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SECTORIAL OPERATORS WITH CONVOLUTION TERM

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Abstract. In the present paper, separability properties of convolution-differential equations with unbounded operator coefficients in Banach valued L_p spaces are investigated. A coercive estimate for resolvent of corresponding realization operator, especially, its R-sectoriality is obtained. Finally, these results applied to establish maximal regularity of Cauchy problem for the abstract parabolic convolution equations and integro-differential equations on infinite dimension state spaces.

1. Introduction, notations and background

In recent years, maximal regularity of differential operator equations, especially parabolic and elliptic type have been studied extensively e. g. in [1–3], [8–10], [14], [17–21], [25] and the references therein. Moreover, convolution-differential equations (CDEs) have been studied in [2], [12–13], [15–16], [22–23] (for comprehensive references see [15]). However, the convolution-differential operator equations (CDOEs) is relatively less investigated subject. In [2] the parabolic type CDEs with bounded operator coefficient was investigated. The main aim of the present paper, is to establish maximal regularity of CDOE

$$\sum_{k=0}^{l} a_k * \frac{d^k u}{dt^k} + A * u = f(t)$$

in E-valued L_p space, where $A = A(t)$ is a possible unbounded operator in a Banach space E, $a_k = a_k(t)$ are complex valued functions on $R = (-\infty, \infty)$. Particularly, we prove that the corresponding realization operator is a generator of analytic semigroup.

Suppose Ω is a measurable subset in R^n and $L_p(\Omega; E)$ denotes the space of all strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_{L_p(\Omega;E)} = \left(\int \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$
$$\|f\|_{L_\infty(\Omega;E)} = \operatorname{ess\,sup}_{x\in\Omega} \left[\|f(x)\|_E \right].$$

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Let $\mathbf{C}$ be the set of complex numbers and

$$S_{\varphi}=\{\lambda;\ \lambda\in\mathbf{C},\ |\arg\lambda|\leqslant\varphi\},\quad 0\leqslant\varphi<\pi.$$

A closed linear operator $A=A(t),\ t\in R$ is said to be uniformly sectorial in a Banach space E, if $D(A(t))$ is dense in E and does not depend on t and there is a positive constant M so that

$$\left\|(A(t)+\lambda I)^{-1}\right\|_{B(E)}\leqslant M|\lambda|^{-1}$$

for every $t\in R$ and $\lambda\in S_{\varphi},\varphi\in[0\,,\pi)$, where I is an identity operator in E and $B(E)$ is the space of all bounded linear operators in E. Sometimes instead of $A+\lambda I$ we will write $A+\lambda$ and denote it by A_{λ}.

Let $E\left(A^{\theta}\right)$ denote the space $D\left(A^{\theta}\right)$ with graphical norm

$$\|u\|_{E\left(A^{\theta}\right)}=\left(\|u\|^{p}+\left\|A^{\theta}u\right\|^{p}\right)^{\frac{1}{p}},\quad 1\leqslant p<\infty,\quad -\infty<\theta<\infty.$$

$C(\Omega;E)$ and $C^{(m)}(\Omega;E)$ will denote the spaces of E-valued bounded, continuous and m-times continuously differentiable functions on Ω, respectively.

$S=S(R^{n};E)$ denotes a Schwartz class i.e. the space of E-valued rapidly decreasing smooth functions on R^{n}, equipped with its usual topology generated by seminorms. Let $S'(R^{n};E)$ denote the space of all continuous linear operators $L:S\rightarrow E$, equipped with the bounded convergence topology. Recall $S(R^{n};E)$ is norm dense in $L_{p}(R^{n};E)$ when $1\leqslant p<\infty$.

Let $\alpha=(\alpha_{1},\alpha_{2},...,\alpha_{n})$, where α_{i} are integers. An E-valued generalized function $D^{\alpha}f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f\in S'(R^{n},E)$, if the equality

$$\langle D^{\alpha}f,\varphi\rangle=(-1)^{|\alpha|}\langle f,D^{\alpha}\varphi\rangle$$

holds for all $\varphi\in S$.

Let $Fu=\hat{u}$ and $F^{-1}u=\check{u}$ denote the Fourier and inverse Fourier transformations of u, respectively. It is known that

$$F\left(D_{x}^{\alpha}f\right)=(i\xi_{1})^{\alpha_{1}}...(i\xi_{n})^{\alpha_{n}}\hat{f},\ D_{\xi}^{\alpha}\left(F\left(f\right)\right)=F\left[(-ix_{n})^{\alpha_{1}}...(-ix_{n})^{\alpha_{n}}f\right]$$

for all $f\in S'(R^{n};E)$.

Let E_{1} and E_{2} be two Banach spaces. A function $\Psi\in L_{\infty}(R^{n};B(E_{1},E_{2}))$ is called a multiplier from $L_{p}(R^{n};E_{1})$ to $L_{p}(R^{n};E_{2})$ for $p\in(1,\infty)$ if the map $u\rightarrow Tu=F^{-1}\Psi(\xi)Fu,\ u\in S(R^{n};E_{1})$ are well defined and extends to a bounded linear operator

$$T:L_{p}(R^{n};E_{1})\rightarrow\ L_{p}(R^{n};E_{2}).$$

The space of all Fourier multipliers from $L_{p}(R^{n};E_{1})$ to $L_{p}(R^{n};E_{2})$ will be denoted by $M_{p}^{p}(E_{1},E_{2})$.

Let $T_h = \left\{\Psi_h \in M_p^p(E_1, E_2), h \in P_0\right\}$ where P_0 is a space containing our parameters (generally, it will be some sectors in the complex domain, through this paper). We say that T_h is a uniformly bounded collection of multipliers (UBM) if there exists a positive constant M independent on $h \in P_0$ such that

$$\left\|F^{-1}\Psi_h F u\right\|_{L_p(R^n;E_2)} \leqslant M \|u\|_{L_p(R^n;E_1)}$$

for all $h \in P_0$ and $u \in S(R^n; E_1)$.

