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FURTHER DEVELOPMENTS OF FURUTA INEQUALITY OF INDEFINITE TYPE

N. BEBIANO, R. LEMOS, J. DA PROVIDÊNCIA AND G. SOARES

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Abstract. A selfadjoint involutive matrix J endows $\mathbb{C}^n$ with an indefinite inner product $[\cdot,\cdot]$ given by $[x,y] := \langle Jx,y\rangle$, $x,y\in\mathbb{C}^n$. We study matrix inequalities for J-selfadjoint matrices with nonnegative eigenvalues. Namely, Furuta inequality of indefinite type is revisited. Characterizations of the J-chaotic order and of the J-relative entropy are obtained via Furuta inequality. The parallelism between the definite versions of the inequalities on Hilbert spaces and the corresponding indefinite versions on Krein spaces is pointed out.

1. Introduction

Let M_n denote the algebra of $n\times n$ complex matrices and let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{C}^n$. For a selfadjoint involution $J\in M_n$, that is, $J=J^*$ and $J^2=I$, we consider $\mathbb{C}^n$ with the Krein structure endowed by the indefinite inner product $[\cdot,\cdot]$

$$[x,y] := \langle Jx,y\rangle, \quad x,y\in\mathbb{C}^n.$$

The J-*adjoint* matrix $A^\#$ of $A\in M_n$ is defined by

$$[Ax,y] = [x,A^\# y], \quad x,y\in\mathbb{C}^n,$$

or equivalently, $A^\# = JA^*J$. A matrix $A\in M_n$ is said to be J-*selfadjoint* if $A^\# = A$, that is, if JA is selfadjoint. For a pair of J-selfadjoint matrices A,B, the J-*order relation* $A\geqslant^J B$ is defined as

$$[Ax,x]\geqslant[Bx,x], \quad x\in\mathbb{C}^n,$$

where this order relation means that the selfadjoint matrix $JA-JB$ is positive semidefinite (notation: $JA-JB\geqslant 0$). A matrix $A\in M_n$ is said to be a J-*contraction* if $I\geqslant^J A^\# A$, or equivalently,

$$[x,x]\geqslant[Ax,Ax], \quad x\in\mathbb{C}^n.$$

It is well known that the spectrum of a J-selfadjoint matrix is symmetric relatively to the real axis. If A is J-selfadjoint and $I\geqslant^J A$, then all the eigenvalues of A are real. In fact, in this case $I-A$ is the product of the selfadjoint matrix J and a positive

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semidefinite matrix. If A is a J-contraction, by a Theorem of Potapov-Ginzburg [1, Chapter 2, Section 4] all the eigenvalues of $A^{\#}A$ are nonnegative.

A real valued continuous function $f(t)$ defined on a real (finite or infinite) interval (α,β) is said to be *operator monotone* if $f(A) \geqslant f(B)$, whenever $A \geqslant B$, that is, $A - B \geqslant 0$. If A is a J-selfadjoint matrix with real spectrum $\sigma(A)$, and $\sigma(A) \subset (\alpha,\beta)$, then for any operator monotone function $f(t)$ on (α,β), we can define $f(A)$ by the Dunford-Riesz integral

$$f(A) := \frac{1}{2\pi i}\int_{\Gamma} f(\xi)(\xi I - A)^{-1} d\xi,$$

where Γ is a closed rectifiable contour in the domain of analytic continuation of $f(t)$, surrounding $\sigma(A)$ positively in its interior. Moreover, $f(A)$ is J-selfadjoint. Ando [2, Theorem 4] proved that if A and B are J-selfadjoint matrices with real spectra contained in the interval (α,β), then $A \geqslant^J B$ implies $f(A) \geqslant^J f(B)$ for any operator monotone function $f(t)$ on (α,β). Thus, having in mind that $f(t) = -\frac{1}{t}$ is operator monotone on $(0,+\infty)$, if A,B are J-selfadjoint matrices with positive eigenvalues and $A \geqslant^J B$, then $B^{-1} \geqslant^J A^{-1}$. Since the principal branch of the logarithm, denoted by $\mathrm{Log}(t)$, is operator monotone, Sano [8, Corollary 2] concluded that if A, B are J-selfadjoint matrices with positive eigenvalues and $A \geqslant^J B$, then $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$. This inequality relation is called the *J-chaotic order* and is weaker than the usual J-order relation $A \geqslant^J B$.

