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MULTIPLICATIVE PERTURBATION BOUNDS FOR WEIGHTED UNITARY POLAR FACTOR

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Abstract. The multiplicative perturbation bounds for weighted unitary polar factor are considered in the weighted unitary invariant norm, weighted spectral norm, and weighted Frobenius norm in this paper. As the special cases, new bounds for subunitary and unitary polar factor are also derived. These new bounds improve the corresponding results published recently to some extent.

1. Introduction

Let $\mathbb{C}^{m\times n}$, $\mathbb{C}_r^{m\times n}$, $\mathbb{C}_{\geqslant}^m$, and $\mathbb{C}_>^m$ denote the set of $m\times n$ complex matrices, subset of $\mathbb{C}^{m\times n}$ comprising matrices with rank r, set of Hermitian positive semidefinite matrices of order m, and subset of $\mathbb{C}_{\geqslant}^m$ consisting of positive definite matrices, respectively. Let I_r be the identity matrix of order r. Given $A\in\mathbb{C}^{m\times n}$, the symbols A^*, $A^{\#}_{MN}$, $R(A)$, $\|A\|_2$, $\|A\|_F$, and $\|A\|$ stand for the conjugate transpose, weighted conjugate transpose, range, spectral norm, Frobenius norm, and unitarily invariant norm of A, respectively. The definition of $A^{\#}_{MN}$ can be found in detail in [19, 23]. Moreover, without specification, in this paper we always assume that $m>n>r$ and the weight matrices $M\in\mathbb{C}_>^m$, $N\in\mathbb{C}_>^n$.

For a matrix $A\in\mathbb{C}_r^{m\times n}$, there are an (M,N) weighted partial isometric matrix Q [26, 27] and a matrix H satisfying $NH\in\mathbb{C}_{\geqslant}^n$ such that

$$A = QH. \tag{1.1}$$

Decomposition (1.1) is called the (M,N) weighted polar decomposition [25, 26] (MN-WPD) of A, and Q and H are called the (M,N) weighted unitary polar factor and generalized nonnegative polar factor, respectively, of this decomposition. In general, the MN-WPD is not unique, while it has been proved that it is unique if the decomposition satisfies

$$R(Q^{\#}_{MN}) = R(H). \tag{1.2}$$

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This condition was given by Yang and Li [25]. In this paper, we assume that the condition (1.2) always holds. Under this condition, the MN-WPD (1.1) can be calculated from the (M,N) singular value decomposition (MN-SVD) (see Lemma 1.1) by

$$Q = U_1 V_1^*, \; H = N^{-1} V_1 \Sigma V_1^*, \tag{1.3}$$

where U_1, V_1, and Σ are as in Lemma 1.1.

When $M = I_m$ and $N = I_n$, the MN-WPD reduces to the generalized polar decomposition (see, e.g., [1, 21]), and Q and H reduce to the subunitary polar factor and nonnegative polar factor. If, in addition, $rank(A) = n$, then the decomposition (1.1) is the polar decomposition, and Q and H are the unitary polar factor and positive polar factor. Therefore, the MN-WPD can be considered as a generalization of the (generalized) polar decomposition. Like the two useful decompositions, it may also have some important applications, see [24] for detailed introduction. Furthermore, some algorithms to compute this decomposition were given in [26].

The perturbation bounds for (generalized) polar decomposition under multiplicative perturbation have been studied by some authors in various norms [2, 5, 10, 12]. The multiplicative perturbation refers to the situation when the perturbed matrix is expressed as $\widetilde{A} = D_1^* A D_2$, where D_1 and D_2 are nonsingular matrices and typically close to the identity matrices of appropriate sizes. In this paper, we focus on studying the multiplicative perturbation bounds for weighted unitary polar factor of WPD. Listed are several bounds for (generalized) polar decomposition, which can be used to compare with the results given in this paper.

Let $A = QH, \widetilde{A} = D_1^* A D_2 = \widetilde{Q}\widetilde{H}$ be the (generalized) polar decomposition of $A, \widetilde{A}$, respectively. In the unitarily invariant norm, for unitary polar factor, i.e., when $A, \widetilde{A} \in \mathbb{C}_n^{m\times n}$, Chen and Li presented a perturbation bound in [2] as follows

$$\begin{aligned}\left\|\widetilde{Q} - Q\right\| \leqslant &\frac{\widetilde{\sigma}_1}{\sigma_n + \widetilde{\sigma}_n}\left(\left\|I_m - D_1^{-1}\right\| + \left\|I_n - D_2^{-1}\right\|\right) + \frac{\sigma_1}{\sigma_n + \widetilde{\sigma}_n}\left(\|D_1 - I_m\| + \|D_2 - I_n\|\right)\\ &+ \min\left\{\left\|I_m - D_1^{-1}\right\|, \|D_1 - I_m\|\right\},\end{aligned} \tag{1.4}$$

where $\sigma_1, \widetilde{\sigma}_1$ are the biggest singular values of $A, \widetilde{A}$ and $\sigma_n, \widetilde{\sigma}_n$ are the smallest singular values of $A, \widetilde{A}$, respectively.

In the Frobenius norm, a bound for subunitary or unitary polar factor is described in the following

$$\left\|\widetilde{Q} - Q\right\|_F \leqslant \sqrt{\left(\left\|I_m - D_1^{-1}\right\|_F + \left\|I_n - D_2^{-1}\right\|_F\right)^2 + \left(\|D_1 - I_m\|_F + \|D_2 - I_n\|_F\right)^2}. \tag{1.5}$$

This bound was obtained by Li for $A, \widetilde{A} \in \mathbb{C}_n^{m\times n}$ in [10] and by Chen, Li, and Sun for $A, \widetilde{A} \in \mathbb{C}_r^{m\times n}$ in [5].

In order to make this paper more self-contained, we now introduce the definitions of weighted norms (see Definition 1.1), and MN-SVD [19, 22] (see Lemma 1.1). Two other lemmas needed in this paper are also listed, where Lemma 1.2 can be found in [6] and Lemma 1.3 can be found in [11].

DEFINITION 1.1. Let $A \in \mathbb{C}_r^{m\times n}$. We call the norms $\|A\|_{(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|$, $\|A\|_{2(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|_2$, and $\|A\|_{F(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|_F$ the weighted unitary invariant norm, weighted spectral norm, and weighted Frobenius norm of A, respectively.

It is worth pointing out that the weighted spectral norm of A is synonymous with the weighted norm of A defined as $\|A\|_{MN} = \left\|M^{1/2}AN^{-1/2}\right\|_2$ in [23] and weighted unitary invariant norm is equivalent to the (M, N)-invariant norm defined by Rao and Rao [20] in essence.