A Banach space E is called a UMD-space [6–7] if the Hilbert operator $(Hf)(x) = \lim\limits_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y}dy$ is bounded in $L_p(R, E)$, $p \in (1, \infty)$ (see. e.g. [9]). UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, p, $q \in (1, \infty)$.

A set $K \subset B(E_1, E_2)$ is called R-bounded (see [9], [11], [24]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in N$

$$\int\limits_0^1 \left\|\sum_{j=1}^m r_j(y) T_j u_j\right\|_{E_2} dy \leqslant C \int\limits_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_1} dy,$$

where $\left\{r_j\right\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$. The smallest C for which the above estimate holds is called a R-bound of the collection K and denoted by $R(K)$.

A set $T_h \subset B(E_1, E_2)$ depending on parameter $h \in Q$ is called uniformly R-bounded with respect to h if there is a constant C, independent of $h \in Q$, such that for all $T_1(h), T_2(h), ..., T_m(h) \in T_h$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbf{N}$

$$\int\limits_0^1 \left\|\sum_{j=1}^m r_j(y) T_j(h) u_j\right\|_{E_2} dy \leqslant C \int\limits_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_1} dy$$

that implies

$$\sup_{h\in Q} R(T_h) \leqslant C.$$

DEFINITION 1. A Banach space E is said to be a space satisfying a multiplier condition if, for any $\Psi \in L_\infty(R; B(E))$, the R-boundedness of the set

$$\left\{|\xi|^k D^k \Psi(\xi) : \xi \in R \setminus 0,\ \{0,1\},\ k = 0, 1\right\}$$

implies that Ψ is a Fourier multiplier, i.e. $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.

REMARK 1. Note that, if E is UMD space then by virtue of [9], [11], [24] the space E satisfies the multiplier condition.

DEFINITION 2. A sectorial operator A is said to be an R-sectorial in a Banach space E if there exists $\varphi \in [0\ , \pi)$ such that the set

$$L_A = \left\{A(A + \xi)^{-1} : \xi \in S_\varphi\right\}$$

is R-bounded.

An operator $A(t)$ is said to be an uniform R-sectorial in a Banach space E if

$$\sup_t R\left(\left\{A(t)(A(t)+\xi)^{-1} : \xi \in S_\varphi\right\}\right) \leqslant M.$$

Note that, in Hilbert spaces every norm bounded set is R-bounded. Therefore, in Hilbert spaces all sectorial operators are R-sectorial (see e.g. [9]).

DEFINITION 3. Let $A = A(t)$, $t \in R$ be closed linear operator in E with domain definition $D(A)$ independent of t. Let $u \in S'(R;E(A))$. Then, the Fourier transformation of $A(t)$ is defined as

$$\langle \hat{A}u, \varphi\rangle = \langle Au, \hat{\varphi}\rangle, \quad u \in D(A), \quad \varphi \in S(R).$$

(For details see [2, p. 7]).

DEFINITION 4. Let $A = A(t)$, $t \in R$ be closed linear operator in E with domain definition $D(A)$ independent of t. Then it is differentiable if for all $u \in\ E(A)$ the following equality holds

$$\left(\frac{d}{dt}A\right)u = A'(t)u = \lim_{h\to 0}\frac{\|A(t+h)u - A(t)u\|_E}{h}.$$

DEFINITION 5. Let $A = A(t)$, $t \in R$ be closed linear operator in E with domain definition $D(A)$ independent of t and $u \in S(R;E(A))$. Then we define:

$$(A * u)(t) = \int_R A(t-y)u(y)dy.$$

For the case $u \in S'(R;E(A))$ see [2, p. 10-11].

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Let l be a positive integer. $W_p^l(R;E_0,E)$ denotes the space of all functions $u \in L_p(R;E_0)$ possess the generalized derivatives $D^l u \in L_p(R;E)$ with the norm

$$\|u\|_{W_p^l(R;E_0,E)} = \|u\|_{L_p(R;E_0)} + \left\|D^l u\right\|_{L_p(R;E)} < \infty.$$

2. Convolution-differential operator equations

Consider the CDOE

$$(L+\lambda)u = \sum_{k=0}^{l} a_k * \frac{d^k u}{dt^k} + A_\lambda * u = f(t), \quad t \in R \tag{1}$$

in $L_p(R;E)$, where $A_\lambda = A+\lambda$, $A = A(t)$ is a possible unbounded operator in a Banach space E, $a_k = a_k(t)$ are complex valued functions.

CONDITION 2.1.. Let $\hat{A}(\xi)$ be uniformly sectorial operator in a Banach space E with $\varphi \in [0,\pi)$. Suppose $a_k \in S'(R)$ and $\hat{a}_k$ be continuous complex valued functions. Moreover:

(i) there exist $M > 0$ and $m \in \{0,1,\dots,l\}$ such that

$$|\hat{a}_k(\xi)| \leqslant M\,|\hat{a}_m(\xi)| \text{ for all } k = 0,1\dots,l \text{ and } \xi \in R.$$

(ii)

$$L(\xi) = \sum_{k=0}^{l} \hat{a}_k(\xi)\,(i\xi)^k \in S(\varphi_1)\,, \text{ and } \lambda \in S(\varphi_2)\,,$$

so that $\varphi_1 + \varphi_2 < \pi$ and

$$L(\xi) + \lambda \in S_\varphi.$$

(iii) there is a positive constant C so that

$$|L(\xi)| \geqslant C\,|\hat{a}_m|\left|\xi^l\right|, \quad \xi \in R\backslash\{0\}. \tag{2}$$

Through this section $\frac{d}{d\xi}\hat{A}(\xi)$ will be denoted as $\hat{A}'(\xi)$.

