Mathematical
Inequalities
& Applications
Volume 13, Number 3 (2010), 579–599

WEIGHTED VERSION OF GENERAL INTEGRAL FORMULA OF EULER TYPE

S. KOVAČ AND J. PEČARIĆ

(*Communicated by I. Perić*)

Abstract. The weighted generalization of the integral formula with m nodes is introduced, and some sharp and the best possible inequalities for the functions whose higher order derivatives belong to L_p spaces are given. Specially, the general one-point integral formula is established. Special cases of the well known weights are considered and generalizations of the Gaussian quadrature formulae with one node are obtained.

1. Introduction

J.Pečarić and S.Varošanec considered in [8] the following situation. Let $m,n \in \mathbb{N}$ be fixed and let $\{P_{jk}\}_{k\in\mathbb{N}}$ be harmonic sequences of polynomials, i.e. $P'_{jk} = P_{j,k-1}$, $k \in \mathbb{N}$, $P_{j0} = 1$, $j = 1,\dots,m$. Let $\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}$ be a subdivision of the interval $[a,b]$. Set

$$S_n(t,\sigma) = \begin{cases} P_{1n}(t), \ t \in [a,x_1] \\ P_{2n}(t), \ t \in (x_1,x_2] \\ \vdots \\ P_{mn}(t), t \in (x_{m-1},b]. \end{cases} \tag{1.1}$$

By successive integration by parts they proved the following formula:

$$(-1)^n \int_a^b S_n(t,\sigma)df^{(n-1)}(t) = \int_a^b f(t)dt + \sum_{k=1}^n (-1)^k \Big[P_{mk}(b) f^{(k-1)}(b) + \sum_{j=1}^{m-1} (P_{jk}(x_j) - P_{j+1,k}(x_j)) f^{(k-1)}(x_j) - P_{1k}(a) f^{(k-1)}(a) \Big]. \tag{1.2}$$

The aim of this paper is to establish the general weighted version of the identity (1.2). Further, we observe functions f whose higher order derivatives belong to L_p spaces, and for such functions we obtain error estimates.

Mathematics subject classification (2010): 26D15, 65D30, 65D32.
Keywords and phrases: harmonic polynomials, weighted integral formula, one-point formula, w-Appell sequences, w-harmonic sequences, weight functions, L^p inequalities, Gauss rule, Euler identity.

Paper MIA-13-40

Now, let $f:[a,b]\to\mathbb{R}$ be an arbitrary function such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$ and let $x\in[a,b]$. The following two identities hold ([3]):

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+T_n(x)+R_n^1(x), \tag{1.3}$$

and

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+T_{n-1}(x)+R_n^2(x), \tag{1.4}$$

where

$$\begin{aligned}
T_l(x)&=\sum_{k=1}^{l}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left(f^{(k-1)}(b)-f^{(k-1)}(a)\right),\quad l\in\mathbb{N}\\
T_0(x)&=0\\
R_n^1(x)&=-\frac{(b-a)^{n-1}}{n!}\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)df^{(n-1)}(t),\\
R_n^2(x)&=-\frac{(b-a)^{n-1}}{n!}\int_a^b\left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right]df^{(n-1)}(t).
\end{aligned} \tag{1.5}$$

Here $B_k(t)$, $k\geqslant 0$ are Bernoulli polynomials, $B_k=B_k(0)$, $k\geqslant 0$ Bernoulli numbers, and $B_k^*(t)$, $k\geqslant 0$ are periodic function of period 1, related to Bernoulli polynomials as

$$B_k^*(t)=B_k(t),\quad 0\leqslant t<1,\quad B_k^*(t+1)=B_k^*(t),\quad t\in\mathbf{R}.$$

From the properties of Bernoulli polynomials $B_0^*(t)=1$, $B_1^*(t)$ is a discontinuous function with the jump -1 at each $t\in\mathbb{N}$, and $B_k^*(t)$ are continuous functions, for $k\geqslant 2$. Further, $B_{2k+1}=B_{2k+1}(0)=0$, for $k\geqslant 1$ and $B_1=-\frac{1}{2}$ (see [6]).

Identities (1.3) and (1.4) are called extended Euler identities. In this paper we will show that these identities can be obtained from a more general result (1.2).