LEMMA 1.1. *Let $A \in \mathbb{C}_r^{m\times n}$. There exist matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ satisfying $U^*MU = I_m$ and $V^*N^{-1}V = I_n$ such that*

$$A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{1.6}$$

where $\Sigma = \operatorname{diag}(\sigma_1,\cdots,\sigma_r)$, $\sigma_i = \sqrt{\lambda_i}$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_r > 0$ are the nonzero eigenvalues of $A^{\#}_{MN}A = (N^{-1}A^*M)A$. The decomposition (1.6) *is called the (M,N) singular value decomposition of A and $\sigma_1 \geqslant \cdots \geqslant \sigma_r > 0$ are called the nonzero (M,N) singular values of A. Further, let $U = (U_1,U_2)$ and $V = (V_1,V_2)$, where $U_1 \in \mathbb{C}^{m\times r}$ and $V_1 \in \mathbb{C}^{n\times r}$. Then*

$$U_1^*MU_1 = V_1^*N^{-1}V_1 = I_r, \quad A = U_1\Sigma V_1^*. \tag{1.7}$$

LEMMA 1.2. *Let $\Omega \in \mathbb{C}^{s\times s}$ and $\Gamma \in \mathbb{C}^{t\times t}$ be two Hermitian matrices, and $S \in \mathbb{C}^{s\times t}$, and*

$$\Delta = [\alpha,\beta] \subset R, \quad \Delta' = R\setminus[\alpha-\delta,\beta+\delta], \quad \delta > 0.$$

Let $\lambda(\Omega)$ and $\lambda(\Gamma)$ denote the eigenvalues sets of Ω and Γ, respectively. If

$$\lambda(\Omega) \subset \Delta, \quad \lambda(\Gamma) \subset \Delta',$$

then the equation $\Omega X - X\Gamma = S$ has a unique solution $X \in \mathbb{C}^{s\times t}$, and moreover, $\|X\| \leqslant \frac{\|S\|}{\delta}$ for any unitarily invariant norm.

LEMMA 1.3. *Let $\Omega \in \mathbb{C}^{s\times s}$ and $\Gamma \in \mathbb{C}^{t\times t}$ be two Hermitian matrices, and let $\lambda(\Omega)$ and $\lambda(\Gamma)$ denote the sets of eigenvalues of Ω and Γ, respectively. If $\lambda(\Omega) \cap \lambda(\Gamma) = \emptyset$, then for any $E,F \in \mathbb{C}^{s\times t}$, the equation $\Omega X - X\Gamma = \Omega E + F\Gamma$ has a unique solution $X \in \mathbb{C}^{s\times t}$, and moreover,*

$$\|X\|_F \leqslant \frac{1}{\eta}\sqrt{\|E\|_F^2 + \|F\|_F^2},$$

where $\eta = \min_{\omega\in\lambda(\Omega),\gamma\in\lambda(\Gamma)} \frac{|\omega-\gamma|}{\sqrt{|\omega|^2+|\gamma|^2}}$. If, in addition, $F = 0$, we have a better bound

$$\|X\|_F \leqslant \frac{1}{\eta}\|E\|_F,$$

where $\eta = \min_{\omega\in\lambda(\Omega),\gamma\in\lambda(\Gamma)} \frac{|\omega-\gamma|}{|\omega|}$.

2. Main results

Let the perturbed matrix $\widetilde{A} \in \mathbb{C}_r^{m\times n}$. Similar to (1.1) and (1.6), we let

$$\widetilde{A} = \widetilde{Q}\widetilde{H} \text{ and } \widetilde{A} = \widetilde{U}\begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix}\widetilde{V}^* \tag{2.1}$$

be the MN-WPD and MN-SVD of $\widetilde{A}$, respectively, in which

$$\widetilde{Q} = \widetilde{U}_1\widetilde{V}_1^*, \quad \widetilde{H} = N^{-1}\widetilde{V}_1\widetilde{\Sigma}\widetilde{V}_1^*, \tag{2.2}$$

where $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2) \in \mathbb{C}^{m\times m}$ and $\widetilde{V} = (\widetilde{V}_1, \widetilde{V}_2) \in \mathbb{C}^{n\times n}$ satisfy $\widetilde{U}^*M\widetilde{U} = I_m$ and $\widetilde{V}^*N^{-1}\widetilde{V} = I_n$, $\widetilde{U}_1 \in \mathbb{C}^{m\times r}$, $\widetilde{V}_1 \in \mathbb{C}^{n\times r}$, $\widetilde{\Sigma} = \mathrm{diag}(\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_r)$ and $\widetilde{\sigma}_1 \geqslant \cdots \geqslant \widetilde{\sigma}_r > 0$ are the nonzero (M,N) singular values of $\widetilde{A}$. Here, we also assume that the uniqueness condition for MN-WPD of $\widetilde{A}$, i.e., $R(\widetilde{Q}^{\#}_{MN}) = R(\widetilde{H})$, is always satisfied.

Furthermore, analogous to (1.7), we have

$$\widetilde{U}_1^*M\widetilde{U}_1 = \widetilde{V}_1^*N^{-1}\widetilde{V}_1 = I_r, \quad \widetilde{A} = \widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^*. \tag{2.3}$$

Next, we first study the multiplicative perturbation bounds for weighted unitary polar factor, and then present the corresponding perturbation bounds for subunitary and unitary polar factors as the special cases.

THEOREM 2.1. *Let $A, \widetilde{A} = D_1^*AD_2 \in \mathbb{C}_r^{m\times n}$, where $D_1 \in \mathbb{C}_m^{m\times m}, D_2 \in \mathbb{C}_n^{n\times n}$. If $A, \widetilde{A}$ have the MN-WPDs according to* (1.1) *and* (2.1)*, respectively, then*

$$\begin{aligned}
\left\|\widetilde{Q} - Q\right\|_{(MN)} \leqslant\ & \frac{\widetilde{\sigma}_1}{\sigma_r + \widetilde{\sigma}_r}\left(\left\|I_m - D_1^{*-1}\right\|_{(MM)} + \left\|I_n - D_2^{-1}\right\|_{(NN)}\right) \\
&+ \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\|D_1^* - I_m\|_{(MM)} + \|D_2 - I_n\|_{(NN)}\right) \\
&+ \min\left\{\left(\|D_1^* - I_m\|_{(MM)} + \left\|I_n - D_2^{-1}\right\|_{(NN)}\right),\right. \\
&\qquad \left.\left(\left\|I_m - D_1^{*-1}\right\|_{(MM)} + \|D_2 - I_n\|_{(NN)}\right)\right\},
\end{aligned} \tag{2.4}$$

where $\sigma_1, \widetilde{\sigma}_1$ are the biggest (M,N) singular values of $A, \widetilde{A}$ and $\sigma_r, \widetilde{\sigma}_r$ are the smallest (M,N) singular values of $A, \widetilde{A}$, respectively.

Proof. According to the MN-SVDs of A and $\widetilde{A}$, we have

$$\widetilde{U}^*M(\widetilde{A} - A)N^{-1}V = \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix}\widetilde{V}^*N^{-1}V - \widetilde{U}^*MU\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.5}$$

Further, taking $\widetilde{A} = D_1^*AD_2$ into account, we have

$$\begin{aligned}
\widetilde{U}^*M(\widetilde{A} - A)N^{-1}V &= \widetilde{U}^*M(D_1^*AD_2 - A)N^{-1}V \\
&= \widetilde{U}^*M(D_1^*AD_2 - D_1^*A + D_1^*A - A)N^{-1}V
\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{U}^* M \left(\widetilde{A}(I_n - D_2^{-1}) + (D_1^* - I_m)A \right) N^{-1} V \\
&= \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \widetilde{V}^* (I_n - D_2^{-1}) N^{-1} V + \widetilde{U}^* M (D_1^* - I_m) U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2.6}
$$

Thus, (2.5) and (2.6) together reveal that

$$
\begin{aligned}
&\begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \widetilde{V}^* N^{-1} V - \widetilde{U}^* M U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \widetilde{V}^* (I_n - D_2^{-1}) N^{-1} V + \widetilde{U}^* M (D_1^* - I_m) U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2.7}
$$