First let us prove the uniformly boundedness of operator functions

$$\sigma_0(\xi,\lambda) = \lambda\left[\hat{A}(\xi) + \lambda + L(\xi)\right]^{-1}$$
$$\sigma_1(\xi,\lambda) = \hat{A}(\xi)\left[\hat{A}(\xi) + \lambda + L(\xi)\right]^{-1}$$

and

$$\sigma_2(\xi,\lambda) = \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\,\hat{a}_k(\xi)\,(i\xi)^k\left[\hat{A}(\xi) + \lambda + L(\xi)\right]^{-1}.$$

LEMMA 2.1. *If Condition 2.1 holds, then the operators* $\sigma_i(\xi,\lambda)$, $i = 0,1,2$ *are uniformly bounded.*

Proof. Let us note that, for the sake of simplicity we shall not change constants in every step. By virtue of [8, Lemma 2.3] for $L(\xi) \in S_{\varphi_1}$, $\lambda \in S_\varphi$ and $\varphi_1 + \varphi < \pi$ there is a positive constant C such that

$$|\lambda + L(\xi)| \geqslant C\,(|\lambda| + |L(\xi)|). \tag{3}$$

In view of uniformly sectoriality of operator $\hat{A}(\xi)$ and by (3) we have the uniform estimate

$$\|\sigma_0(\xi,\lambda)\|_{B(E)} \leqslant M\,|\lambda|\,[1 + |\lambda| + |L(\xi)|]^{-1} \leqslant M.$$

Moreover, by using the resolvent properties of sectorial operators we obtain

$$\begin{aligned}
\|\sigma_1(\xi,\lambda)\|_{B(E)} &= \left\| I - (\lambda + L(\xi))\left[\hat{A}(\xi) + \lambda + L(\xi)\right]^{-1} \right\|_{B(E)} \\
&\leqslant 1 + |\lambda + L(\xi)| \left\| \left[\hat{A}(\xi) + \lambda + L(\xi)\right]^{-1} \right\|_{B(E)} \\
&\leqslant 1 + M\,|\lambda + L(\xi)|\,(1 + |\lambda + L(\xi)|)^{-1} \leqslant 1 + M.
\end{aligned}$$

Next, let us consider σ_2. It is clear to see that

$$\|\sigma_2(\xi,\lambda)\|_{B(E)} \leqslant C\sum_{k=0}^{l}|\hat{a}_k||\lambda|\left(|\xi||\lambda|^{-\frac{1}{l}}\right)^k|\lambda+L(\xi)|^{-1}$$
$$\times|\lambda+L(\xi)|\left\|\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1}\right\|_{B(E)}.$$

Then setting $y=\left(|\lambda|^{-\frac{1}{l}}|\xi|\right)$ in the following well known inequality

$$y^k \leqslant C\left(1+y^l\right),\ y_k \geqslant 0,\ k \leqslant l$$

due to sectoriality of A we have

$$\|\sigma_2(\xi,\lambda)\|_{B(E)} \leqslant C\sum_{k=0}^{l}|\hat{a}_k|\left[|\lambda|+|\xi|^l\right][|\lambda+L(\xi)|]^{-1}.$$

Due to boundedness of $\sum_{k=0}^{l}|\hat{a}_k|$, by using the estimates (2) and (3) we get

$$\|\sigma_2(\xi,\lambda)\|_{B(E)} \leqslant C. \qquad \square$$

LEMMA 2.2. *Suppose Condition 2.1 holds and $\hat{A}(\xi)$ is an uniformly R-sectorial operator. Then, the following sets*

$$S_0(\xi,\lambda) = \left\{\lambda\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1};\ \xi\in R\setminus\{0\}\right\},$$
$$S_1(\xi,\lambda) = \left\{\hat{A}(\xi)\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1};\ \xi\in R\setminus\{0\}\right\},$$
$$S_2(\xi,\lambda) = \left\{\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\hat{a}_k(\xi)(i\xi)^k\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1};\ \xi\in R\setminus\{0\}\right\}$$

are uniformly R-bounded.

Proof. Due to R-sectoriality of A we obtain that $S_1(\xi,\lambda)$ is R bounded. Let

$$\sigma(\xi,\lambda)=[\lambda+L(\xi)]\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1}.$$

Since,

$$I-[\lambda+L(\xi)]\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1}=\hat{A}\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1},$$

it clearly follows from R-boundedness of $S_1(\lambda,\xi)$ that the set

$$S(\xi,\lambda)=\{\sigma(\xi,\lambda);\ \xi\in R\setminus\{0\}\}$$

is R-bounded. Moreover, by Condition 2.1 and by estimate (3) there is a positive constant M so that

$$|\lambda|\,|\lambda+L(\xi)|^{-1}\leqslant M.$$

So, due to R-boundedness of $S(\xi,\lambda)$ and by virtue of Kahane's contraction principle for collection of R-bounded operators [9, Lemma 3.5] we obtain the R-boundedness of set $S_0(\xi,\lambda)$. Therefore, by Lemma 2.1 we obtain the uniformly R-boundedness of $\sigma_0(\xi,\lambda)$, $\sigma_1(\xi,\lambda)$. I.e.

$$\sup_{\lambda} R\{S_0(\xi,\lambda)\}\leqslant M_0,\qquad \sup_{\lambda} R\{S_1(\xi,\lambda)\}\leqslant M_1. \tag{4}$$

Thanks to R-boundedness of the set $S(\xi,\lambda)$ for all $\xi_1,\xi_2,\ldots,\xi_m\in R$, $\sigma(\xi_1,\lambda)$, $\sigma(\xi_2,\lambda)$, ..., $\sigma(\xi_m,\lambda)$, $u_1,u_2,\ldots,u_m\in E$, $m\in N$ we have

$$\int_0^1\left\|\sum_{j=1}^m r_j(y)\,\sigma(\xi_j,\lambda)\,u_j\right\|_E dy\leqslant C\int_0^1\left\|\sum_{j=1}^m r_j(y)\,u_j\right\|_E dy, \tag{5}$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. Moreover,

$$\sigma_2(\xi,\lambda)=\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\hat{a}_k(\xi)\,(i\xi)^k\,[\lambda+L(\xi)]^{-1}\,\sigma(\lambda,\xi). \tag{6}$$

Then by virtue of (4), (6) and in view of Kahane's contraction principle, we get from (5)

$$\int_0^1\left\|\sum_{j=1}^m r_j(y)\,\sigma_2(\xi_j,\lambda)\,u_j\right\|_E dy\leqslant C\int_0^1\left\|\sum_{j=1}^m r_j(y)\,\sigma(\xi_j,\lambda)\,u_j\right\|_E dy.$$

