Mathematical
Inequalities
& Applications
Volume 13, Number 2 (2010), 339–347

ON THE TWO–POINT OSTROWSKI INEQUALITY

JOSIP PEČARIĆ AND ŠIME UNGAR

(*Communicated by M. Matić*)

Abstract. We prove the L_p-version of an inequality similar to the two-point Ostrowski inequality of Matić and Pečarić [4].

1. Introduction

For a function $f\colon [a,b]\to\mathbb{R}$ satisfying the Lipschitz condition with constant $M>0$, and $a\leqslant c<d\leqslant b$, Matić and Pečarić [4] proved the following two-point Ostrowski inequality:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x\right|\leqslant\frac{(c-a)^2+(b-d)^2}{2(c-a+b-d)}\cdot M.$$

This result was generalized by Pečarić, Perić and Vukelić in [6]. Further generalizations were done by Aglić Aljinović, Pečarić and Perić in [1], where they consider also the L_p-cases, $1\leqslant p\leqslant\infty$, as well as the general case when $[c,d]\not\subseteq[a,b]$. For instance, they proved that for $a\leqslant c<b\leqslant d$ and for a function f such that $|f'|^p$ is R-integrable on $[a,d]$, the following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t\right|$$
$$\leqslant\left(\frac{1}{(q+1)(a-b+d-c)}\cdot\left(\frac{(d-b)^{q+1}}{(d-c)^{q-1}}-\frac{(c-a)^{q+1}}{(b-a)^{q-1}}\right)\right)^{\frac{1}{q}}\cdot\|f'\|_p.$$

Recently Dragomir [3] proved the following Ostrowski type inequality for a continuous function $f\colon[a,b]\to\mathbb{R}$ which is differentiable on (a,b):

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant\frac{b-a}{|x|}\left(\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right)\cdot\|f-\iota f'\|_\infty,$$

where $\iota(t)=t$, $t\in[a,b]$. This was generalized to the L_p-case by Pečarić and Ungar [7]. Here we will, in a similar manner, obtain an estimate of the two-point Ostrowski type, which will, in special cases, reduce to results from [7] and [3].

Mathematics subject classification (2010): 26D15, 26D10.
Keywords and phrases: Integral inequality, two-point Ostrowski, p-norm.

Paper MIA-13-27

2. The main result

We will first consider the case of a function $f\colon [a,b] \to \mathbb{R}$ and a sub-segment $[c,d] \subseteq [a,b]$. The general case of 'overlapping' intervals, i. e. when the intersection $[a,b] \cap [c,d]$ equals $[c,b]$ or $[a,d]$, will be dealt with in Section 4.

Now we state our main result:

THEOREM 1. *Let the function $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0 < a < b$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leqslant p,q \leqslant \infty$, and numbers $a \leqslant c < d \leqslant b$, the following inequality holds:*

$$\begin{aligned}
&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x - (d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right| \\
&\leqslant 2\,(b-a)^{\frac{1}{p}} \cdot (d-c)^{\frac{1}{p}} \cdot \|f - \iota f'\|_p \cdot \\
&\quad \cdot \left(\left(\frac{d^3-c^3}{3(1+q)(2-q)} + \frac{a^{2-q}(d^{1+q}-c^{1+q}) - a^{1+q}(d^{2-q}-c^{2-q})}{(1-2q)(1+q)(2-q)}\right)^{\frac{1}{q}} + \right. \\
&\quad \left. + \left(\frac{d^3-c^3}{3(1+q)(2-q)} + \frac{b^{2-q}(d^{1+q}-c^{1+q}) - b^{1+q}(d^{2-q}-c^{2-q})}{(1-2q)(1+q)(2-q)}\right)^{\frac{1}{q}}\right) \qquad (1)
\end{aligned}$$

where $\iota(t) = t$, $t \in [a,b]$.

First we state a simple lemma (for the proof see [7]):

LEMMA 2. *Let the function $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $a \cdot b > 0$. Then*

$$t\,f(x) - x\,f(t) = xt\int_x^t \big(f(u) - u\,f'(u)\big)\,\frac{1}{u^2}\,\mathrm{d}u \qquad (2)$$

for all $x,t \in [a,b]$. □

Proof of Theorem 1. We first prove the theorem for $1 < p,q < \infty$, $p,q \neq 2$. These limit cases will be discussed in the next section.