Using the similar argument, we have

$$
\begin{aligned}
&U^* M \widetilde{U} \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* N^{-1} \widetilde{V} \\
&\qquad = U^* M (I_m - D_1^{*-1}) \widetilde{U} \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* (D_2 - I_n) N^{-1} \widetilde{V}.
\end{aligned} \tag{2.8}
$$

The equality (2.7) implies that

$$
\begin{aligned}
&\begin{pmatrix} \widetilde{\Sigma}\widetilde{V}_1^* N^{-1} V_1 & \widetilde{\Sigma}\widetilde{V}_1^* N^{-1} V_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \widetilde{U}_1^* M U_1 \Sigma & 0 \\ \widetilde{U}_2^* M U_1 \Sigma & 0 \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{\Sigma}\widetilde{V}_1^* (I_n - D_2^{-1}) N^{-1} V_1 & \widetilde{\Sigma}\widetilde{V}_1^* (I_n - D_2^{-1}) N^{-1} V_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \widetilde{U}_1^* M (D_1^* - I_m) U_1 \Sigma & 0 \\ \widetilde{U}_2^* M (D_1^* - I_m) U_1 \Sigma & 0 \end{pmatrix},
\end{aligned} \tag{2.9}
$$

and the equality (2.8) implies that

$$
\begin{aligned}
&\begin{pmatrix} U_1^* M \widetilde{U}_1 \widetilde{\Sigma} & 0 \\ U_2^* M \widetilde{U}_1 \widetilde{\Sigma} & 0 \end{pmatrix} - \begin{pmatrix} \Sigma V_1^* N^{-1} \widetilde{V}_1 & \Sigma V_1^* N^{-1} \widetilde{V}_2 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} U_1^* M (I_m - D_1^{*-1}) \widetilde{U}_1 \widetilde{\Sigma} & 0 \\ U_2^* M (I_m - D_1^{*-1}) \widetilde{U}_1 \widetilde{\Sigma} & 0 \end{pmatrix} + \begin{pmatrix} \Sigma V_1^* (D_2 - I_n) N^{-1} \widetilde{V}_1 & \Sigma V_1^* (D_2 - I_n) N^{-1} \widetilde{V}_2 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2.10}
$$

Then, according to (2.9), we have

$$
\widetilde{\Sigma}\widetilde{V}_1^* N^{-1} V_1 - \widetilde{U}_1^* M U_1 \Sigma = \widetilde{\Sigma}\widetilde{V}_1^* (I_n - D_2^{-1}) N^{-1} V_1 + \widetilde{U}_1^* M (D_1^* - I_m) U_1 \Sigma, \tag{2.11}
$$

$$
\widetilde{V}_1^* N^{-1} V_2 = \widetilde{V}_1^* (I_n - D_2^{-1}) N^{-1} V_2, \tag{2.12}
$$

$$
-\widetilde{U}_2^* M U_1 = \widetilde{U}_2^* M (D_1^* - I_m) U_1, \tag{2.13}
$$

and according to (2.10), we have

$$
U_1^* M \widetilde{U}_1 \widetilde{\Sigma} - \Sigma V_1^* N^{-1} \widetilde{V}_1 = U_1^* M (I_m - D_1^{*-1}) \widetilde{U}_1 \widetilde{\Sigma} + \Sigma V_1^* (D_2 - I_n) N^{-1} \widetilde{V}_1, \tag{2.14}
$$

$$U_2^* M\widetilde{U}_1 = U_2^* M(I_m - D_1^{*-1})\widetilde{U}_1, \tag{2.15}$$
$$-V_1^* N^{-1}\widetilde{V}_2 = V_1^*(D_2 - I_n)N^{-1}\widetilde{V}_2. \tag{2.16}$$

Subtracting (2.11) from the conjugate transpose of (2.14) leads to

$$\begin{aligned}\widetilde{\Sigma}\left(\widetilde{U}_1^* MU_1 - \widetilde{V}_1^* N^{-1}V\right) &+ \left(\widetilde{U}_1^* MU_1 - \widetilde{V}_1^* N^{-1}V_1\right)\Sigma \\ &= \widetilde{\Sigma}\left(\widetilde{U}_1^*(I_m - D_1^{-1})MU_1 - \widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_1\right) \\ &\quad + \left(\widetilde{V}_1^* N^{-1}(D_2^* - I_n)V_1 - \widetilde{U}_1^* M(D_1^* - I_m)U_1\right)\Sigma. \end{aligned}\tag{2.17}$$

Applying Lemma 1.2 to (2.17) with $\Omega = \widetilde{\Sigma}, \Gamma = -\Sigma$, and

$$X = \widetilde{U}_1^* MU_1 - \widetilde{V}_1^* N^{-1}V_1, \tag{2.18}$$
$$\begin{aligned} S = \widetilde{\Sigma}&\left(\widetilde{U}_1^*(I_m - D_1^{-1})MU_1 - \widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_1\right) \\ &+ \left(\widetilde{V}_1^* N^{-1}(D_2^* - I_n)V_1 - \widetilde{U}_1^* M(D_1^* - I_m)U_1\right)\Sigma \end{aligned}$$

gives

$$\|X\| = \left\|\widetilde{U}_1^* MU_1 - \widetilde{V}_1^* N^{-1}V_1\right\| \leqslant \frac{1}{\delta}\|S\|, \tag{2.19}$$

where $\delta = \sigma_r + \widetilde{\sigma}_r$.

Since

$$\begin{aligned}\widetilde{U}^* M(\widetilde{Q} - Q)N^{-1}V &= \begin{pmatrix}\widetilde{U}_1^* \\ \widetilde{U}_2^*\end{pmatrix} M(\widetilde{U}_1\widetilde{V}_1^* - U_1V_1^*)N^{-1}(V_1, V_2) \\ &= \begin{pmatrix}\widetilde{V}_1^* N^{-1}V_1 - \widetilde{U}_1^* MU_1 & \widetilde{V}_1^* N^{-1}V_2 \\ -\widetilde{U}_2^* MU_1 & 0\end{pmatrix},\end{aligned}\tag{2.20}$$

noting (2.18), we can obtain

$$\left\|\widetilde{U}^* M(\widetilde{Q} - Q)N^{-1}V\right\| \leqslant \|X\| + \left\|\widetilde{V}_1^* N^{-1}V_2\right\| + \left\|\widetilde{U}_2^* MU_1\right\|,$$

which together with (2.19) says that

$$\left\|\widetilde{U}^* M(\widetilde{Q} - Q)N^{-1}V\right\| \leqslant \frac{1}{\delta}\|S\| + \left\|\widetilde{V}_1^* N^{-1}V_2\right\| + \left\|\widetilde{U}_2^* MU_1\right\|. \tag{2.21}$$

Therefore, it follows from (2.21), (2.12) and (2.13) that

$$\left\|\widetilde{U}^* M(\widetilde{Q} - Q)N^{-1}V\right\| \leqslant \frac{1}{\delta}\|S\| + \left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_2\right\| + \left\|\widetilde{U}_2^* M(D_1^* - I_m)U_1\right\|. \tag{2.22}$$