Throughout we always consider the principal branch of any matrix function. The celebrated Löwner-Heinz inequality establishes that $A \geqslant B \geqslant 0$ implies $A^{\alpha} \geqslant B^{\alpha}$ for any $0 \leqslant \alpha \leqslant 1$. Sano [8, Theorem 2.6] obtained the following indefinite version of Löwner-Heinz inequality. For J-selfadjoint matrices A,B with nonnegative eigenvalues such that $I \geqslant^J A \geqslant^J B$ and $0 < \alpha < 1$, then the J-selfadjoint powers A^{α}, B^{α} are well defined and $I \geqslant^J A^{\alpha} \geqslant^J B^{\alpha}$. The particular case of this result when the exponent $\alpha = 1/2$ is due to Ando [2, Theorem 6], being the cases $\alpha = 0$, $\alpha = 1$ trivially satisfied. The famous Furuta [4] inequality, which is an extension of Löwner-Heinz inequality, states that if $A \geqslant B \geqslant 0$ then, for each $r \geqslant 0$,

$$\left(A^{\frac{r}{2}} A^{p} A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{1}$$

and

$$\left(B^{\frac{r}{2}} A^{p} B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(B^{\frac{r}{2}} B^{p} B^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{2}$$

hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$. Recently, Sano [8, Theorem 3.4] obtained the indefinite version of Furuta inequality (1), which yields Löwner-Heinz inequality of indefinite type as a particular case (cf. Theorem 2.1 and take $r = 0$).

This note surveys previous results in the area and complements some of them. It is organized as follows. In Section 2, the indefinite version of Furuta inequality (2) is obtained. A characterization of the J-chaotic order due to Sano [9, Theorem 7] is complemented and is applied to estimate the J-relative entropy. In Section 3, additional characterizations of the J-chaotic order are obtained. The proofs of the results here presented for Krein spaces follow analogous steps to the corresponding ones concerning spaces with definite metric. They are included for the sake of completeness, being enhanced the specificities inherent to Krein spaces.

2. On the indefinite Furuta inequality and applications

To obtain the indefinite version of Furuta inequality (2), we use the following lemmas.

LEMMA 2.1. [8] *Let A be a J-selfadjoint matrix with nonnegative eigenvalues and $I \geqslant^J A$. Then $I \geqslant^J A^\lambda$ for all $\lambda > 0$.*

LEMMA 2.2. [8] *Let A,B be J-selfadjoint matrices with positive eigenvalues and $I \geqslant^J A$, $I \geqslant^J B$. Then*

$$(ABA)^\lambda = AB^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^2B^{\frac{1}{2}}\right)^{\lambda-1}B^{\frac{1}{2}}A, \qquad \lambda \in \mathbb{R}.$$

LEMMA 2.3. *Let $A,B \in M_n$ be J-selfadjoint matrices. Then $X^\# AX \geqslant^J X^\# BX$ for all $X \in M_n$ if and only if $A \geqslant^J B$.*

Proof. Observe that $X^\# AX \geqslant^J X^\# BX$ for all $X \in M_n$. That is, $X^\#(A-B)X \geqslant^J 0$ for all $X \in M_n$ if and only if $X^*J(A-B)X \geqslant 0$ for all $X \in M_n$, which occurs if and only if $J(A-B) \geqslant 0$, or equivalently, $A \geqslant^J B$. □

THEOREM 2.1. *Let A,B be J-selfadjoint matrices with nonnegative eigenvalues and $\mu I \geqslant^J A \geqslant^J B$ for some $\mu > 0$. For each $r \geqslant 0$,*

$$\text{(i)} \quad \left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

and

$$\text{(ii)} \quad \left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

hold for all $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

Proof. The part (i) was obtained by Sano [8], so we prove (ii). Without loss of generality, we may suppose that $\mu = 1$ (otherwise, replace A and B by $\frac{1}{\mu}A$ and $\frac{1}{\mu}B$, respectively). By [8, Proposition 3.3] we may assume that A, B are invertible. If $0 \leqslant p \leqslant 1$, the result is obvious by Löwner-Heinz inequality of indefinite type and Lemma 2.3. We only have to consider $p > 1$ and $q = \frac{p+r}{1+r}$, since the case $q > \frac{p+r}{1+r}$ follows by Löwner-Heinz inequality of indefinite type. So, we prove the following *essential part* of (ii):

$$\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \geqslant^J B^{1+r} \tag{3}$$

for all $p > 1$ and $r \geqslant 0$.