Thanks to R-boundedness of the set $S_1(\xi,\lambda)$ we get from above

$$\begin{aligned}\int_0^1\left\|\sum_{j=1}^m r_j(y)\,\sigma_2(\xi_j,\lambda)\,u_j\right\|_E dy&\leqslant C\int_0^1\left\|\sum_{j=1}^m \sigma_1(\xi_j,\lambda)\,r_j(y)\,u_j\right\|_E dy\\&\leqslant C\int_0^1\left\|\sum_{j=1}^m r_j(y)\,u_j\right\|_E dy.\end{aligned} \tag{7}$$

The estimate (7) implies the R-boundedness of set $S_2(\xi,\lambda)$. Moreover, from Lemma 2.1 and (7) we obtain

$$\sup_{\lambda} R\{S_2(\xi,\lambda)\}\leqslant C. \tag{8}$$

Namely, the estimates (4), (8) imply that the sets $S_0(\xi,\lambda)$, $S_1(\xi,\lambda)$, $S_2(\xi,\lambda)$ are R-bounded and it's R-bounds are independent on λ. □

LEMMA 2.3. *Let all conditions of Lemma 2.2 be hold and*

$$\hat{a}_k \in C^{(1)}(R), \quad k = 0, 1, ..., l, \quad \hat{A}'(\xi)\hat{A}^{-1}(\xi) \in C(R; B(E)).$$

Suppose, there are positive constants C_1 *and* C_2 *such that*

$$R\left(\left\{\xi \hat{A}'(\xi)\left(\hat{A}+\xi\right)^{-1} : \xi \in S_\varphi\right\}\right) \leqslant C_1, \tag{9}$$

$$\left|\xi \frac{d}{d\xi}\hat{a}_k(\xi)\right| \leqslant C_2. \tag{10}$$

Then, the following sets

$$G_0(\xi,\lambda) = \left\{\xi \frac{d}{d\xi}\left(\lambda\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1}\right);\ \xi \in R \setminus \{0\}\right\},$$
$$G_1(\xi,\lambda) = \left\{\xi \frac{d}{d\xi}\left(\hat{A}(\xi)\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1}\right);\ \xi \in R \setminus \{0\}\right\},$$
$$G_2(\xi,\lambda) = \left\{\xi \frac{d}{d\xi}\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\hat{a}_k(\xi)(i\xi)^k\left[\hat{A}(\xi)+\lambda+L(\xi))\right]^{-1};\ \xi \in R \setminus \{0\}\right\},$$

are uniformly R*-bounded i.e.*

$$\sup_\lambda R\{G_i(\xi,\lambda)\} \leqslant M_i, \quad i = 0, 1, 2.$$

Proof. It is easy to see that

$$\xi \frac{d}{d\xi}[\hat{A}(\xi)B(\xi,\lambda)] = \xi\hat{A}'(\xi)B(\xi,\lambda) - \hat{A}(\xi)B(\xi,\lambda)[\xi\hat{A}'(\xi)B(\xi,\lambda)+D(\xi,\lambda)B(\xi,\lambda)],$$

where

$$B(\xi,\lambda) = \left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-1},$$
$$D(\xi,\lambda) = \sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\left(\xi\frac{d\hat{a}_k}{d\xi}(i\xi)^k + k\hat{a}_k(\xi)(i\xi)^k\right).$$

Due to R-sectoriality of $\hat{A}$ and by condition (9) we get that the sets

$$\left\{\hat{A}(\xi)B(\xi,\lambda);\ \xi \in R \setminus \{0\}\right\},\ \left\{\xi\hat{A}'(\xi)B(\xi,\lambda);\ \xi \in R \setminus \{0\}\right\}$$

are R-bounded. Moreover, it is clear that

$$D(\xi,\lambda)B(\xi,\lambda) = D(\xi,\lambda)|\lambda+L(\xi)|^{-1}|\lambda+L(\xi)|B(\xi,\lambda).$$

By condition (10) and in view of Condition 2.1we obtain that

$$\left|D(\xi,\lambda)|\lambda+L(\xi)|^{-1}\right| \leqslant M.$$

Then by Kahane's contraction principle we have the R-boundedness of set

$$\{D(\xi,\lambda)B(\xi,\lambda);\ \xi\in R\setminus\{0\}\}.$$

Then by virtue of additional and product properties of R-bounded operators (see e.g [9], Proposition 3.4) we obtain R-boundedness of $G_1(\xi,\lambda)$. In a similar way the R-boundedness of $G_0(\xi,\lambda)$ is derived. Moreover, it is clear that

$$\begin{aligned}
&\xi\frac{d}{d\xi}\sum_{k=0}^{l}|\lambda|_k^{1-\frac{k}{l}}\,\hat{a}(\xi)(i\xi)^kB(\xi,\lambda)\\
&\quad=\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\Big\{\Big[\xi\tfrac{d\hat{a}_k}{d\xi}(i\xi)^k+k\hat{a}_k(\xi)(i\xi)^k\Big]B(\xi,\lambda)\\
&\qquad+\hat{a}_k(\xi)(i\xi)^k\left[\xi\hat{A}'(\xi)+\sum_{k=0}^{l}\left(\frac{d\hat{a}_k}{d\xi}(i\xi)^k+ik\hat{a}_k(\xi)(i\xi)^{k-1}\right)\right](-1)[B(\xi,\lambda)]^2\Big\}\\
&\quad=D(\xi,\lambda)B(\xi,\lambda)+\left[\sum_{k=0}^{l}|\lambda|_k^{1-\frac{k}{l}}\,\hat{a}(\xi)(i\xi)^kB(\xi,\lambda)\right]\\
&\qquad\times\left[\hat{A}'(\xi)B(\xi,\lambda)+D(\xi,\lambda)B(\xi,\lambda)\right].
\end{aligned}$$

By using the R-boundedness of set $\{D(\xi,\lambda)B(\xi,\lambda);\ \xi\in R\setminus\{0\}\}$, by virtue of additional and product properties of R-bounded operators in a similar way we obtain the R-boundedness of the set $G_2(\xi,\lambda)$. □

Then from Lemma 2.2 and Lemma 2.3 we obtain

RESULT 2.1. Let all conditions of Lemma 2.3 are satisfied and E is a Banach space satisfying the multiplier condition. Then, operator-functions $\sigma_i(\xi,\lambda)$, $i=0,1,2$ are uniformly bounded collections of multipliers in $L_p(R;E)$.