Note that

$$\begin{aligned}\|S\| &\leqslant \left\|\left(\widetilde{V}_1^* N^{-1}(D_2^* - I_n)V_1 - \widetilde{U}_1^* M(D_1^* - I_m)U_1\right)\Sigma\right\| \\ &\quad + \left\|\widetilde{\Sigma}\left(\widetilde{U}_1^*(I_m - D_1^{-1})MU_1 - \widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_1\right)\right\| \\ &\leqslant \|\Sigma\|_2 \left\|\widetilde{V}_1^* N^{-1}(D_2^* - I_n)V_1 - \widetilde{U}_1^* M(D_1^* - I_m)U_1\right\| \\ &\quad + \left\|\widetilde{\Sigma}\right\|_2 \left\|\widetilde{U}_1^*(I_m - D_1^{-1})MU_1 - \widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_1\right\| \\ &\leqslant \sigma_1 \left(\left\|\widetilde{V}_1^* N^{-1}(D_2^* - I_n)V_1\right\| + \left\|\widetilde{U}_1^* M(D_1^* - I_m)U_1\right\|\right) \\ &\quad + \widetilde{\sigma}_1 \left(\left\|\widetilde{U}_1^*(I_m - D_1^{-1})MU_1\right\| + \left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V\right\|\right) \\ &\leqslant \sigma_1 \left(\left\|N^{-1/2}(D_2^* - I_n)N^{1/2}\right\| + \left\|M^{1/2}(D_1^* - I_m)M^{-1/2}\right\|\right) \\ &\quad + \widetilde{\sigma}_1 \left(\left\|M^{-1/2}(I_m - D_1^{-1})M^{1/2}\right\| + \left\|N^{1/2}(I_n - D_2^{-1})N^{-1/2}\right\|\right) \\ &= \sigma_1 \left(\|D_2 - I_n\|_{(NN)} + \|D_1^* - I_m\|_{(MM)}\right) \\ &\quad + \widetilde{\sigma}_1 \left(\left\|I_m - D_1^{*-1}\right\|_{(MM)} + \left\|I_n - D_2^{-1}\right\|_{(NN)}\right)\end{aligned} \tag{2.23}$$

and

$$\left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_2\right\| \leqslant \left\|N^{1/2}(I_n - D_2^{-1})N^{-1/2}\right\| = \left\|I_n - D_2^{-1}\right\|_{(NN)}, \tag{2.24}$$

$$\left\|\widetilde{U}_2^* M(D_1^* - I_m)U_1\right\| \leqslant \left\|M^{1/2}(D_1^* - I_m)M^{-1/2}\right\| = \|D_1^* - I_m\|_{(MM)}. \tag{2.25}$$

Then, together with (2.22), (2.23), (2.24), (2.25) and the fact that

$$\left\|\widetilde{U}^* M(\widetilde{Q} - Q)N^{-1}V\right\| = \left\|M^{1/2}(\widetilde{Q} - Q)N^{-1/2}\right\| = \left\|\widetilde{Q} - Q\right\|_{(MN)}, \tag{2.26}$$

we have

$$\begin{aligned}\left\|\widetilde{Q} - Q\right\|_{(MN)} &\leqslant \frac{\widetilde{\sigma}_1}{\sigma_r + \widetilde{\sigma}_r}\left(\left\|I_m - D_1^{*-1}\right\|_{(MM)} + \left\|I_n - D_2^{-1}\right\|_{(NN)}\right) \\ &\quad + \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\|D_1^* - I_m\|_{(MM)} + \|D_2 - I_n\|_{(NN)}\right) \\ &\quad + \left(\|D_1^* - I_m\|_{(MM)} + \left\|I_n - D_2^{-1}\right\|_{(NN)}\right).\end{aligned} \tag{2.27}$$

Moreover, we also have

$$U^* M(\widetilde{Q} - Q)N^{-1}\widetilde{V} = \begin{pmatrix} U_1^* M\widetilde{U}_1 - V_1^* N^{-1}\widetilde{V}_1 & -V_1^* N^{-1}\widetilde{V}_2 \\ U_2^* M\widetilde{U}_1 & 0 \end{pmatrix}, \tag{2.28}$$

which implies

$$\left\|U^* M(\widetilde{Q} - Q)N^{-1}\widetilde{V}\right\| \leqslant \|X^*\| + \left\|V_1^* N^{-1}\widetilde{V}_2\right\| + \left\|U_2^* M\widetilde{U}_1\right\|. \tag{2.29}$$

From (2.14) and (2.15), we can obtain

$$\begin{aligned}\left\|U_2^* M \widetilde{U}_1\right\| &= \left\|U_2^* M (I_m - D_1^{*-1}) \widetilde{U}_1\right\| \\ &\leqslant \left\|M^{1/2}(I_m - D_1^{*-1}) M^{-1/2}\right\| = \|I_m - D_1^{*-1}\|_{(MM)}, \qquad (2.30)\\ \left\|V_1^* N^{-1} \widetilde{V}_2\right\| &= \left\|V_1^*(D_2 - I_n) N^{-1} \widetilde{V}_2\right\| \\ &\leqslant \left\|N^{1/2}(D_2 - I_n) N^{-1/2}\right\| = \|D_2 - I_n\|_{(NN)}. \qquad (2.31)\end{aligned}$$

Therefore, according to (2.29), (2.19), (2.23), (2.30), (2.31), and noting the fact

$$\left\|U^* M(\widetilde{Q} - Q) N^{-1} \widetilde{V}\right\| = \left\|\widetilde{Q} - Q\right\|_{(MN)}, \qquad (2.32)$$

we have

$$\begin{aligned}\left\|\widetilde{Q} - Q\right\|_{(MN)} \leqslant & \frac{\widetilde{\sigma}_1}{\sigma_r + \widetilde{\sigma}_r}\left(\|I_m - D_1^{*-1}\|_{(MM)} + \|I_n - D_2^{-1}\|_{(NN)}\right) \\ & + \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\|D_1^* - I_m\|_{(MM)} + \|D_2 - I_n\|_{(NN)}\right) \\ & + \left(\|I_m - D_1^{*-1}\|_{(MM)} + \|D_2 - I_n\|_{(NN)}\right). \qquad (2.33)\end{aligned}$$

Consequently, the proof is completed combining (2.27) with (2.33).

If the weighted unitary invariant norm in Theorem 2.1 is replaced with the weighted spectral norm, i.e., weighted norm, we have the following smaller perturbation bound.

THEOREM 2.2. *Assume that the conditions of Theorem* 2.1 *hold. Then*

$$\begin{aligned}\left\|\widetilde{Q} - Q\right\|_{MN} \leqslant & \frac{\widetilde{\sigma}_1}{\sigma_r + \widetilde{\sigma}_r}\left(\|I_m - D_1^{*-1}\|_{MM} + \|I_n - D_2^{-1}\|_{NN}\right) \\ & + \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\|D_1^* - I_m\|_{MM} + \|D_2 - I_n\|_{NN}\right) \\ & + \min\left\{\max\left\{\|D_1^* - I_m\|_{MM}, \|I_n - D_2^{-1}\|_{NN}\right\},\right. \\ & \left.\max\left\{\|I_m - D_1^{*-1}\|_{MM}, \|D_2 - I_n\|_{NN}\right\}\right\}. \qquad (2.34)\end{aligned}$$

Proof. Observe that (2.20) can be rewritten as

$$\widetilde{U}^* M(\widetilde{Q} - Q) N^{-1} V = \begin{pmatrix} \widetilde{V}_1^* N^{-1} V_1 - \widetilde{U}_1^* M U_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \widetilde{V}_1^* N^{-1} V_2 \\ -\widetilde{U}_2^* M U_1 & 0 \end{pmatrix}.$$