If $0 \leqslant r \leqslant 1$, $A \geqslant^J B$ implies $B^{-r} \geqslant^J A^{-r}$. Thus, by Lemmas 2.1, 2.2, 2.3 and taking into account that $f(t) = t^{\frac{p-1}{p+r}}$ is operator monotone on $(0, +\infty)$, we successively obtain

$$\begin{aligned}\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} &= B^{\frac{r}{2}} A^{\frac{p}{2}} \left(A^{-\frac{p}{2}} B^{-r} A^{-\frac{p}{2}}\right)^{\frac{p-1}{p+r}} A^{\frac{p}{2}} B^{\frac{r}{2}} \\ &\geqslant^J B^{\frac{r}{2}} A^{\frac{p}{2}} \left(A^{-\frac{p}{2}} A^{-r} A^{-\frac{p}{2}}\right)^{\frac{p-1}{p+r}} A^{\frac{p}{2}} B^{\frac{r}{2}} \\ &= B^{\frac{r}{2}} A B^{\frac{r}{2}} \\ &\geqslant^J B^{1+r}.\end{aligned}$$

Now, let $A_1 = \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}$ and $B_1 = B^{1+r}$. Since $I \geqslant^J A_1 \geqslant^J B_1$, by Lemmas 2.1, 2.3 and the above procedure, we get

$$\left(B_1^{\frac{1}{2}} A_1^{\frac{p+r}{1+r}} B_1^{\frac{1}{2}}\right)^{\frac{1+1}{\frac{p+r}{1+r}+1}} \geqslant^J B_1^{1+1},$$

which can be written as follows when $s = 2r+1$

$$\left(B^{\frac{s}{2}} A^p B^{\frac{s}{2}}\right)^{\frac{1+s}{p+s}} \geqslant^J B^{1+s}.$$

This is (3) with $1 \leqslant s \leqslant 3$ instead of r. Repeating this process, we obtain (3) for any $r \geqslant 0$. $\square$

REMARK 1. Consider Furuta *magic box* "$\square$" and define

$$f(\square) := \left(A^{\frac{r}{2}} \square A^{\frac{r}{2}}\right)^{\frac{1}{q}} \qquad \text{and} \qquad g(\square) := \left(B^{\frac{r}{2}} \square B^{\frac{r}{2}}\right)^{\frac{1}{q}}.$$

Although $I \geqslant^J A \geqslant^J B$ does not always ensure $I \geqslant^J A^p \geqslant^J B^p$ for $p > 1$ and for J-selfadjoint matrices A, B with nonnegative eigenvalues, Theorem 2.1 asserts the validity of the following J-order preserving operator inequalities:

$$f(A^p) \geqslant^J f(B^p) \qquad \text{and} \qquad g(A^p) \geqslant^J g(B^p)$$

whenever $I \geqslant^J A \geqslant^J B$, under the assumptions on p, q and r in Theorem 2.1.

REMARK 2. If A, B are J-selfadjoint matrices with positive eigenvalues and $A \geqslant^J B \geqslant^J \mu I$ for some $\mu > 0$, then (i) and (ii) in Theorem 2.1 also hold. In fact, since $\frac{1}{\mu} I \geqslant^J B^{-1} \geqslant^J A^{-1}$, we can apply Theorem 2.1 and for each $r \geqslant 0$, we have

$$\left(B^{-\frac{r}{2}} B^{-p} B^{-\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(B^{-\frac{r}{2}} A^{-p} B^{-\frac{r}{2}}\right)^{\frac{1}{q}} \tag{4}$$

and

$$\left(A^{-\frac{r}{2}}B^{-p}A^{-\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{-\frac{r}{2}}A^{-p}A^{-\frac{r}{2}}\right)^{\frac{1}{q}} \tag{5}$$

for all $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$. Taking inverses in (4) and (5), we obtain (ii) and (i), respectively.

REMARK 3. If $0<q<1$ or $(1+r)q<p+r$ with $p>0$, $q>0$ and $r>0$, then there exist J-selfadjoint matrices A, B with nonnegative eigenvalues and $I \geqslant^J A \geqslant^J B$, such that

$$\left(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}}\right)^{\frac{1}{q}} \not\geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{1}{q}}.$$

Indeed, for $J = \operatorname{diag}(1,-1)$ let

$$A = \operatorname{diag}\left(\frac{1}{2},2\right) \qquad \text{and} \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix}.$$