Let $w:[a,b]\to\mathbb{R}_+$ be some probability density function, i.e. integrable function satisfying $\int_a^b w(t)dt=1$. A. Aglić Aljinović and J. Pečarić ([2]) have proved the following two identities

$$\begin{aligned}
f(x)=&\int_a^b w(t)f(t)dt+\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{k!}\times\\
&\times\left(B_k\left(\frac{x-a}{b-a}\right)-\int_a^b w(t)B_k\left(\frac{t-a}{b-a}\right)dt\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]\\
&-\frac{(b-a)^{n-1}}{n!}\int_a^b\left(B_n^*\left(\frac{x-t}{b-a}\right)-\int_a^b w(s)B_n^*\left(\frac{s-t}{b-a}\right)ds\right)df^{(n-1)}(t)
\end{aligned} \tag{1.6}$$

and

$$\begin{aligned} f(x) &= \int_a^b w(t)f(t)dt + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} \times \qquad (1.7) \\ &\times \left(B_k\left(\frac{x-a}{b-a}\right) - \int_a^b w(t)B_k\left(\frac{t-a}{b-a}\right)dt \right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\ &- \frac{(b-a)^{n-1}}{n!} \int_a^b \left(B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right) \right. \\ &\left. - \int_a^b w(s)\left(B_n^*\left(\frac{s-t}{b-a}\right) - B_n\left(\frac{s-a}{b-a}\right)\right) ds \right) df^{(n-1)}(t), \end{aligned}$$

which are the weighted generalization of (1.3) and (1.4).

In this paper we will show that identities (1.6) and (1.7) can be obtained as special cases of the general $m-$point weighted integral formula, for the particular three-point subdivision $\{a < x < b\}$. Also, we will consider the general one-point formulae of Matić, Pečarić and Ujević [7] where in the approximation of the integral, the values of the higher order derivatives in the node x are used. From this general formula we will get the generalizations of the quadrature formulae of the Gaussian type (see [6]) for the special choices of the weight function w and the node x.

2. The weighted generalization of the integral formula via w-harmonic sequences and related inequalities

Let us assume that $w_k : [a,b] \to \mathbb{R}$, $k = 1,\dots,n$ are absolutely continuous functions and $w : [a,b] \to \mathbb{R}$ is integrable functions. We say that $\{w_k\}_{k=1,\dots,n}$ is w-harmonic sequence of functions (see [1]) if the following conditions are satisfied:

$$\begin{aligned} w_1'(t) &= w(t), \quad \text{for } t \in [a,b], \qquad (2.1) \\ w_k'(t) &= w_{k-1}(t), \quad \text{for } t \in [a,b], \text{ for all } k = 2,\dots,n,. \end{aligned}$$

LEMMA 1. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function on $[a,b]$ and $\{w_k\}_{k=1,\dots,n}$ be w-harmonic sequence of functions on $[a,b]$. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$, then the following identity holds*

$$\begin{aligned} \int_a^b w(t)g(t)dt &= A_n(w,g;a,b) + R_n(w,g;a,b), \qquad (2.2) \\ A_n(w,g;a,b) &= \sum_{k=1}^{n} (-1)^{k-1} [w_k(b)g^{(k-1)}(b) - w_k(a)g^{(k-1)}(a)] \\ R_n(w,g;a,b) &= (-1)^n \int_a^b w_n(t)g^{(n)}(t)dt. \end{aligned}$$

Proof. We prove (2.2) by mathematical induction. For $n = 1$ integration by parts gives

$$\int_a^b w(t)g(t)dt = w_1(b)g(b) - w_1(a)g(a) - \int_a^b w_1(t)g'(t)dt. \qquad (2.3)$$

Let us assume that for $l = 1, \ldots, n-1$ we have

$$\int_a^b w(t)g(t)dt = \sum_{k=1}^{l}(-1)^{k-1}[w_k(b)g^{(k-1)}(b) - w_k(a)g^{(k-1)}(a)] + (-1)^l \int_a^b w_l(t)g^{(l)}dt. \tag{2.4}$$

Further, integration by parts yields

$$\int_a^b w_l(t)g^{(l)}(t)dt = w_{l+1}(b)g^{(l)}(b) - w_{l+1}(a)g^{(l)}(a) - \int_a^b w_{l+1}(t)g^{(l+1)}(t)dt. \tag{2.5}$$

Finally, we impose identity (2.5) to the relation (2.4) and obtain

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{k=1}^{l}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b) - w_k(a)g^{(k-1)}(a)\right] \\ &\quad + (-1)^l\Big[w_{l+1}(b)g^{(l)}(b) - w_{l+1}(a)g^{(l)}(a) \\ &\quad - \int_a^b w_{l+1}(t)g^{(l+1)}(t)dt\Big] \\ &= \sum_{k=1}^{l+1}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b) - w_k(a)g^{(k-1)}(a)\right] \\ &\quad + (-1)^{l+1}\int_a^b w_{l+1}(t)g^{(l+1)}(t)dt,\end{aligned}$$

so the assertion is valid for $l+1$. □

REMARK 1. Dragomir proved in [4] a similar result as (2.2) for the case of w-Appell sequences of functions.