Applying Lemma 2 to our function f and integrating on t over $[a,b]$, gives

$$\frac{b^2-a^2}{2}\,f(x) - x\int_a^b f(t)\,\mathrm{d}t = x\int_a^b \left(t\int_x^t \big(f(u) - u\,f'(u)\big)\frac{1}{u^2}\,\mathrm{d}u\right)\mathrm{d}t.$$

Integrating this identity on x over $[c,d]$, gives

$$\begin{aligned}
&\frac{b^2-a^2}{2}\int_c^d f(x)\,\mathrm{d}x - \frac{d^2-c^2}{2}\int_a^b f(t)\,\mathrm{d}t \\
&\qquad = \int_c^d \left(x\int_a^b \left(t\int_x^t \big(f(u) - u\,f'(u)\big)\frac{1}{u^2}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x. \qquad (3)
\end{aligned}$$

Taking the absolute value gives

$$\begin{aligned}
&\frac{1}{2}\cdot\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right| \\
&\quad=\left|\int_c^d\left(\int_a^b\left(\int_x^t\big(f(u)-u f'(u)\big)\frac{xt}{u^2}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right| \\
&\quad\leqslant\int_c^d\left|\int_a^b\left|\int_x^t\left|\big(f(u)-u f'(u)\big)\frac{xt}{u^2}\right|\mathrm{d}u\right|\mathrm{d}t\right|\mathrm{d}x \\
&\quad=\int_c^d\left(\int_a^x\left(\int_t^x\left|\big(f(u)-u f'(u)\big)\frac{xt}{u^2}\right|\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x+ \\
&\qquad\qquad+\int_c^d\left(\int_x^b\left(\int_x^t\left|\big(f(u)-u f'(u)\big)\frac{xt}{u^2}\right|\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x. \qquad (4)
\end{aligned}$$

Application of the Hölder's inequality shows that the right hand side of (4) is

$$\begin{aligned}
&\leqslant\left(\int_c^d\left(\int_a^x\left(\int_t^x|f(u)-uf'(u)|^p\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{p}}\cdot\left(\int_c^d\left(\int_a^x\left(\int_t^x\frac{x^q t^q}{u^{2q}}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{q}}+ \\
&\quad+\left(\int_c^d\left(\int_x^b\left(\int_x^t|f(u)-uf'(u)|^p\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{p}}\cdot\left(\int_c^d\left(\int_x^b\left(\int_x^t\frac{x^q t^q}{u^{2q}}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\leqslant\left(\int_c^d\left(\int_a^b\left(\int_a^b|f(u)-uf'(u)|^p\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{p}}\cdot \\
&\quad\cdot\left(\left(\int_c^d\left(\int_a^x\left(\int_t^x\frac{x^q t^q}{u^{2q}}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{q}}+\left(\int_c^d\left(\int_x^b\left(\int_x^t\frac{x^q t^q}{u^{2q}}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{q}}\right) \\
&=(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}\|f-\iota f'\|_p\cdot \\
&\quad\cdot\left(\left(\int_c^d\left(\int_a^x\left(\int_t^x\frac{x^q t^q}{u^{2q}}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{q}}+\left(\int_c^d\left(\int_x^b\left(\int_x^t\frac{x^q t^q}{u^{2q}}\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{q}}\right). \qquad (5)
\end{aligned}$$

The first triple integral in the last line of (5) equals

$$\frac{d^3-c^3}{3(1+q)(2-q)}+\frac{a^{2-q}(d^{1+q}-c^{1+q})-a^{1+q}(d^{2-q}-c^{2-q})}{(1-2q)(1+q)(2-q)} \qquad (6)$$

and similarly for the second integral (b replaces a).