The above matrices are J-selfadjoint and have nonnegative eigenvalues. It is easy to see that $I \geqslant^J A \geqslant^J B$. In fact,

$$J(I-A) = \operatorname{diag}\left(\frac{1}{2},1\right) \geqslant 0, \quad J(I-B) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \geqslant 0, \quad J(A-B) = \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix} \geqslant 0.$$

Considering $0<q=\frac{1}{2}<1$, $r=2$ and $p=1$, then

$$A^{\frac{r+p}{q}} \not\geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{1}{q}},$$

because the eigenvalues of

$$J\left(A^6-(ABA)^2\right) = \begin{bmatrix} \frac{65}{64} & 16 \\ 16 & 191 \end{bmatrix}$$

are -0.32243 and 192.338, and so $J\left(A^6-(ABA)^2\right) \not\geqslant 0$.

For Theorem 2.1 (ii), an analogous situation occurs. For $J = \operatorname{diag}(1,-1)$, let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{3} & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix},$$

which are J-selfadjoint matrices with nonnegative eigenvalues. It is easy to see that $I \geqslant^J A \geqslant^J B$, since

$$J(I-A) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & 2 \end{bmatrix} \geqslant 0, \quad J(I-B) = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \geqslant 0, \quad J(A-B) = \begin{bmatrix} \frac{1}{2} & -\frac{2}{3} \\ -\frac{2}{3} & 1 \end{bmatrix} \geqslant 0.$$

Considering $q=r=2$ and $p=8$, then $(1+r)q<p+r$ and

$$\left(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\right)^{\frac{1}{q}} \not\geqslant^J B^{\frac{r+p}{q}},$$

because the eigenvalues of the matrix

$$J\left((BA^8B)^{\frac{1}{2}} - B^5\right) = \begin{bmatrix} 35.4137 & -129.36 \\ -129.36 & 472.23 \end{bmatrix}$$

are -0.020707 and 507.664, and so $J\left((BA^8B)^{\frac{1}{2}} - B^5\right) \ngeq 0$.

REMARK 4. The following question remains open: What is the best possible domain for p,q and r in Theorem 2.1 that ensures the validity of inequalities (i) and (ii)?

As an application of Furuta inequality of indefinite type, the following characterization of the J-chaotic order has been obtained.

(Sano [9, Theorem 7]) *If A,B are J-selfadjoint matrices with positive eigenvalues and $I \geqslant^J A$, $I \geqslant^J B$, then the following statements are equivalent:*

(i) $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$*;*

(ii) $A^r \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ *for all* $p > 0$ *and* $r > 0$*.*

THEOREM 2.2. *Let A, B be J-selfadjoint matrices with positive eigenvalues and $I \geqslant^J A$, $I \geqslant^J B$. Then the following statements are mutually equivalent:*

(i) $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$*;*

(ii) $A^p \geqslant^J \left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{1}{2}}$ *for all* $p \geqslant 0$*;*

(iii) $A^r \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$*;*

(iv) $\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *for all* $p \geqslant 0$, $r \geqslant 0$ *and* $q \geqslant 1$ *with* $rq \geqslant p + r$*.*

Proof. By [9, Theorem 7] (i) $\Rightarrow$ (iii) for all $p > 0$ and $r > 0$. If $r = 0$ or $p = 0$, the implication is trivially satisfied.

(iii) $\Rightarrow$ (ii) Consider $r = p$ in (iii).

(ii) $\Rightarrow$ (i) It follows from (ii) that

$$\frac{A^p - I}{p} \geqslant^J \frac{\left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{1}{2}} - I}{p} \tag{6}$$

for all $p > 0$. Since $X^{\frac{1}{2}} - I = (X - I)\left(X^{\frac{1}{2}} + I\right)^{-1}$ for any $X \in M_n$, we may write the right hand side of (6) as follows:

$$\frac{A^{\frac{p}{2}} B^p A^{\frac{p}{2}} - I}{p} \left(\left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{1}{2}} + I\right)^{-1} = \left(A^{\frac{p}{2}} \frac{B^p - I}{p} A^{\frac{p}{2}} + \frac{A^p - I}{p}\right) \left(\left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{1}{2}} + I\right)^{-1}.$$

We observe that A^p and B^p are given by the Dunford-Riesz integral (and coincide with the definitions of functional calculus). Moreover,

$$\frac{\xi^p-1}{p}\to \operatorname{Log}(\xi) \qquad \text{as} \qquad p\to 0^+,$$

uniformly for ξ. Hence, letting $p\to 0^+$ in (6), we have

$$\operatorname{Log}(A)\geqslant^J \frac{1}{2}\big(\operatorname{Log}(B)+\operatorname{Log}(A)\big)$$

and (i) is obtained.