Then, considering (2.18) and (2.26), we have

$$\begin{aligned}\left\|\widetilde{U}^* M(\widetilde{Q} - Q) N^{-1} V\right\|_2 = \left\|\widetilde{Q} - Q\right\|_{MN} &\leqslant \|X\|_2 + \left\|\begin{pmatrix} 0 & \widetilde{V}_1^* N^{-1} V_2 \\ -\widetilde{U}_2^* M U_1 & 0 \end{pmatrix}\right\|_2 \\ &\leqslant \|X\|_2 + \max\left\{\left\|\widetilde{V}_1^* N^{-1} V_2\right\|_2, \left\|-\widetilde{U}_2^* M U_1\right\|_2\right\},\end{aligned}$$

which together with (2.19), (2.23), (2.12), (2.13), (2.24), and (2.25) implies

$$\begin{aligned}\left\|\widetilde{Q}-Q\right\|_{MN} \leqslant \frac{\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_r}\left(\left\|I_m-D_1^{*-1}\right\|_{MM}+\left\|I_n-D_2^{-1}\right\|_{NN}\right)\\ +\frac{\sigma_1}{\sigma_r+\widetilde{\sigma}_r}\left(\|D_1^*-I_m\|_{MM}+\|D_2-I_n\|_{NN}\right)\\ +\max\left\{\|D_1^*-I_m\|_{MM},\left\|I_n-D_2^{-1}\right\|_{NN}\right\}.\end{aligned} \tag{2.35}$$

Similarly, from (2.28), (2.18), and (2.32), we have

$$\begin{aligned}\left\|U^*M(\widetilde{Q}-Q)N^{-1}\widetilde{V}\right\|_2 = \left\|\widetilde{Q}-Q\right\|_{MN} \leqslant \|X^*\|_2 + \left\|\begin{pmatrix}0 & -V_1^*N^{-1}\widetilde{V}_2\\ U_2^*M\widetilde{U}_1 & 0\end{pmatrix}\right\|_2\\ \leqslant \|X\|_2+\max\left\{\left\|-V_1^*N^{-1}\widetilde{V}_2\right\|_2,\left\|U_2^*M\widetilde{U}_1\right\|_2\right\},\end{aligned}$$

which combined with (2.19), (2.23), (2.30), and (2.31) gives

$$\begin{aligned}\left\|\widetilde{Q}-Q\right\|_{MN} \leqslant \frac{\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_r}\left(\left\|I_m-D_1^{*-1}\right\|_{MM}+\left\|I_n-D_2^{-1}\right\|_{NN}\right)\\ +\frac{\sigma_1}{\sigma_r+\widetilde{\sigma}_r}\left(\|D_1^*-I_m\|_{MM}+\|D_2-I_n\|_{NN}\right)\\ +\max\left\{\left\|I_m-D_1^{*-1}\right\|_{MM},\|D_2-I_n\|_{NN}\right\}.\end{aligned} \tag{2.36}$$

Thus, (2.35) and (2.36) together yield the proof.

If we replace the weighted unitary invariant norm in Theorem 2.1 by the weighted Frobenius norm, an alternative perturbation bound can be derived as follows.

THEOREM 2.3. *Assume that the conditions of Theorem* 2.1 *hold and add a condition that* $\eta = \min_{1\leqslant i,j\leqslant r}\frac{\widetilde{\sigma}_i+\sigma_j}{\sqrt{\widetilde{\sigma}_i^2+\sigma_j^2}} \geqslant \sqrt{2-\varepsilon}$, *where* $\widetilde{\sigma}_i$ *is the* i*-th* (M,N) *singular value of* $\widetilde{A}$, σ_j *is the* j*-th* (M,N) *singular value of* A, *and* $0\leqslant\varepsilon\leqslant 1$, *then*

$$\begin{aligned}\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2 \leqslant \left(1+\frac{\varepsilon}{\eta^2}\right)\Big(\left\|I_m-D_1^{*-1}\right\|_{F(MM)}^2+\|I_m-D_1^*\|_{F(MM)}^2\\ +\left\|I_n-D_2^{-1}\right\|_{F(NN)}^2+\|I_n-D_2\|_{F(NN)}^2\Big)\\ -\left(1-\frac{2-\varepsilon}{\eta^2}\right)\max\Big\{\left(\left\|I_m-D_1^{*-1}\right\|_{F(MM)}^2+\|I_n-D_2\|_{F(NN)}^2\right),\\ \left(\left\|I_n-D_2^{-1}\right\|_{F(NN)}^2+\|I_m-D_1^*\|_{F(MM)}^2\right)\Big\}\end{aligned} \tag{2.37}$$

$$\begin{aligned}\leqslant \frac{2}{2-\varepsilon}\Big(\left\|I_m-D_1^{*-1}\right\|_{F(MM)}^2+\|I_m-D_1^*\|_{F(MM)}^2\\ +\left\|I_n-D_2^{-1}\right\|_{F(NN)}^2+\|I_n-D_2\|_{F(NN)}^2\Big).\end{aligned} \tag{2.38}$$

Proof. Applying Lemma 1.3 to (2.17) with $\Omega = \widetilde{\Sigma}$, $\Gamma = -\Sigma$, X as in (2.18), and

$$E = \widetilde{U}_1^*(I_m - D_1^{-1})MU_1 - \widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_1,$$
$$F = -\widetilde{V}_1^* N^{-1}(D_2^* - I_n)V_1 + \widetilde{U}_1^* M(D_1^* - I_m)U_1,$$
$$\eta = \min_{1\leqslant i,j\leqslant r} \frac{\widetilde{\sigma}_i + \sigma_j}{\sqrt{\widetilde{\sigma}_i^2 + \sigma_j^2}} \geqslant \sqrt{2-\varepsilon}$$

gives

$$\|X\|_F \leqslant \frac{1}{\eta}\sqrt{\|E\|_F^2 + \|F\|_F^2}. \tag{2.39}$$

From (2.20), (2.18), and the properties of the Frobenius norm, we have

$$\left\|\widetilde{U}^* M(\widetilde{Q} - Q)N^{-1}V\right\|_F^2 = \|X\|_F^2 + \left\|\widetilde{V}_1^* N^{-1}V_2\right\|_F^2 + \left\|\widetilde{U}_2^* MU_1\right\|_F^2. \tag{2.40}$$

Thus, it follows from (2.26), (2.40), (2.39), (2.12), and (2.13) that

$$\begin{aligned}\left\|\widetilde{Q} - Q\right\|_{F(MN)}^2 \leqslant \frac{1}{\eta^2}\left(\|E\|_F^2 + \|F\|_F^2\right) + \left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_2\right\|_F^2 \\ + \left\|\widetilde{U}_2^* M(D_1^* - I_m)U_1\right\|_F^2.\end{aligned} \tag{2.41}$$