Plugging this two integrals into (5), appending the result to (4) and multiplying by 2, gives the required inequality (1), and proves the theorem for $1<p,q<\infty$, $p,q\neq 2$. $\square$

3. Limit cases

Let us now have a look at the limit cases $(p,q)=(\infty,1)$, $(1,\infty)$, and $(2,2)$. Examining the calculations in the proof of Theorem 1, one readily sees that everything goes through also in the case $p=\infty$, $q=1$, so inequality (1) holds in this case too. For reference, we will state this result separately:

COROLLARY 3. *Let the function $f\colon [a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$, and let $a\leqslant c<d\leqslant b$. Then*

$$\begin{aligned}&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|\\&\qquad\leqslant (d^2-c^2)\cdot\left(\frac{a^2+b^2}{c+d}-(a+b)+\frac{2}{3}\cdot\frac{c^2+cd+d^2}{c+d}\right)\cdot\|f-\iota f'\|_\infty,\end{aligned}\tag{7}$$

or equivalently, resembling more the form of the results in [3] *and* [7],

$$\begin{aligned}&\left|\frac{a+b}{c+d}\cdot\frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\\&\qquad\leqslant\frac{1}{b-a}\left(\frac{a^2+b^2}{c+d}-(a+b)+\frac{2}{3}\cdot\frac{c^2+cd+d^2}{c+d}\right)\cdot\|f-\iota f'\|_\infty,\end{aligned}\tag{7a}$$

where $\iota(t)=t$, $t\in[a,b]$. □

In the case $p=1$, $q=\infty$, the last line in (5) has to be interpreted as

$$\begin{aligned}\max_{\substack{t\leqslant u\leqslant x\\ a\leqslant t\leqslant x\\ c\leqslant x\leqslant d}}\frac{xt}{u^2}+\max_{\substack{x\leqslant u\leqslant t\\ x\leqslant t\leqslant b\\ c\leqslant x\leqslant d}}\frac{xt}{u^2}&\leqslant\max_{c\leqslant x\leqslant d}\left(x\cdot\max_{\substack{t\leqslant u\leqslant x\\ a\leqslant t\leqslant x}}\frac{t}{u^2}\right)+\max_{c\leqslant x\leqslant d}\left(x\cdot\max_{\substack{x\leqslant u\leqslant t\\ x\leqslant t\leqslant b}}\frac{t}{u^2}\right)\\&\leqslant\max_{c\leqslant x\leqslant d}\left(x\cdot\frac{1}{a}\right)+\max_{c\leqslant x\leqslant d}\left(x\cdot\frac{b}{x^2}\right)=\frac{d}{a}+\frac{b}{c}.\end{aligned}\tag{8}$$

Putting (8) into (5) and appending the result to (4), gives

COROLLARY 4. *Let the functions f and ι be as in Corollary 3. Then*

$$\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|\leqslant 2(b-a)(d-c)\left(\frac{d}{a}+\frac{b}{c}\right)\cdot\|f-\iota f'\|_1.\qquad\square$$

It is not difficult to see that the constant $2(b-a)(d-c)\left(\frac{d}{a}+\frac{b}{c}\right)$ in Corollary 4 is equal to the limit of the constant on the right hand side in (1) as $q\to\infty$, proving Theorem 1 for $p=1$ and $q=\infty$.

Finally, for $p=q=2$, the integrals in the last line of (5) have to be calculated separately 'by hand', so for the first of these integrals, instead of (6) we get

$$\frac{1}{27}\Big(2\,(c^3-d^3)+3(c^3-d^3)\ln a-3(a^3+c^3)\ln c+3(a^3+d^3)\ln d\Big),$$

and similarly for the second integral (b replaces a). Putting this into (5), appending the result to (4) and multiplying by 2, gives

COROLLARY 5. *Let the functions f and $\imath$ be as in Corollary 3. Then*

$$\begin{aligned}&\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|\\&\leqslant \frac{2}{3\sqrt{3}}(b-a)^{\frac{1}{2}}\cdot(d-c)^{\frac{1}{2}}\cdot\|f-\imath f'\|_2\cdot\\&\cdot\Big(\big(2(c^3-d^3)+3(c^3-d^3)\ln a-3(a^3+c^3)\ln c+3(a^3+d^3)\ln d\big)^{\frac{1}{2}}+\\&+\big(2(c^3-d^3)+3(c^3-d^3)\ln b-3(b^3+c^3)\ln c+3(b^3+d^3)\ln d\big)^{\frac{1}{2}}\Big).\end{aligned}$$

□

Again, it is not difficult to see that the constant on the right hand side in Corollary 5 is equal to the limit of the constant on the right hand side in (1) as $q\to 2$, proving the required inequality (1) for $p=q=2$. This completes the proof of Theorem 1.