(iii) $\Rightarrow$ (iv) Let $p\geqslant 0$, $r\geqslant 0$ and $q\geqslant 1$ with $rq\geqslant p+r$. By Lemma 2.1 and having in mind the hypothesis, we have

$$I\geqslant^J A^r\geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}}.$$

Hence, we can apply Löwner-Heinz inequality of indefinite type with $\alpha=\frac{p+r}{rq}$ and we get

$$A^{\frac{p+r}{q}}\geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}.$$

(iv) $\Rightarrow$ (iii) The case $r=0$ is trivially satisfied. If $r>0$, take $q=\frac{p+r}{r}$ in (iv). □

REMARK 5. If $rq<p+r$ with $p>0$, $q>0$ and $r>0$, then there exist J-selfadjoint matrices A and B with positive eigenvalues, $I\geqslant^J A$, $I\geqslant^J B$ such that $\operatorname{Log}(A)\geqslant^J\operatorname{Log}(B)$ and

$$\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}\not\geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}.$$

Indeed, for $J=\operatorname{diag}(1,-1)$, let

$$A=\begin{bmatrix}\frac{1}{2} & 1\\ -1 & 5\end{bmatrix} \qquad \text{and} \qquad B=\begin{bmatrix}\frac{1}{10} & 1\\ -1 & 6\end{bmatrix},$$

which are J-selfadjoint matrices with positive eigenvalues. It is easy to see that $I\geqslant^J A$ and $A\geqslant^J B$. We also have

$$J\big(\operatorname{Log}(A)-\operatorname{Log}(B)\big)=\begin{bmatrix}0.970917 & -0.0863738\\ -0.0863738 & 0.188158\end{bmatrix}\geqslant 0.$$

Considering $q=r=2$ and $p=8$, then $rq<p+r$ and

$$A^{\frac{r+p}{q}}\not\geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}},$$

since the eigenvalues of the matrix

$$J\left(A^5-(AB^8A)^{\frac{1}{2}}\right)=\begin{bmatrix} 151.396 & -695.55 \\ -695.55 & 3191.49 \end{bmatrix}$$

are -0.182424 and 3343.07, and so $J\left(A^5-(AB^8A)^{\frac{1}{2}}\right)\not\geqslant 0$.

THEOREM 2.3. *Let A,B be J-selfadjoint matrices with nonnegative eigenvalues and $I\geqslant^J A$, $I\geqslant^J B$. For a fixed $\delta>0$, the following statements are equivalent:*

(i) $A^\delta \geqslant^J B^\delta$*;*

(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *for all* $p\geqslant 0$, $r\geqslant 0$ *and* $q\geqslant 1$ *with* $(\delta+r)q\geqslant p+r$.

Proof. (i) $\Rightarrow$ (ii) Let $p\geqslant 0$, $r\geqslant 0$ and $q\geqslant 1$ with $(\delta+r)q\geqslant p+r$. Consider $p_1=p/\delta$, $r_1=r/\delta$, $A_1=A^\delta$ and $B_1=B^\delta$. Hence, the inequality $(1+r_1)q\geqslant p_1+r_1$ is satisfied. By Lemma 2.1 and (i), we have $I\geqslant^J A_1\geqslant^J B_1$. Thus, we can apply Theorem 2.1 and we obtain

$$\left(A_1^{\frac{r_1}{2}}A_1^{p_1}A_1^{\frac{r_1}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}}\right)^{\frac{1}{q}},$$

and so (ii) follows.

(ii) $\Rightarrow$ (i) Put $p=\delta$, $r=0$ and $q=1$ in (ii). $\square$

REMARK 6. If $\delta>0$ and $A^\delta\geqslant^J B^\delta$, then

$$\frac{A^\delta-I}{\delta}\geqslant^J\frac{B^\delta-I}{\delta}$$

and letting $\delta\to 0^+$, we have $\operatorname{Log}(A)\geqslant^J\operatorname{Log}(B)$. The equivalence between (i) and (iv) in Theorem 2.2 can be interpreted as the case $\delta\to 0^+$ in Theorem 2.3. The case $\delta=1$ in Theorem 2.3 is just Furuta inequality of indefinite type.