Note that

$$\begin{aligned}
&\frac{1}{\eta^2}\|E\|_F^2 + \left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_2\right\|_F^2 \\
&\quad\leqslant \frac{2}{\eta^2}\left(\left\|\widetilde{U}_1^*(I_m - D_1^{-1})MU_1\right\|_F^2 + \left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_1\right\|_F^2\right) \\
&\qquad + \left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_2\right\|_F^2 \\
&\quad= \frac{2}{\eta^2}\left\|\widetilde{U}_1^*(I_m - D_1^{-1})MU_1\right\|_F^2 + \frac{2-\varepsilon}{\eta^2}\left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}(V_1, V_2)\right\|_F^2 \\
&\qquad + \frac{\varepsilon}{\eta^2}\left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_1\right\|_F^2 + \left(1 - \frac{2-\varepsilon}{\eta^2}\right)\left\|\widetilde{V}_1^*(I_n - D_2^{-1})N^{-1}V_2\right\|_F^2 \\
&\quad\leqslant \frac{2}{\eta^2}\left(\|I_m - D_1^{*-1}\|_{F(MM)}^2 + \|I_n - D_2^{-1}\|_{F(NN)}^2\right) + \left(1 - \frac{2-\varepsilon}{\eta^2}\right)\|I_n - D_2^{-1}\|_{F(NN)}^2 \\
&\quad= \frac{2}{\eta^2}\|I_m - D_1^{*-1}\|_{F(MM)}^2 + \left(1 + \frac{\varepsilon}{\eta^2}\right)\|I_n - D_2^{-1}\|_{F(NN)}^2.
\end{aligned} \tag{2.42}$$

Similarly,

$$\frac{1}{\eta^2}\|F\|_F^2 + \left\|\widetilde{U}_2^* M(D_1^* - I_m)U_1\right\|_F^2 \leqslant \left(1 + \frac{\varepsilon}{\eta^2}\right)\|I_m - D_1^*\|_{F(MM)}^2 + \frac{2}{\eta^2}\|I_n - D_2\|_{F(NN)}^2,$$

which together with (2.41) and (2.42) leads to

$$\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2 \leqslant \frac{2}{\eta^2}\|I_m-D_1^{*-1}\|_{F(MM)}^2+\left(1+\frac{\varepsilon}{\eta^2}\right)\|I_n-D_2^{-1}\|_{F(NN)}^2 + \left(1+\frac{\varepsilon}{\eta^2}\right)\|I_m-D_1^*\|_{F(MM)}^2+\frac{2}{\eta^2}\|I_n-D_2\|_{F(NN)}^2. \tag{2.43}$$

Furthermore, (2.28) combined with the properties of Frobenius norm implies

$$\left\|U^*M(\widetilde{Q}-Q)N^{-1}\widetilde{V}\right\|_F^2=\|X^*\|_F^2+\left\|V_1^*N^{-1}\widetilde{V}_2\right\|_F^2+\left\|U_2^*M\widetilde{U}_1\right\|_F^2,$$

which together with (2.32), (2.39), (2.15), and (2.16) gives

$$\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2 \leqslant \frac{1}{\eta^2}\left(\|E\|_F^2+\|F\|_F^2\right)+\left\|U_2^*M(I_m-D_1^{*-1})\widetilde{U}_1\right\|_F^2 + \left\|V_1^*(D_2-I_n)N^{-1}\widetilde{V}_2\right\|_F^2. \tag{2.44}$$

Similar to (2.42), we can obtain

$$\frac{1}{\eta^2}\|E\|_F^2+\left\|U_2^*M(I_m-D_1^{*-1})\widetilde{U}_1\right\|_F^2=\frac{1}{\eta^2}\|F^*\|_F^2+\left\|U_2^*M(I_m-D_1^{*-1})\widetilde{U}_1\right\|_F^2 \leqslant \left(1+\frac{\varepsilon}{\eta^2}\right)\|I_m-D_1^{*-1}\|_{F(MM)}^2+\frac{2}{\eta^2}\|I_n-D_2^{-1}\|_{F(NN)}^2 \tag{2.45}$$

and

$$\frac{1}{\eta^2}\|F\|_F^2+\left\|V_1^*(D_2-I_n)N^{-1}\widetilde{V}_2\right\|_F^2=\frac{1}{\eta^2}\left\|\widetilde{F}^*\right\|_F^2+\left\|V_1^*(D_2-I_n)N^{-1}\widetilde{V}_2\right\|_F^2 \leqslant \left(1+\frac{\varepsilon}{\eta^2}\right)\|I_n-D_2\|_{F(NN)}^2+\frac{2}{\eta^2}\|I_m-D_1^*\|_{F(MM)}^2. \tag{2.46}$$

Thus, (2.44), (2.45), and (2.46) together says that

$$\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2 \leqslant \left(1+\frac{\varepsilon}{\eta^2}\right)\|I_m-D_1^{*-1}\|_{F(MM)}^2+\frac{2}{\eta^2}\|I_n-D_2^{-1}\|_{F(NN)}^2 + \frac{2}{\eta^2}\|I_m-D_1^*\|_{F(MM)}^2+\left(1+\frac{\varepsilon}{\eta^2}\right)\|I_n-D_2\|_{F(NN)}^2. \tag{2.47}$$

Then, combining with (2.43), (2.47), and the condition on η, we have (2.37) and (2.38).

When $A,\widetilde{A}\in\mathbb{C}_n^{m\times n}$, the MN-SVDs of A and $\widetilde{A}$ are reduced to

$$A=U\begin{pmatrix}\Sigma\\0\end{pmatrix}V^*=U_1\Sigma V^* \text{ and } \widetilde{A}=\widetilde{U}\begin{pmatrix}\widetilde{\Sigma}\\0\end{pmatrix}\widetilde{V}^*=\widetilde{U}_1\widetilde{\Sigma}\widetilde{V}^*.$$

In this case, the MN-WPDs of A and $\widetilde{A}$ can be computed by

$$Q = U_1V^*,\ H = N^{-1}V\Sigma V^* \text{ and } \widetilde{Q} = \widetilde{U}_1\widetilde{V}^*,\ \widetilde{H} = N^{-1}\widetilde{V}\widetilde{\Sigma}\widetilde{V}^*.$$

Thus, X, S, E, F, $\widetilde{U}^*M(\widetilde{Q}-Q)N^{-1}V$, and $U^*M(\widetilde{Q}-Q)N^{-1}\widetilde{V}$ appearing in the proofs of Theorem 2.1 and Theorem 2.3 are reduced to

$$\begin{aligned}
X &= \widetilde{U}_1^*MU_1 - \widetilde{V}^*N^{-1}V,\\
S &= \widetilde{\Sigma}\left(\widetilde{U}_1^*(I_m - D_1^{-1})MU_1 - \widetilde{V}^*(I_n - D_2^{-1})N^{-1}V\right)\\
&\qquad + \left(\widetilde{V}^*N^{-1}(D_2^* - I_n)V - \widetilde{U}_1^*M(D_1^* - I_m)U_1\right)\Sigma,\\
E &= \widetilde{U}_1^*(I_m - D_1^{-1})MU_1 - \widetilde{V}^*(I_n - D_2^{-1})N^{-1}V,\\
F &= -\widetilde{V}^*N^{-1}(D_2^* - I_n)V + \widetilde{U}_1^*M(D_1^* - I_m)U_1,\\
\widetilde{U}^*M(\widetilde{Q}-Q)N^{-1}V &= \begin{pmatrix} \widetilde{V}^*N^{-1}V - \widetilde{U}_1^*MU_1 \\ -\widetilde{U}_2^*MU_1 \end{pmatrix},\\
U^*M(\widetilde{Q}-Q)N^{-1}\widetilde{V} &= \begin{pmatrix} U_1^*M\widetilde{U}_1 - V^*N^{-1}\widetilde{V} \\ U_2^*M\widetilde{U}_1 \end{pmatrix}.
\end{aligned}$$

In terms of above discussions and the methods to prove Theorem 2.1 and Theorem 2.3, we can get the following theorems.