By Lemma 2.2, Lemma 2.3 and by virtue [24, Theorem 3.4] we obtain

RESULT 2.2. Let all conditions of Lemma 2.3 are satisfied and E is an UMD space. Then, operator-functions $\sigma_i(\xi,\lambda)$ are uniformly bounded collections of multipliers in $L_p(R;E)$.

THEOREM 2.1. *Let E be a Banach space satisfying the multiplier condition and $\hat{A}(\xi)$ is uniformly R-sectorial in E. Suppose, the Condition 2.1 and* (9)–(10) *hold. Then for each $f\in L_p(R;E)$, $p\in(1,\infty)$ the problem* (1) *has a unique solution and the following coercive uniform estimate holds*

$$\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\left\|a_k*\frac{d^ku}{dt^k}\right\|_{L_p(R;E)}+\|A*u\|_{L_p(R;E)}+|\lambda|\,\|u\|_{L_p(R;E)}\leqslant C\|f\|_{L_p(R;E)}\qquad(11)$$

for $\lambda\in S(\varphi)$, $\varphi\in[0,\pi)$.

Proof. By applying the Fourier transform to equation (1) we obtain

$$\left[\hat{A}(\xi)+L(\xi)+\lambda\right]\hat{u}(\xi)=\hat{f}(\xi).$$

Since $L(\xi) \in S(\varphi_1)$ for all $\xi \in R$ and $\hat{A}$ is sectorial, the operator $\hat{A}(\xi) + L(\xi) + \lambda$ is invertible in E. So we obtain that the solution of the equation (1) can be represented in the form

$$u(t) = F^{-1}\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1} \hat{f}.$$

Then there are positive numbers C_1 and C_2 such that

$$C_1 |\lambda| \|u\|_{L_p(R;E)} \leqslant \left\|F^{-1}\left[\sigma_0(\xi,\lambda)\hat{f}\right]\right\|_{L_p(R;E)} \leqslant C_2 |\lambda| \|u\|_{L_p(R;E)},$$

$$C_1 \|A * u\|_{L_p(R;E)} \leqslant \left\|F^{-1}\left[\sigma_1(\xi,\lambda)\hat{f}\right]\right\|_{L_p(R;E)} \leqslant C_2 \|A * u\|_{L_p(R;E)},$$

$$C_1 \left\|F^{-1}\left[\sigma_2(\xi,\lambda)\hat{f}\right]\right\|_{L_p(R;E)} \leqslant \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \left\|\hat{a}_k * \frac{d^k u}{dx^k}\right\|_{L_p(R;E)} \leqslant C_2 \left\|F^{-1}\left[\sigma_2(\xi,\lambda)\hat{f}\right]\right\|_{L_p(R;E)},$$

where $\sigma_i(\xi,\lambda)$, $i = 0,1,2$ are operator functions defined in Lemma 2.1. By Result 2.1 operator-functions $\sigma_i(\xi,\lambda)$ are uniformly bounded collections of multipliers in $L_p(R;E)$. It implies that for $f \in L_p(R;E)$,

$$|\lambda| \|u\|_{L_p(R;E)} \leqslant C_0 \|f\|_{L_p(R;E)}, \|A * u\|_{L_p(R;E)} \leqslant C_1 \|f\|_{L_p(R;E)},$$

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \left\|a_k * \frac{d^k u}{dt^k}\right\|_{L_p(R;E)} \leqslant C_2 \|f\|_{L_p(R;E)}.$$

I.e. we obtain that for all $f \in L_p(R;E)$, there is a unique solution of the equation (1) in the form $u(x) = F^{-1}[A + \lambda + L(\xi)]^{-1} \hat{f}$ and the estimate (11) holds.

Let Q denote the operator in $F = L_p(R;E)$ generated by problem (1) i. e.

$$Qu = \sum_{k=0}^{l} a_k * \frac{d^k u}{dt^k} + A_\lambda * u. \qquad \square$$

From Theorem 2.1 and Result 2.2 we have

RESULT 2.3. Let conditions of Theorem 3.1 hold for the Banach spaces $E \in UMD$. Then the assertion of Theorem 3.1 is valid.

RESULT 2.4. Assume all conditions of the Theorem 2.1 hold. Then, for all $\lambda \in S(\varphi)$ the resolvent of operator Q exist and the following sharp estimate holds

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \left\|a_k * \left[\frac{d^k}{dx^k}(Q+\lambda)^{-1}\right]\right\|_{B(F)} + \left\|A * (Q+\lambda)^{-1}\right\|_{B(F)} + \left\|A * (Q+\lambda)^{-1}\right\|_{B(F)} + \left\||\lambda|(Q+\lambda)^{-1}\right\|_{B(F)} \leqslant C.$$

REMARK 2.1. The Result 2.4 particularly, implies that the operator Q is sectorial in $L_p(R;E)$. I.e. if $\hat{A}$ is uniformly R-sectorial for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$. Then (see e.g. [22, §1.14.5] the operator Q is a generator of analytic semigroup in $L_p(R;E)$.

REMARK 2.2. Note that (2) (in Condition 2.1) is not required if we consider our problem without spectral parameter. In this case we will have the following estimate

$$\sum_{k=0}^{l}\left\|a_k * \frac{d^k u}{dt^k}\right\|_{L_p(R;E)} + \|A * u\|_{L_p(R;E)} \leqslant C_2 \|f\|_{L_p(R;E)}.$$

3. The Cauchy problem for parabolic CDOE

In this section, we shall consider the following Cauchy problem for convolution parabolic equation

$$\frac{\partial u}{\partial t}(t,x) + Ou = f(t,x), \quad u(0,x) = 0, \tag{12}$$

where

$$Ou = \sum_{k=0}^{l} a_k * \frac{\partial^k u}{\partial x^k} + A * u.$$

Note that $A = A(x)$ is a possible unbounded operator in E and $a_k = a_k(x)$ are complex-valued functions. Applying Theorem 2.1 we establish in this section the maximal regularity of the problem (12). First we show O is an R-sectorial in $X = L_p(R;E)$.