Let J be a selfadjoint involution and let A,B be J-selfadjoint matrices with positive eigenvalues and $A\geqslant^J B$. We define the J*-relative entropy* of A and B by

$$S(A|B)=A^{\frac{1}{2}}\operatorname{Log}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

This concept extends the concept of relative operator entropy in Hilbert spaces introduced by Fujii and Kamei [3]. Since $I\geqslant^J A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, it is easy to see that $S(A|B)\leqslant^J 0$. If $A=B$, then $S(A|B)=0$ and conversely.

The following theorem surveys some results valid for Krein spaces which are formally similar to the corresponding ones for Hilbert spaces [5, 7].

THEOREM 2.4. *Let A,B,C be J-selfadjoint matrices with positive eigenvalues, such that $I \geqslant^J A$, $I \geqslant^J B$, $I \geqslant^J C$. Then the following statements are mutually equivalent:*

(i) $\mathrm{Log}(C) \geqslant^J \mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$*;*

(ii) $\left(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geqslant^J A^r \geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$*;*

(iii) $\left(A^{\frac{r}{2}}C^{p_0}A^{\frac{r}{2}}\right)^{\frac{r}{p_0+r}} \geqslant^J A^r \geqslant^J \left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{r}{p_0+r}}$ *for a fixed positive number* p_0 *and for all* r *such that* $r \in [0,r_0]$*, where* r_0 *is a fixed positive number;*

(iv) $\mathrm{Log}\left(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}\right) \geqslant^J \mathrm{Log}\left(A^{p+r}\right) \geqslant^J \mathrm{Log}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$*;*

(v) $\mathrm{Log}\left(A^{\frac{r}{2}}C^{p_0}A^{\frac{r}{2}}\right) \geqslant^J \mathrm{Log}\left(A^{p_0+r}\right) \geqslant^J \mathrm{Log}\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)$ *for a fixed positive number* p_0 *and for all* r *such that* $r \in [0,r_0]$*, where* r_0 *is a fixed positive number;*

(vi) $S(A^{-r}|C^p) \geqslant^J S(A^{-r}|A^p) \geqslant^J S(A^{-r}|B^p)$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$*;*

(vii) $S(A^{-r}|C^{p_0}) \geqslant^J S(A^{-r}|A^{p_0}) \geqslant^J S(A^{-r}|B^{p_0})$ *for a fixed positive number* p_0 *and for all* r *such that* $r \in [0,r_0]$*, where* r_0 *is a fixed positive number.*

Proof. By [9, Theorem 7], $\mathrm{Log}(C) \geqslant^J \mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$ if and only if

$$C^p \geqslant^J \left(C^{\frac{p}{2}}A^rC^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \qquad \text{and} \qquad A^r \geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \tag{7}$$

for all $p \geqslant 0$ and $r \geqslant 0$. Applying Lemma 2.2 to the right hand side of the inequality in (7) on the left, we have

$$C^{\frac{p}{2}}A^{\frac{r}{2}}A^{-r}A^{\frac{r}{2}}C^{\frac{p}{2}} \geqslant^J C^{\frac{p}{2}}A^{\frac{r}{2}}\left(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}\right)^{-\frac{r}{p+r}}A^{\frac{r}{2}}C^{\frac{p}{2}}.$$

By Lemma 2.3, this is equivalent to

$$A^{-r} \geqslant^J \left(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}\right)^{-\frac{r}{p+r}},$$

that is,

$$\left(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geqslant^J A^r.$$

Hence (7) is equivalent to

$$\left(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geqslant^J A^r \geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$$

and (i) $\Leftrightarrow$ (ii) is proved. Since (iii) and (iv) are particular cases of (ii) and (v), respectively, we have (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v). Noting that the principal branch of the logarithm is operator monotone, we have (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v). Considering $r = 0$

in (v), we have (v) $\Rightarrow$ (i). We prove that (iv) $\Leftrightarrow$ (vi). Under the hypothesis $I \geqslant^J A$, $I \geqslant^J B$, $I \geqslant^J C$, and by Lemma 2.1 we have $A^{-r} \geqslant^J I \geqslant^J C^p$ and $A^{-r} \geqslant^J I \geqslant^J B^p$ for all $p \geqslant 0$, $r \geqslant 0$. Now, we show that (iv) $\Leftrightarrow$ (vi). From the definition of J-relative entropy and replacing in Lemma 2.3 X by $A^{-r/2}$, A by $\operatorname{Log}\left(A^{\frac{r}{2}}C^pA^{\frac{r}{2}}\right)$ and B by $\operatorname{Log}(A^{p+r})$, and then substituting in the same lemma A by $\operatorname{Log}(A^{p+r})$ and B by $\operatorname{Log}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)$, the result follows. The equivalence (v) $\Leftrightarrow$ (vii) is analogously proved. $\square$