THEOREM 2.4. *Let* $A, \widetilde{A} = D_1^*AD_2 \in C_n^{m\times n}$, *where* $D_1 \in C_m^{m\times m}, D_2 \in C_n^{n\times n}$. *If* $A, \widetilde{A}$ *have the MN-WPDs according to* (1.1) *and* (2.1)*, respectively, then*

$$\begin{aligned}
\left\|\widetilde{Q}-Q\right\|_{(MN)} &\leqslant \frac{\widetilde{\sigma}_1}{\sigma_n + \widetilde{\sigma}_n}\left(\|I_m - D_1^{*-1}\|_{(MM)} + \|I_n - D_2^{-1}\|_{(N,N)}\right)\\
&\quad + \frac{\sigma_1}{\sigma_n + \widetilde{\sigma}_n}\left(\|D_1^* - I_m\|_{(MM)} + \|D_2 - I_n\|_{(NN)}\right)\\
&\quad + min\left\{\|I_m - D_1^{*-1}\|_{(MM)}, \|D_1^* - I_m\|_{(MM)}\right\},
\end{aligned} \tag{2.48}$$

where $\sigma_1, \widetilde{\sigma}_1$ *are the biggest* (M,N) *singular values of* $A, \widetilde{A}$, *and* $\sigma_n, \widetilde{\sigma}_n$ *are the smallest* (M,N) *singular values of* $A, \widetilde{A}$, *respectively.*

THEOREM 2.5. *Assume that the conditions in Theorem* 2.4 *hold and add a condition that* $\eta = \min\limits_{1\leqslant i,j\leqslant n} \frac{\widetilde{\sigma}_i + \sigma_j}{\sqrt{\widetilde{\sigma}_i^2 + \sigma_j^2}} \geqslant \sqrt{2-\varepsilon}$, *where* $\widetilde{\sigma}_i$ *is the* i*-th* (M,N) *singular value of* $\widetilde{A}$, σ_j *is the* j*-th* (M,N) *singular value of* A, *and* $0 \leqslant \varepsilon \leqslant 1$, *then*

$$\begin{aligned}
\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2 &\leqslant \left(1 + \frac{\varepsilon}{\eta^2}\right)\left(\|I_m - D_1^{*-1}\|_{F(MM)}^2 + \|I_m - D_1^*\|_{F(MM)}^2\right)\\
&\quad + \frac{2}{\eta^2}\left(\|I_n - D_2^{-1}\|_{F(NN)}^2 + \|I_n - D_2\|_{F(NN)}^2\right)
\end{aligned}$$

$$-\left(1-\frac{2-\varepsilon}{\eta^2}\right)\max\left\{\left\|I_m-D_1^{*-1}\right\|_{F(MM)}^2,\|I_m-D_1^*\|_{F(MM)}^2\right\} \quad (2.49)$$

$$\leqslant \frac{2}{2-\varepsilon}\left(\left\|I_m-D_1^{*-1}\right\|_{F(MM)}^2+\|I_m-D_1^*\|_{F(MM)}^2\right.$$
$$\left.+\left\|I_n-D_2^{-1}\right\|_{F(NN)}^2+\|I_n-D_2\|_{F(NN)}^2\right). \quad (2.50)$$

Four corollaries of above theorems are given as follows when $M=I_m$ and $N=I_n$. These results give some new perturbation bounds for subunitary and unitary polar factors.

COROLLARY 2.1. *Let $A,\widetilde{A}=D_1^*AD_2\in C_r^{m\times n}$, where $D_1\in C_m^{m\times m},D_2\in C_n^{n\times n}$. If $A,\widetilde{A}$ have the generalized polar decompositions $A=QH,\widetilde{A}=\widetilde{Q}\widetilde{H}$, respectively, then*

$$\left\|\widetilde{Q}-Q\right\|\leqslant\frac{\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_r}\left(\left\|I_m-D_1^{-1}\right\|+\left\|I_n-D_2^{-1}\right\|\right)+\frac{\sigma_1}{\sigma_r+\widetilde{\sigma}_r}\left(\|D_1-I_m\|+\|D_2-I_n\|\right)$$
$$+\min\left\{\left(\|D_1-I_m\|+\left\|I_n-D_2^{-1}\right\|\right),\left(\left\|I_m-D_1^{-1}\right\|+\|D_2-I_n\|\right)\right\}, \quad (2.51)$$

where $\sigma_1,\widetilde{\sigma}_1$ are the biggest singular values of $A,\widetilde{A}$, and $\sigma_n,\widetilde{\sigma}_n$ are the smallest singular values of $A,\widetilde{A}$, respectively.

COROLLARY 2.2. *Assume that the conditions of Corollary* 2.1 *hold. Then*

$$\left\|\widetilde{Q}-Q\right\|_2\leqslant\frac{\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_r}\left(\left\|I_m-D_1^{-1}\right\|_2+\left\|I_n-D_2^{-1}\right\|_2\right)+\frac{\sigma_1}{\sigma_r+\widetilde{\sigma}_r}\left(\|D_1-I_m\|_2+\|D_2-I_n\|_2\right)$$
$$+\min\left\{\max\left\{\|D_1-I_m\|_2,\left\|I_n-D_2^{-1}\right\|_2\right\},\max\left\{\left\|I_m-D_1^{-1}\right\|_2,\|D_2-I_n\|_2\right\}\right\}. \quad (2.52)$$

COROLLARY 2.3. *Assume that the conditions of Corollary* 2.1 *hold and add a condition that $\eta=\min\limits_{1\leqslant i,j\leqslant r}\frac{\widetilde{\sigma}_i+\sigma_j}{\sqrt{\widetilde{\sigma}_i^2+\sigma_j^2}}\geqslant\sqrt{2-\varepsilon}$, where $\widetilde{\sigma}_i$ is the i-th singular value of $\widetilde{A}$, σ_j is the j-th singular value of A, and $0\leqslant\varepsilon\leqslant1$, then*

$$\left\|\widetilde{Q}-Q\right\|_F^2\leqslant\left(1+\frac{\varepsilon}{\eta^2}\right)\left(\left\|I_m-D_1^{-1}\right\|_F^2+\|I_m-D_1\|_F^2+\left\|I_n-D_2^{-1}\right\|_F^2+\|I_n-D_2\|_F^2\right)$$
$$-\left(1-\frac{2-\varepsilon}{\eta^2}\right)\max\left\{\left\|I_m-D_1^{-1}\right\|_F^2+\|I_n-D_2\|_F^2,\left\|I_n-D_2^{-1}\right\|_F^2+\|I_m-D_1\|_F^2\right\} \quad (2.53)$$

$$\leqslant\frac{2}{2-\varepsilon}\left(\left\|I_m-D_1^{-1}\right\|_F^2+\|I_m-D_1\|_F^2+\left\|I_n-D_2^{-1}\right\|_F^2+\|I_n-D_2\|_F^2\right). \quad (2.54)$$