THEOREM 3.1. *Suppose Condition 2.1 holds, E is an UMD space and the operator $\hat{A}(\xi)$ is uniformly R-sectorial in E for φ with $0 \leqslant \varphi < \pi - \varphi_1$. Then operator O is R-sectorial in $L_p(R;E)$.*

Proof. From the Result 2.4 we obtain that the operator O is sectorial in $L_p(R;E)$. We have to prove the R-boundedness of the set

$$\sigma(\xi,\lambda) = \left\{\lambda (O+\lambda)^{-1} : \lambda \in S_\varphi\right\}.$$

From the proof of Theorem 2.1 we have

$$\lambda (O+\lambda)^{-1} f = F^{-1}\Phi(\xi,\lambda)\hat{f}, \quad f \in L_p(R^n;E),$$

where

$$\Phi(\xi,\lambda) = \lambda\left(\hat{A}(\xi) + L(\xi) + \lambda\right)^{-1}.$$

By definition of R-boundedness, it is enough to show that the operator function $\lambda[\hat{A}(\xi) + L(\xi) + \lambda]^{-1}$ (depended on variable λ and parameter ξ) is UBM in $L_p(R;E)$. In a similar manner as in Lemma 2.3 one can easily show that $\Phi(\xi,\lambda)$ is UBM in $L_p(R;E)$. Then, by definition of R-boundedness we have

$$\int_0^1 \left\|\sum_{j=1}^{m} r_j(y)\lambda_j (O+\lambda_j)^{-1} f_j\right\|_X dy = \int_0^1 \left\|\sum_{j=1}^{m} r_j(y) F^{-1}\Phi(\xi,\lambda_j)\hat{f}_j\right\|_X dy$$

$$= \int_0^1 \left\|F^{-1}\sum_{j=1}^{m} r_j(y)\Phi(\xi,\lambda_j)\hat{f}_j\right\|_X dy \leqslant C\int_0^1 \left\|\sum_{j=1}^{m} r_j(y) f_j\right\|_X dy$$

for all $\xi \in R$, $\lambda_1, \lambda_2 ... \lambda_m \in S_\varphi$, $f_1, f_2, ..., f_m \in X$, $m \in N$, where $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. Hence, the set $\sigma(\xi,\lambda)$ is R-bounded.

Let E be a Banach space. For R_+^2, $\mathbf{p} = (p, p_1)$, $Y = L_{\mathbf{p}}\left(R_+^2; E\right)$ will be denote the space of all $\mathbf{p}$-summable E-valued functions with mixed norm (see e.g. [6, §4] for complex-valued case), i.e. the space of all measurable E-valued functions f defined on R_+^2, for which

$$\|f\|_{L_{\mathbf{p}}\left(R_+^2;E\right)} = \left(\int\limits_R \left(\int\limits_{R_+} \|f(x,t)\|_E^p \, dx \right)^{\frac{p_1}{p}} dt \right)^{\frac{1}{p_1}} < \infty.$$

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Suppose, l is an integer and $W_{\mathbf{p}}^{1,l}\left(R_+^2; E_0, E\right)$ denotes the space of all functions $u \in Y$ possess the generalized derivatives $D_t u$, $D_x^k u \in Y$, with the norm

$$\|u\|_{W_{\mathbf{p}}^{1,l}\left(R_+^2;E_0,E\right)} = \|u\|_Y + \|D_t u\|_Y + \sum_{k=1}^{n} \left\|D_x^k u\right\|_Y . \qquad \square$$

Now we are ready to state the main result of this section.

THEOREM 3.2. *Assume the Condition 2.1 holds for* $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, $s > 0$, $E \in UMD$ *and the operator* $\hat{A}(\xi)$ *is uniformly* R*-sectorial in* E. *Then for all* $f \in Y$ *the equation* (12) *has a unique solution* $u \in W_{\mathbf{p}}^1\left(R_+; E(O), E\right)$ *satisfying*

$$\|D_t u\|_Y + \sum_{k=0}^{l} \left\|a_\alpha * D_x^k u\right\|_Y + \|A * u\|_Y \leqslant C \|f\|_Y .$$

Proof. It is clear to see that

$$Y = L_{p_1}\left(R_+; X\right).$$

Therefore, the problem (12) can be expressed as

$$\frac{du}{dt} + Ou(t) = f(t), \; u(0) = 0, \; t \in R_+. \tag{13}$$

By virtue of [1, Theorem 4.5.2], $X \in UMD$ implies $E \in UMD$, for $p \in (1,\infty)$. Then due to R-sectoriality of O with $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, by virtue of [1, Proposition 8.10] we obtain that for $f \in L_{p_1}\left(R_+; X\right)$ the equation (13) has a unique solution $u \in W_{p_1}^1\left(R_+; D(O), X\right)$ satisfying

$$\|D_t u\|_{L_{p_1}(R_+;X)} + \|Ou\|_{L_{p_1}(R_+;X)} \leqslant C \|f\|_{L_{p_1}(R_+;X)} .$$

In view of Theorem 2.1 from the above estimate we get the assertion. $\square$

4. Boundary value problems for integro-differential equations

Let $\tilde{\Omega} = R \times \Omega$, where $\Omega \subset R^{\mu}$ is an open connected set with compact C^{2m}-boundary $\partial\Omega$. Consider the BVP for integro-differential equation

$$(L+\lambda)u = \sum_{k=0}^{l} a_k * \frac{\partial^k u}{\partial t^k} + \sum_{|\alpha|\leqslant 2m} \left(b_\alpha a_\alpha D_y^\alpha + \lambda\right) * u = f(t,y), \quad t \in R,\ y \in \Omega, \quad (14)$$

$$B_j u = \sum_{|\beta|\leqslant m_j} b_{j\beta}(y) D_y^\beta u(t,y) = 0, \quad y \in \partial\Omega,\ j = 1,2,...,m \quad (15)$$

where

$$D_j = -i\frac{\partial}{\partial y_j}, \quad y = (y_1,...,y_\mu), \quad b_\alpha = b_\alpha(t), \quad a_\alpha = a_\alpha(y),$$
$$\alpha = (\alpha_1, \alpha_2, ..., \quad \alpha_n), \quad a_k = a_k(t), \quad u = u(t,y).$$