Let us now consider the limit case $d=c=:x$. It is reasonable to assume that $\frac{1}{d-c}\int_c^d f(s)\,\mathrm{d}s$ has the value $f(x)$. It will be more convenient, both for taking the required limits and for comparing the results with those in [7], to rewrite the inequality (1) in the following form:

$$\begin{aligned}&\left|\frac{a+b}{c+d}\cdot\frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\\&\leqslant 2(b-a)^{\frac{1}{p}-1}\cdot(d-c)^{\frac{1}{p}-1}\cdot(c+d)^{-1}\cdot\|f-\imath f'\|_p\cdot\\&\cdot\left(\left(\frac{d^3-c^3}{3(1+q)(2-q)}+\frac{a^{2-q}(d^{1+q}-c^{1+q})-a^{1+q}(d^{2-q}-c^{2-q})}{(1-2q)(1+q)(2-q)}\right)^{\frac{1}{q}}+\right.\\&\left.+\left(\frac{d^3-c^3}{3(1+q)(2-q)}+\frac{b^{2-q}(d^{1+q}-c^{1+q})-b^{1+q}(d^{2-q}-c^{2-q})}{(1-2q)(1+q)(2-q)}\right)^{\frac{1}{q}}\right).\end{aligned}\tag{1a}$$

Taking the appropriate limits in $(1a)$ and using Corollaries 3, 4, and 5, we obtain

COROLLARY 6. *Let the functions f and $\imath$ be as in Corollary 3. Then for every*

$x \in [a,b]$ we have the following inequalities:

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{1}{b-a}\left(\frac{a^2+b^2}{2x}-(a+b)+x\right)\cdot\|f-\iota f'\|_\infty \tag{9}$$

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \left(\frac{1}{a}+\frac{b}{x^2}\right)\cdot\|f-\iota f'\|_1 \tag{10}$$

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{1}{3(b-a)^{\frac{1}{2}}}\left(\left(\ln\left(\frac{x}{a}\right)^3+\frac{a^3}{x^3}-1\right)^{\frac{1}{2}}+\left(\ln\left(\frac{x}{b}\right)^3+\frac{b^3}{x^3}-1\right)^{\frac{1}{2}}\right)\cdot\|f-\iota f'\|_2. \tag{11}$$

The above results coincide with those in [3] and [7].

4. Case of overlapping intervals

We turn now to the case when the line segments $[a,b]$ and $[c,d]$ overlap, i. e. $[a,b]\cap[b,d]$ equals $[c,b]$ or $[a,d]$. It suffices to consider the first case, $a\leqslant c<b\leqslant d$. The other one is obtained by interchanging $a\leftrightarrow c$ and $b\leftrightarrow d$.

First let us introduce a notation. For real numbers $\alpha\leqslant\gamma<\delta\leqslant\beta$ and a real function $\varphi\in L_p[\alpha,\beta]$, $1\leqslant p\leqslant\infty$, denote by

$$\|\varphi\|_{p,[\gamma,\delta]} := \left(\int_\gamma^\delta |\varphi(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

the L_p-norm of the restriction of φ to the sub-interval $[\gamma,\delta]\subseteq[\alpha,\beta]$. Obviously, for $[\gamma',\delta']\subseteq[\gamma,\delta]$, the following holds:

$$\|\varphi\|_{p,[\gamma',\delta']}\leqslant\|\varphi\|_{p,[\gamma,\delta]}\,. \tag{12}$$

We can now state our main result for overlapping intervals:

THEOREM 7. *Let $0<a\leqslant c<b\leqslant d$ and let the function $f\colon[a,d]\to\mathbb{R}$ be continuous on $[a,d]$ and differentiable on (a,d). Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1\leqslant p,q\leqslant\infty$, and $\iota(t)=t$, $t\in[a,d]$, the following inequality holds:*

$$\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{2\,(b-a)^{\frac{1}{p}}}{(1+q)^{\frac{1}{q}}\,(2-q)^{\frac{1}{q}}}\cdot \cdot\left(A_{p,[a,b]}\cdot\|f-\iota f'\|_{p,[a,b]}+B_{p,[a,d]}\cdot\|f-\iota f'\|_{p,[a,d]}\right) \tag{13}$$

where the constants $A_{p,[a,b]}$ and $B_{p,[a,d]}$ are given by the following formulas:
For $1<p,q<\infty,\ p,q\neq 2$

$$A_{p,[a,b]}=(b-c)^{\frac{1}{p}}\cdot\left(\left(\frac{b^3-c^3}{3}+\frac{a^{2-q}(b^{1+q}-c^{1+q})-a^{1+q}(b^{2-q}-c^{2-q})}{1-2q}\right)^{\frac{1}{q}}+\right.$$
$$\left.+\left(\frac{b^3-c^3}{3}+\frac{b^{1+q}c^{2-q}-b^{2-q}c^{1+q}}{1-2q}\right)^{\frac{1}{q}}\right)$$

$$B_{p,[a,d]}=(d-b)^{\frac{1}{p}}\cdot\left(\frac{(b^{1+q}-a^{1+q})(d^{2-q}-b^{2-q})-(b^{2-q}-a^{2-q})(d^{1+q}-b^{1+q})}{1-2q}\right)^{\frac{1}{q}};$$

for $p=\infty,\ q=1$

$$A_{\infty,[a,b]}=\frac{1}{3}(b-c)(3a^2+2b^2+2c^2-3ab-3ac-bc)$$
$$B_{\infty,[a,d]}=(b-a)(d-b)(d-a);$$

for $p,q=2$

$$A_{2,[a,b]}=\frac{(b-c)^{\frac{1}{2}}}{\sqrt{3}}\cdot\left(\left(a^3\ln\frac{b}{c}+b^3\ln\frac{b}{a}-c^3\ln\frac{c}{a}-\frac{2}{3}(b^3-c^3)\right)^{\frac{1}{2}}+\right.$$
$$\left.+\left((b^3+c^3)\ln\frac{b}{c}-\frac{2}{3}(b^3-c^3)\right)^{\frac{1}{2}}\right)$$
$$B_{2,[a,d]}=\frac{(d-b)^{\frac{1}{2}}}{\sqrt{3}}\cdot\left((d^3-b^3)\ln\frac{b}{a}-(b^3-a^3)\ln\frac{d}{b}\right)^{\frac{1}{2}};$$

and for $p=1,\ q=\infty$

$$A_{1,[a,b]}=(b-c)\cdot\left(\frac{b}{a}+\frac{b}{c}\right)$$
$$B_{1,[a,d]}=(d-b)\cdot\frac{d}{a}.$$

Proof. We proceed as in the proof of Theorem 1, except that in (4) we split the triple integral $\int_c^d\left|\int_a^b\left|\int_x^t|g(u,t,x)|\,\mathrm{d}u\right|\mathrm{d}t\right|\mathrm{d}x$, where we have denoted $g(u,t,x):=(f(u)-uf'(u))\frac{xt}{u^2}$, into three terms

$$\int_c^b\left|\int_a^x\left|\int_x^t|g|\,\mathrm{d}u\right|\mathrm{d}t\right|\mathrm{d}x+\int_c^b\left|\int_x^b\left|\int_x^t|g|\,\mathrm{d}u\right|\mathrm{d}t\right|\mathrm{d}x+\int_b^d\left|\int_a^b\left|\int_x^t|g|\,\mathrm{d}u\right|\mathrm{d}t\right|\mathrm{d}x$$
$$=\int_c^b\left(\int_a^x\left(\int_t^x|g|\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x+\int_c^b\left(\int_x^b\left(\int_x^t|g|\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x+\int_b^d\left(\int_a^b\left(\int_t^x|g|\,\mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x \tag{14}$$

For the first two terms we proceed as in (5) with b instead of d, giving the term in (13) containing $A_{p,[a,b]}$, and integrating the third term in (14) we obtain the term in (13) containing $B_{p,[a,d]}$.