COROLLARY 2.4. *Let $A,\widetilde{A}=D_1^*AD_2\in\mathbb{C}_n^{m\times n}$, where $D_1\in\mathbb{C}_m^{m\times m}$ and $D_2\in\mathbb{C}_n^{n\times n}$. If $A,\widetilde{A}$ have the polar decompositions $A=QH$, $\widetilde{A}=\widetilde{Q}\widetilde{H}$, and $\eta=\min\limits_{1\leqslant i,j\leqslant n}\frac{\widetilde{\sigma}_i+\sigma_j}{\sqrt{\widetilde{\sigma}_i^2+\sigma_j^2}}\geqslant\sqrt{2-\varepsilon}$, where $\widetilde{\sigma}_i$ is the i-th singular value of $\widetilde{A}$, σ_j is the j-th singular value of A, and $0\leqslant\varepsilon\leqslant 1$, then*

$$\begin{aligned}\left\|\widetilde{Q}-Q\right\|_F^2 &\leqslant \left(1+\frac{\varepsilon}{\eta^2}\right)\left(\left\|I_m-D_1^{-1}\right\|_F^2+\|I_m-D_1\|_F^2\right)+\frac{2}{\eta^2}\left(\left\|I_n-D_2^{-1}\right\|_F^2+\|I_n-D_2\|_F^2\right)\\ &\quad-\left(1-\frac{2-\varepsilon}{\eta^2}\right)\max\left\{\left\|I_m-D_1^{-1}\right\|_F^2,\|I_m-D_1\|_F^2\right\} \qquad (2.55)\\ &\leqslant\frac{2}{2-\varepsilon}\left(\left\|I_m-D_1^{-1}\right\|_F^2+\|I_m-D_1\|_F^2+\left\|I_n-D_2^{-1}\right\|_F^2+\|I_n-D_2\|_F^2\right). \qquad (2.56)\end{aligned}$$

REMARK 2.1. When $M=I_m$ and $N=I_n$ in Theorem 2.4, the result (2.48) is reduced to the corresponding bound for unitary polar factor, i.e., (1.4) in this paper.

REMARK 2.2. It is not difficult to find that if the conditions in Corollary 2.3 and Corollary 2.4 hold, i.e., $\eta\geqslant\sqrt{2-\varepsilon}$ and $0\leqslant\varepsilon\leqslant 1$, and ε is set to be a suitable value, the new bounds (2.53), (2.54), (2.55), and (2.56) may be smaller than the corresponding one (1.5). In fact, if $0\leqslant\varepsilon\leqslant\frac{4\left(\left\|I_m-D_1^{-1}\right\|_F\left\|I_n-D_2^{-1}\right\|_F+\|D_1-I_m\|_F\|D_2-I_n\|_F\right)}{\left(\left\|I_m-D_1^{-1}\right\|_F+\left\|I_n-D_2^{-1}\right\|_F\right)^2+\left(\|D_1-I_m\|_F+\|D_2-I_n\|_F\right)^2}$ and $\eta\geqslant\sqrt{2-\varepsilon}$, the bounds (2.54) and (2.56) are not greater than the one (1.5). In addition, it is easy to see that the bounds (2.53) and (2.55) are smaller than the corresponding ones (2.54) and (2.56). However, they seem to be a little complicated in form.

REMARK 2.3. As we know, if the perturbed matrix is expressed as $\widetilde{A}=A+E$, then such perturbation is called additive perturbation. For this perturbation, many authors studied various perturbation bounds for the (generalized) polar decomposition in various norms, see e.g., [2, 3, 4, 5, 7, 8, 9, 13, 14, 15, 16, 17, 18, 21]. Here we only introduce two bounds which are given in the spectral norm and Frobenius norm, respectively, since they can be used to compare with the results obtained in this paper in the following examples. The one in the special norm was derived by Li [13] recently for subunitary polar factor, i.e., for $A,\widetilde{A}\in\mathbb{C}_r^{m\times n}$,

$$\left\|\widetilde{Q}-Q\right\|_2\leqslant\frac{1+\sqrt{3}}{\sigma_r+\widetilde{\sigma}_r}\|E\|_2, \qquad (2.57)$$

where $\sigma_r,\widetilde{\sigma}_r$ are the smallest singular values of $A,\widetilde{A}$, respectively. The one in Frobenius norm was obtained by Li and Sun [16] for $A,\widetilde{A}\in\mathbb{C}_r^{m\times n}$,

$$\left\|\widetilde{Q}-Q\right\|_F\leqslant\frac{2}{\sigma_r+\widetilde{\sigma}_r}\|E\|_F. \qquad (2.58)$$

In general, this bound was claimed to be the current best perturbation bound in Frobenius norm without assuming that $\|E\|_F$ is tiny [16, 17].

When the Frobenius norm in the bound (2.58) is replaced with the unitarily invariant norm, Li [9] showed that it also holds for $A, \widetilde{A} \in \mathbb{C}_n^{n\times n}$. However, in this case, it is not satisfied for more general cases [13], for example, for $A, \widetilde{A} \in \mathbb{C}_r^{m\times n}$ or $\mathbb{C}_n^{m\times n}$.

REMARK 2.4. From Remark 2.1 in [5], we know the multiplicative perturbation implies the additive perturbation. However, in general neither of the perturbation bounds of such two perturbations for (generalized) polar decomposition is uniformly better than the other. Two examples are provided in the following. For them, the bounds obtained in this paper are a little better in comparison. Furthermore, it is not difficult to see that the conclusion on (generalized) polar decomposition introduced above is also valid for WPD.

EXAMPLE 2.1. Let

$$A = \begin{pmatrix} 1000 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}_2^{4\times 3}, \quad D_1 = I_4, \quad D_2 = \operatorname{diag}(1.0001, 0.9999, 1).$$

Then, we have

$$\frac{1+\sqrt{3}}{\sigma_r + \widetilde{\sigma}_r} \|E\|_2 = 1.366094 \times 10^2$$

and the value of bound (2.52) 1.000151×10^2. For this example, the multiplicative perturbation bound (2.52) is better than the additive perturbation bound (2.57).

EXAMPLE 2.2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 2.75 & 0 \end{pmatrix} \in \mathbb{C}_2^{4\times 3}, \quad D_1 = \operatorname{diag}(0.9998, 1, 1.0001, 0.9999),$$

$$D_2 = \operatorname{diag}(1.0002, 0.9999, 1).$$

Then, after some computation, we can get $\eta = 1.1628$ and the perturbation bounds (1.5) and (2.58), respectively, in the following:

$$\left\|\widetilde{Q} - Q\right\|_F \leqslant 6.6264 \times 10^{-4} \text{ and } \left\|\widetilde{Q} - Q\right\|_F \leqslant 7.0848 \times 10^{-4}.$$

If the value of ε is set to be 0.7,0.8,0.9, then $\eta \geqslant \sqrt{2-\varepsilon} = 1.1402, 1.0954, 1.0488$, respectively. In this case, the bound (2.54) can be obtained, respectively, as follows:

$$\left\|\widetilde{Q} - Q\right\|_F \leqslant 5.8177 \times 10^{-4}, \quad 6.0553 \times 10^{-4}, \text{ and } 6.3246 \times 10^{-4}.$$

For this example, the new bound (2.54) is better than the ones (1.5) and (2.58) when $\varepsilon = 0.7, 0.8, 0.9$. Moreover, we have

$$\frac{4(\left\|I_m - D_1^{-1}\right\|_F \left\|I_n - D_2^{-1}\right\|_F + \|D_1 - I_m\|_F \|D_2 - I_n\|_F)}{(\left\|I_m - D_1^{-1}\right\|_F + \left\|I_n - D_2^{-1}\right\|_F)^2 + (\|D_1 - I_m\|_F + \|D_2 - I_n\|_F)^2} = 0.9979.$$

Consequently, the bound (2.54) is always better than the one (1.5) if $\varepsilon \leqslant 0.9979$ and $\sqrt{2-\varepsilon} \leqslant 1.1628$, i.e., $0.6479 \leqslant \varepsilon \leqslant 0.9979$ for this example.

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