}+\sup\{n\widetilde{V}_p(L,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{K}_c^n\}\\
&\quad=n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}}+n^{-\frac{p}{n}}\widetilde{\Omega}_p(L)^{\frac{n+p}{n}}.
\end{aligned} \tag{4.3}$$

This shows that

$$\widetilde{\Omega}_p(K\tilde{+}_{n-p}L)^{\frac{n+p}{n}}\leqslant\widetilde{\Omega}_p(K)^{\frac{n+p}{n}}+\widetilde{\Omega}_p(L)^{\frac{n+p}{n}}.$$

The equality of (1.8) holds if and only if $K\tilde{+}_{n-p}L$ are dilates with $K$ and $L$, respectively. That is, $K$ and $L$ are dilates. □

## 5. Monotonicity results

Winternitz gave the monotonicity result for affine surface area as follows:

LEMMA 5.1. [5] *If $E$ is a centered ellipsoid, and $K\in\mathscr{K}_c^n$, and $K\subset E$, then*

$$\Omega(K)\leqslant\Omega(E). \tag{5.1}$$

An important generalization of the above inequality was established by Lutwak as follows:

LEMMA 5.2. [18] *If $E$ is a centered ellipsoid, and $K \in \mathscr{K}_c^n$, and if either $K \subset E$, for $1 \leqslant p < n$, or $E \subset K$, for $p > n$, then*

$$\Omega_p(K) \leqslant \Omega_p(E), \tag{5.2}$$

*with equality if and only if $K = E$.*

In this section, we obtain the monotonicity result for the $L_p$-dual affine surface area.

THEOREM 5.3. *If $E$ is a centered ellipsoid, and $K \in \mathscr{K}_c^n$, and if $K \subset E$, for $0 < p < n$, then*

$$\widetilde{\Omega}_p(K) \leqslant \widetilde{\Omega}_p(E), \tag{5.3}$$

*with equality if and only if $K = E$.*

*Proof.* From Lemma 3.3 and the fact that $K \subset E$, for $0 < p < n$, follows

$$\widetilde{\Omega}_p(K)^{n+p} \leqslant n^{n+p}\omega_n^{2p}V(K)^{n-p} \leqslant n^{n+p}\omega_n^{2p}V(E)^{n-p} = \widetilde{\Omega}_p(E)^{n+p}.$$

This shows that $\widetilde{\Omega}_p(K) = \widetilde{\Omega}_p(E)$, would imply that $V(K) = V(E)$, and hence $K = E$. $\square$

In [18], Lutwak defined the $L_p$ affine area ratio of $K$ as follows: For $K \in \mathscr{K}_0^n$, the $L_p$ affine area ratio of $K$ was defined by

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{1/p}. \tag{5.4}$$

Lutwak proved the $L_p$ affine area ratio of $K$ does not exceed the Santaló product of $K$, he also proved (5.4) is monotone nondecreasing in $p$:

THEOREM 5.4. [18] *If $K \in \mathscr{K}_0^n$, and $1 \leqslant p \leqslant q$, then*

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{\frac{1}{p}} \leqslant \left(\frac{\Omega_q(K)^{n+q}}{n^{n+q}V(K)^{n-q}}\right)^{\frac{1}{q}}. \tag{5.5}$$

For $K \in \mathscr{K}_0^n$, we define the $L_p$-dual affine area ratio of $K$ by

$$\left(\frac{\widetilde{\Omega}_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{1/p}.$$

THEOREM 5.4*. *If $K \in \mathscr{K}_0^n$, and $0 < p \leqslant q < n$, then*

$$\left(\frac{\widetilde{\Omega}_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{\frac{1}{p}} \leqslant \left(\frac{\widetilde{\Omega}_q(K)^{n+q}}{n^{n+q}V(K)^{n-q}}\right)^{\frac{1}{q}}. \tag{5.6}$$

*Proof.* For $K, L \in \mathscr{K}_0^n$, since $0 < p \leqslant q < n$ and

$$\rho(K,u)^{n-p}\rho(L^*,u)^p = (\rho(K,u)^{n-q}\rho(L^*,u)^q)^{\frac{p}{q}}(\rho(K,u)^n)^{\frac{q-p}{q}},$$

the Hölder inequality, together with (2.5), we obtain

$$\begin{aligned}\widetilde{V}_p(K,L^*) &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L^*,u)^p dS(u)\\ &= \frac{1}{n}\int_{S^{n-1}} (\rho(K,u)^{n-q}\rho(L^*,u)^q)^{\frac{p}{q}}(\rho(K,u)^n)^{\frac{q-p}{q}} dS(u)\\ &\leqslant \widetilde{V}_q(K,L^*)^{\frac{p}{q}} V(K)^{\frac{q-p}{q}},\end{aligned}$$

that is

$$[\widetilde{V}_p(K,L^*)/V(K)]^{\frac{1}{p}} \leqslant [\widetilde{V}_q(K,L^*)/V(K)]^{\frac{1}{q}}. \tag{5.7}$$

The definition of $\widetilde{\Omega}_p(K)$ can be written as

$$V(K)\left(\frac{\widetilde{\Omega}_p(K)}{nV(K)}\right)^{\frac{n+p}{p}} = \sup\{[\widetilde{V}_p(K,Q^*)/V(K)]^{\frac{n}{p}} V(Q) : \ Q \in \mathscr{K}_c^n\}.$$

Associated with (5.7), we can get (5.6). □

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