COROLLARY 2.1. *Let A,B,C be J-selfadjoint matrices with positive eigenvalues and $A \geqslant^J I$, $I \geqslant^J B$, $I \geqslant^J C$. If $\operatorname{Log}(C) \geqslant^J -\operatorname{Log}(A) \geqslant^J \operatorname{Log}(B)$, then*

$$S(A|C) \geqslant^J -2A\operatorname{Log}(A) \geqslant^J S(A|B).$$

Proof. Take $r = p = 1$ and replace A by A^{-1} in Theorem 2.4 (vi). Then

$$S(A|C) \geqslant^J S(A|A^{-1}) \geqslant^J S(A|B)$$

and the proof is complete, observing that $S(A|A^{-1}) = -2A\operatorname{Log}(A)$. $\square$

3. Characterizations of the J-chaotic order

We recall that a matrix U is called *J-unitary* if $U^{\#}U = I$, that is, $U^*JU = J$. If T is a J-contraction, let $|T| = (T^{\#}T)^{\frac{1}{2}}$ be the *J-modulus* of T.

LEMMA 3.1. (cf. [6]) *Let A, B be J-selfadjoint matrices with nonnegative eigenvalues and $I \geqslant^J A$, $I \geqslant^J B$. Then there exists a unique J-unitary matrix U, such that*

$$ABA \leqslant^J UBU^{\#}. \tag{8}$$

Moreover, $U = I$ if and only if A commutes with B.

Proof. By [8, Proposition 3.3], we may assume that A and B are invertible. By assumption, A and B are J-contractions, so is $AB^{\frac{1}{2}}$. In fact, since $I \geqslant^J A^2$ by Lemma 2.1 and $I \geqslant^J B$, it follows that

$$I \geqslant^J B = B^{\frac{1}{2}}IB^{\frac{1}{2}} \geqslant^J B^{\frac{1}{2}}A^2B^{\frac{1}{2}} = \left(AB^{\frac{1}{2}}\right)^{\#}\left(AB^{\frac{1}{2}}\right)$$

and so $AB^{\frac{1}{2}}$ is a J-contraction. Let us consider the J-polar decomposition of $AB^{\frac{1}{2}}$, that is, $AB^{\frac{1}{2}} = U\left|AB^{\frac{1}{2}}\right|$, where U is a J-unitary matrix and $\left|AB^{\frac{1}{2}}\right|$ is the J-modulus of $AB^{\frac{1}{2}}$, which is a J-selfadjoint matrix. Since A and B are invertible, then U is uniquely determined. Moreover,

$$ABA = \left(AB^{\frac{1}{2}}\right)\left(AB^{\frac{1}{2}}\right)^{\#} = U\left|AB^{\frac{1}{2}}\right|^2U^{\#} = UB^{\frac{1}{2}}A^2B^{\frac{1}{2}}U^{\#}.$$

Since $A^2 \leqslant^J I$, by Lemma 2.3 with $X = B^{\frac{1}{2}}U^{\#}$, we have

$$ABA \;=\; UB^{\frac{1}{2}}A^2B^{\frac{1}{2}}U^{\#} \;=\; X^{\#}A^2X \leqslant^J X^{\#}X \;=\; UBU^{\#}.$$

Hence, (8) is proved. Now, we have $U = I$ if and only if $AB^{\frac{1}{2}} = \left|AB^{\frac{1}{2}}\right|$, that is

$$\left(AB^{\frac{1}{2}}\right)^2 = \left(AB^{\frac{1}{2}}\right)^{\#}\left(AB^{\frac{1}{2}}\right).$$

Hence, $AB^{\frac{1}{2}}AB^{\frac{1}{2}} = B^{\frac{1}{2}}AAB^{\frac{1}{2}}$. Since A and B are invertible, this is equivalent to $AB^{\frac{1}{2}} = B^{\frac{1}{2}}A$, which holds if and only if $AB = BA$. □

Next, we present further characterizations of the J-chaotic order.

THEOREM 3.1. (cf. [6]) *Let A,B be J-selfadjoint matrices with positive eigenvalues and $I \geqslant^J A$, $I \geqslant^J B$. Then the following conditions are mutually equivalent:*

(i) $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$.