If $\tilde{\Omega} = R \times \Omega$, $\mathbf{p} = (p_1, p)$, $L_{\mathbf{p}}(\tilde{\Omega})$ will be denote the space of all $\mathbf{p}$-summable scalar-valued functions with mixed norm (see e.g. [5]), i.e. the space of all measurable functions f defined on $\tilde{\Omega}$, for which

$$\|f\|_{L_{\mathbf{p}}(\tilde{\Omega})} = \left(\int_R \left(\int_\Omega |f(t,y)|^{p_1} dt\right)^{\frac{p}{p_1}} dy\right)^{\frac{1}{p}} < \infty.$$

Analogously, $W_{\mathbf{p}}^m(\tilde{\Omega})$ denotes the Sobolev space with corresponding mixed norm [5].

Let Q denote the operator generated by BVP (14)–(15). Let

$$F = B\left(L_{\mathbf{p}}\left(\tilde{\Omega}\right)\right).$$

THEOREM 4.1. *Let the following conditions be satisfied;*

(1) $a_\alpha \in C\left(\bar{\Omega}\right)$ *for each* $|\alpha| = 2m$ *and* $a_\alpha \in \left[L_\infty + L_{r_k}\right](\Omega)$ *for each* $|\alpha| = k < 2m$ *with* $r_k \geqslant p_1$, $p_1 \in (1,\infty)$ *and* $2m - k > \frac{l}{r_k}$, $\nu_\alpha \in L_\infty$;

(2) $b_{j\beta} \in C^{2m-m_j}(\partial\Omega)$ *for each* j, β, $m_j < 2m$, $p \in (1,\infty)$;

(3) *for* $y \in \bar{\Omega}$, $\xi \in R^\mu$, λ *with arg* $\lambda = \pi$, $|\xi| + |\lambda| \neq 0$ *let*

$$\lambda + \sum_{|\alpha|=2m} a_\alpha(y)\xi^\alpha \neq 0;$$

(4) *for each* $y_0 \in \partial\Omega$ *local BVP in local coordinates corresponding to* y_0

$$\lambda + \sum_{|\alpha|=2m} a_\alpha(y_0) D^\alpha \vartheta(y) = 0,$$

$$B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta \vartheta(y) = h_j, \quad j = 1,2,...,m$$

has a unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1, h_2, ..., h_m) \in R^m$ *and for* $\xi' \in R^{\mu-1}$ *with* $|\xi'| + |\lambda| \neq 0$;

(5) $\hat{a}_k,\ \hat{b}_\alpha \in C^{(1)}(R)$, *the Condition 2.1 satisfied for* $\hat{a}_k = \hat{a}_k(\xi)$ *and there are positive constants* $C_i,\ i = 1,\ 2$ *so that*

$$\left|\xi \frac{d}{d\xi}\hat{a}_k(\xi)\right| \leqslant C_1, \qquad \left|\xi \frac{d}{d\xi}\hat{b}_\alpha(\xi)\right| \leqslant C_2.$$

Then for all $f \in L_{\mathbf{p}}(\tilde{\Omega})$ *the problem* $(14)-(15)$ *has a unique solution and the following coercive uniform estimate for* $\lambda \in S(\varphi),\ \varphi \in [0, \pi)$ *holds*

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \left\| a_k * \frac{\partial^k u}{\partial t^k} \right\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \| |\lambda| u\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \sum_{|\alpha| \leqslant 2m} \|b_\alpha a_\alpha D^\alpha * u\|_{L_{\mathbf{p}}(\tilde{\Omega})} \leqslant C \|f\|_{L_{\mathbf{p}}(\tilde{\Omega})}.$$

Proof. Let $E = L_{p_1}(\Omega)$. By virtue of [6], $L_{p_1}(\Omega)$ is UMD space for $p_1 \in (1, \infty)$. Consider the operator A defined by

$$D(A) = W_{p_1}^{2m}(\Omega; B_j u = 0), \qquad A(x)u = \sum_{|\alpha| \leqslant 2m} b_\alpha(t) a_\alpha(y) D^\alpha u(y).$$

It is easy to see that $\hat{A}(\xi)$ and $\frac{d}{d\xi}\hat{A}(\xi)$ are operators in $L_{p_1}(\Omega)$ defined by

$$D(\hat{A}) = D\left(\frac{d}{d\xi}\hat{A}\right) = W_{p_1}^{2m}(\Omega; B_j u = 0), \tag{16}$$
$$\hat{A}(\xi)u = \sum_{|\alpha| \leqslant 2m} \hat{b}_\alpha(\xi) a_\alpha(y) D^\alpha u(y),$$
$$\frac{d}{d\xi}\hat{A}(\xi)u = \sum_{|\alpha| \leqslant 2m} \frac{d}{d\xi}\hat{b}_\alpha(\xi) a_\alpha(y) D^\alpha u(y).$$