Starting from (14), the special cases $p=\infty$, $p=2$, and $p=1$ are dealt with in the same manner as in Section 3 for the case $a\leqslant c<b\leqslant d$, so we omit the details. □

For $a\leqslant c<b\leqslant d$ let $\|\ \|_p$ denote $\|\ \|_{p,[a,d]}$ — the norm over the whole domain $[a,b]\cup[c,d]$ of f. Then, using the discrete Hölder's inequality, and because of (12), we have also

COROLLARY 8. *With notations as in Theorem 7, the following inequalities hold:*

$$\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leqslant \frac{2\,(b-a)^{\frac{1}{p}}}{(1+q)^{\frac{1}{q}}\,(2-q)^{\frac{1}{q}}}\cdot\left(A^q_{p,[a,b]}+B^q_{p,[a,d]}\right)^{\frac{1}{q}}\cdot\left(\|f-\iota f'\|^p_{p,[a,b]}+\|f-\iota f'\|^p_{p,[a,d]}\right)^{\frac{1}{p}} \tag{15}$$

$$\leqslant \frac{2\cdot 2^{\frac{1}{p}}\cdot(b-a)^{\frac{1}{p}}}{(1+q)^{\frac{1}{q}}\,(2-q)^{\frac{1}{q}}}\cdot\left(A^q_{p,[a,b]}+B^q_{p,[a,d]}\right)^{\frac{1}{q}}\cdot\|f-\iota f'\|_p. \tag{16}$$

As special cases of Corollary 8, for $p=\infty$ and for $p=1$, we have

COROLLARY 9. *Let $0<a\leqslant c<b\leqslant d$ and let the function $f\colon[a,d]\to\mathbb{R}$ be continuous on $[a,d]$ and differentiable on (a,d). Then the following inequality holds:*

$$\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leqslant\left(\frac{2}{3}(b^3-c^3)+a^2(d-c)-b^2(c+d)+c^2(a+b)+d^2(b-a)\right)\cdot\|f-\iota f'\|_\infty$$

and

$$\left|(b^2-a^2)\int_c^d f(x)\,\mathrm{d}x-(d^2-c^2)\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leqslant 4\,(b-a)(d-c)\cdot\left(\frac{b}{a}+\frac{b}{c}+\frac{d}{a}\right)\cdot\|f-\iota f'\|_1$$

where $\iota(t)=t,\ t\in[a,d]$. □

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, I. PERIĆ, *Estimations of the difference of two weighted integral means via weighted Montgomery identity*, Mathematical Inequalities & Applications, **7** 3 (2004), 315–336.

[2] N. S. BARNETT, S. S. DRAGOMIR, *Issues of estimation in the monitoring of constant flow continuous streams*, RGMIA, Research report collection, **2** 3 (1999), 275–282.

[3] S. S. DRAGOMIR, *An inequality of Ostrowski type via Pompeiu's mean value theorem*, Journal of Inequalities in Pure and Applied Mathematics, Vol. 6, **3** 83 (2005), 1–9.

[4] M. MATIĆ, J. PEČARIĆ, *Two-point Ostrowski inequality*, Mathematical Inequalities & Applications, **4** 2 (2001), 215–221.

[5] A. OSTROWSKI, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Hel., **10** (1938), 226–227.
[6] J. PEČARIĆ, I. PERIĆ, A. VUKELIĆ, *Estimations of the difference of two integral means via Euler-type identities*, Mathematical Inequalities & Applications, **7** 3 (2004), 365–378.
[7] J. PEČARIĆ, Š. UNGAR, *On an inequality of Ostrowski type*, Journal of Inequalities in Pure and Applied Mathematics, Vol. **7**, 4, 151 (2006), 1–5.
[8] D. POMPEIU, *Sur une proposition analogue au théorème des accroissements finis*, Mathematica (Cluj, Romania), **22** (1946), 143–146.

(Received June 17, 2007)

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
e-mail: pecaric@hazu.hr

Šime Ungar
Department of Mathematics
University of Zagreb
e-mail: ungar@math.hr