(ii) *There exists a unique J-unitary matrix U_t for all $t \geqslant 0$, such that $U_t \to I$ as $t \to 0^+$ and*

$$B^t \;\leqslant^J\; U_tA^tU_t^{\#} \qquad t \geqslant 0.$$

(iii) *There exists a unique J-unitary matrix $U_{s,t}$ for all $s \geqslant 0$ and $t \geqslant 0$, such that $U_{s,t} \to I$ as $s \to 0^+$, $t \to 0^+$ and*

$$A^{\frac{s}{2}}B^tA^{\frac{s}{2}} \;\leqslant^J\; U_{s,t}A^{s+t}U_{s,t}^{\#} \qquad s \geqslant 0,\ t \geqslant 0.$$

Proof.
(i) ⇒ (iii) By Theorem 2.2, $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$ if and only if

$$A^u \geqslant^J \left(A^{\frac{u}{2}}B^tA^{\frac{u}{2}}\right)^{\frac{u}{t+u}}$$

for all $t \geqslant 0$ and $u \geqslant 0$. By Lemma 2.2, $I \geqslant^J A^u$. Considering in Theorem 2.1 $A_1 = A^u$ and $B_1 = \left(A^{\frac{u}{2}}B^tA^{\frac{u}{2}}\right)^{\frac{u}{t+u}}$, for each $r \geqslant 0$ we find

$$A^{\frac{u(r+p)}{q}} \geqslant^J \left(A^{\frac{ur}{2}}\left(A^{\frac{u}{2}}B^tA^{\frac{u}{2}}\right)^{\frac{up}{t+u}}A^{\frac{ur}{2}}\right)^{\frac{1}{q}}$$

for $p \geqslant 0$, $q \geqslant 1$ with $(1+r)q \geqslant p+r$. Now, let $q = 2$ and $p = \frac{t+u}{u}$ with $u > 0$. Then

$$A^{\frac{u(r+1)+t}{2}} \geqslant^J \left(A^{\frac{u(r+1)}{2}}B^tA^{\frac{u(r+1)}{2}}\right)^{\frac{1}{2}} \tag{9}$$

holds for each $r \geqslant 0$, $t \geqslant 0$, $u > 0$ with $u(r+1) \geqslant t$. Let $s = \frac{u(r+1)-t}{2}$ and

$$T = A^{-\frac{s+t}{2}} \left(A^{s+\frac{t}{2}} B^t A^{s+\frac{t}{2}}\right)^{\frac{1}{2}} A^{-\frac{s+t}{2}}.$$

We observe that $u(r+1) \geqslant t$ if and only if $s \geqslant 0$. From (9) it follows that $T \leqslant^J I$. Taking the squares of both sides of the equality

$$A^{\frac{s+t}{2}} T A^{\frac{s+t}{2}} = \left(A^{s+\frac{t}{2}} B^t A^{s+\frac{t}{2}}\right)^{\frac{1}{2}},$$

we obtain

$$T A^{s+t} T = A^{\frac{s}{2}} B^t A^{\frac{s}{2}}.$$

Since $I \geqslant^J A^{s+t}$ and $I \geqslant^J T$, by Lemma 3.1 there exists a unique J-unitary matrix $U_{s,t}$, such that $U_{s,t} \to I$ as $s \to 0^+$, $t \to 0^+$ and

$$A^{\frac{s}{2}} B^t A^{\frac{s}{2}} = T A^{s+t} T \leqslant^J U_{s,t} A^{s+t} U_{s,t}^{\#}.$$

(iii) $\Rightarrow$ (ii) Taking $s = 0$ the result trivially follows.

(ii) $\Rightarrow$ (i) Since U_t is J-unitary for any $t > 0$, we have

$$U_t \frac{A^t - I}{t} U_t^{\#} \geqslant^J \frac{B^t - I}{t}.$$

Letting $t \to 0^+$, we have $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$, because $U_t \to I$ as $t \to 0^+$. □

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N. Bebiano
CMUC, Department of Mathematics
University of Coimbra
Coimbra, Portugal
e-mail: bebiano@mat.uc.pt

R. Lemos
MA, Department of Mathematics
University of Aveiro,
Aveiro, Portugal
e-mail: rute@ua.pt

J. da Providência
CFT, Department of Physics
University of Coimbra
Coimbra, Portugal
e-mail: providencia@teor.fis.uc.pt

G. Soares
CM-UTAD, Department of Mathematics
University of Trás-os-Montes and Alto Douro
Vila Real, Portugal
e-mail: gsoares@utad.pt