Therefore, the Fourier transformation of problem (14)–(15) can be rewritten in the form of (1), where $u(t) = u(t,.)$, $f(t) = f(t,.)$ are functions with values in $E = L_{p_1}(\Omega)$. In view of condition (1)–(5) and by virtue of [4] the following problems for $f \in L_{p_1}(\Omega)$ and for $|\lambda| \to \infty$,

$$\lambda u(y) + \sum_{|\beta| \leqslant 2m} \hat{b}_\alpha(\xi) a_\beta(y) D^\beta u(y) = f(y), \tag{17}$$
$$B_j u = \sum_{|\beta| \leqslant m_j} b_{j\beta}(y) D^\beta u(y) = 0, \quad j = 1, 2, ..., m$$
$$\lambda u(y) + \sum_{|\beta| \leqslant 2m} \frac{d}{d\xi}\hat{b}_\alpha(\xi) a_\beta(y) D^\beta u(y) = f(y), \tag{18}$$
$$B_j u = \sum_{|\beta| \leqslant m_j} b_{j\beta}(y) D^\beta u(y) = 0, \quad j = 1, 2, ..., m$$

has unique solutions belong to $W_{p_1}^{2m}(\Omega)$ and the coercive estimates hold

$$\|u\|_{W_{p_1}^{2m}(\Omega)} \leqslant C\left\|\hat{A}u\right\|_{L_{p_1}(\Omega)}, \quad \|u\|_{W_{p_1}^{2m}(\Omega)} \leqslant C\left\|\frac{d}{d\xi}\hat{A}u\right\|_{L_{p_1}(\Omega)} \tag{19}$$

for solutions of the problems (17) and (18), respectively. In view of condition (5) from (16) and (19) we obtain

$$\left\|\frac{d}{d\xi}\hat{A}(\xi)u\right\|_{L_{p_1}(\Omega)} \leqslant C\|u\|_{W_{p_1}^{2m}(\Omega)} \leqslant C\left\|\hat{A}u\right\|_{L_{p_1}(\Omega)},$$

$$\left\|\xi\frac{d}{d\xi}\hat{A}(\xi)u\right\|_{L_{p_1}(\Omega)} \leqslant C\|u\|_{W_{p_1}^{2m}(\Omega)} \leqslant C\left\|\hat{A}u\right\|_{L_{p_1}(\Omega)}.$$

That is $\hat{a}_k$ and $\hat{A}(\xi)$ are hold (9)–(10) conditions. Moreover, by virtue of [9, Theorem 8.2] the operator $\hat{A}$, generated by (16), is uniformly R-sectorial in L_{p_1}. I.e. all conditions of Theorem 2.1 hold and we obtain the assertion. □

5. Infinite systems of integro-differential equations

Consider the following infinity system

$$\sum_{k=0}^{l} a_k * \frac{d^k u_m}{dt^k} + \sum_{j=1}^{\infty}(d_j+\lambda)*u_j(t) = f_m(t), \quad t\in R^n, \quad m=1,2,...,\infty. \tag{20}$$

CONDITION 5.1. There are positive constants C_1 and C_2 so that for $\left\{d_j(t)\right\}_1^{\infty} \in l_q$ for all $t\in R$ and some $t_0\in R$,

$$C_1\left|d_j(t_0)\right| \leqslant \left|d_j(t)\right| \leqslant C_2\left|d_j(t_0)\right|.$$

Suppose $\hat{a}_k$, $\hat{d}_m \in C^{(1)}(R)$ and there are positive constants M_i, $i=1,\,2$ so that

$$\left|\xi\frac{d}{d\xi}\hat{a}_\alpha(\xi)\right| \leqslant M_1, \quad \left|\xi\frac{d}{d\xi}\hat{d}_m(\xi)\right| \leqslant M_2.$$

Let

$$D(t) = \{d_m(t)\},\ d_m>0,\ u=\{u_m\},\ D*u=\{d_m*u_m\},\ m=1,2,...\infty,$$

$$l_q(D) = \left\{u : u\in l_q, \|u\|_{l_q(D)} = \|D*u\|_{l_q} = \left(\sum_{m=1}^{\infty}|d_m*u_m|^q\right)^{\frac{1}{q}} < \infty\right\},\ 1<q<\infty.$$

Let Q be the differential operator in $L_p(R;l_q)$ generated by problem (20). Let

$$B = B(L_p(R;l_q)).$$

THEOREM 5.1. *Suppose the Condition 5.1 satisfied.*
Then:

(a) for $f(t) = \{f_m(t)\}_1^\infty \in L_p(R; l_q(D))$, $\lambda \in S(\varphi)$, $\varphi \in [0, \pi)$ *the problem* (20) *has a unique solution and the coercive uniform estimate*

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \left\| a_k * \frac{d^k u}{dt^k} \right\|_{L_p(R;l_q)} + \|D * u\|_{L_p(R;l_q)} + |\lambda| \|u\|_{L_p(R;l_q)} \leqslant C \|f\|_{L_p(R;l_q)} \quad (21)$$

holds for the solution of the problem (20).

(b) For $\lambda \in S(\varphi)$, $\varphi \in [0, \pi)$ *there exists a resolvent* $(Q+\lambda)^{-1}$ *of operator* Q *and*

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \left\| a_k * \left[\frac{d^k}{dt^k} (Q+\lambda)^{-1} \right] \right\|_B + \left\| D * (Q+\lambda)^{-1} \right\|_B + \left\| |\lambda| (Q+\lambda)^{-1} \right\|_B \leqslant C. \quad (22)$$

Proof. Really, let $E = l_q$, A be infinite matrices, such that

$$A = [d_m(t) \delta_{jm}], \quad m, j = 1, 2, \ldots \infty.$$

Then

$$\hat{A}(\xi) = \left[\hat{d}_m(\xi) \delta_{jm}\right], \quad \frac{d}{d\xi} \hat{A}(\xi) = \left[\frac{d}{d\xi} \hat{d}_m(\xi) \delta_{jm} \right], \quad m, j = 1, 2, \ldots \infty.$$

It is clear to see that (9)–(10) condition are hold for $\hat{a}_k(\xi)$, $\hat{A}(\xi)$ and the operator A is uniformly R-sectorial in l_q. Therefore, by virtue of Theorem 2.1 and Result 2.2 we obtain that, the problem (20) for all $f \in L_p(R; l_q)$, $\lambda \in S(\varphi)$, $\varphi \in (0, \pi)$ and sufficiently large $|\lambda|$ has a unique solution u and estimates (21), (22) are hold. □

REMARK 5.1. There are a lot of sectorial operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of E and concrete sectorial differential, pseudo differential operators, or finite, infinite matrices, etc. instead of operator A on (1), we can obtain the maximal regularity of different class of convolution equations or system of equations by virtue of Theorem 2.1.

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