Let us consider the subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ of the interval $[a,b]$, for some $m \in \mathbb{N}$. Let $w : [a,b] \to \mathbb{R}$ be an integrable function. On each interval $[x_{k-1}, x_k]$, $k = 1, \ldots, m$ we consider different w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$, i.e. we have

$$\begin{aligned} w'_{k1}(t) &= w(t) && \text{for } t \in [x_{k-1}, x_k] \\ (w_{kj})'(t) &= w_{k,j-1}(t) && \text{for } t \in [x_{k-1}, x_k], \text{ for all } j = 2, \ldots, n. \end{aligned} \tag{2.6}$$

Further, let us define

$$W_{n,w}(t,\sigma) = \begin{cases} w_{1n}(t) & \text{for } t \in [a, x_1], \\ w_{2n}(t) & \text{for } t \in (x_1, x_2], \\ \vdots \\ w_{mn}(t) & \text{for } t \in (x_{m-1}, b]. \end{cases} \tag{2.7}$$

THEOREM 1. *If* $g:[a,b]\to\mathbb{R}$ *is such that* $g^{(n-1)}$ *is absolutely continuous on* (x_{k-1},x_k) *for* $k\in\{1,\dots,m\}$*, then the following identity holds*

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{j=1}^{n}(-1)^{j-1}\Big[w_{mj}(b)g^{(j-1)}(b)\\ &\quad+\sum_{k=1}^{m-1}\left[w_{kj}(x_k)-w_{k+1,j}(x_k)\right]g^{(j-1)}(x_k)-w_{1j}(a)g^{(j-1)}(a)\Big]\\ &\quad+(-1)^n\int_a^b W_{n,w}(t,\sigma)g^{(n)}(t)dt.\end{aligned}\tag{2.8}$$

Proof. Using relation (2.2) on each interval $(x_{k-1},x_k]$ for appropriate w-harmonic sequence, we get the following identity

$$\int_{x_{k-1}}^{x_k} w(t)g(t)dt = A_n(w,g;x_{k-1},x_k)+R_n(w,g;x_{k-1},x_k).\tag{2.9}$$

By summing relation (2.9) from $k=1$ to m we obtain

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{j=1}^{n}(-1)^{j-1}\Big[w_{mj}(b)g^{(j-1)}(b)\\ &\quad+\sum_{k=1}^{m-1}\left[w_{kj}(x_k)-w_{k+1,j}(x_k)\right]g^{(j-1)}(x_k)-w_{1j}(a)g^{(j-1)}(a)\Big]\\ &\quad+\sum_{k=1}^{m}R_n(w,g;x_{k-1},x_k)\\ &= \sum_{j=1}^{n}(-1)^{j-1}\Big[w_{mj}(b)g^{(j-1)}(b)\\ &\quad+\sum_{k=1}^{m-1}\left[w_{kj}(x_k)-w_{k+1,j}(x_k)\right]g^{(j-1)}(x_k)-w_{1j}(a)g^{(j-1)}(a)\Big]\\ &\quad+(-1)^n\int_a^b W_{n,w}(t,\sigma)g^{(n)}(t)dt. \qquad \square\end{aligned}\tag{2.10}$$

The next theorem establishes more general identity (2.8).

THEOREM 2. *Let* $g:[a,b]\to\mathbb{R}$ *be such that* $g^{(n-1)}$ *is continuous function of bounded variation on* $[a,b]$*, for some* $n\in\mathbb{N}$*. Then the following identity holds*

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{j=1}^{n}(-1)^{j-1}\Big[w_{mj}(b)g^{(j-1)}(b)\\ &\quad+\sum_{k=1}^{m-1}\left[w_{kj}(x_k)-w_{k+1,j}(x_k)\right]g^{(j-1)}(x_k)-w_{1j}(a)g^{(j-1)}(a)\Big]\\ &\quad+(-1)^n\int_a^b W_{n,w}(t,\sigma)dg^{(n-1)}(t).\end{aligned}\tag{2.11}$$

Proof. Let us fix $k=1,\dots,m$. For $j=1,\dots,n$ we define

$$I_{kj}=(-1)^j\int_{x_{k-1}}^{x_k} w_{kj}(t)df^{(j-1)}(t)$$

and let

$$I_{k0}=\int_{x_{k-1}}^{x_k} w(t)f(t)dt.$$

By integration by parts formula we obtain

$$I_{kj}=(-1)^j\left(w_{kj}(x_k)f^{(j-1)}(x_k)-w_{kj}(x_{k-1})f^{(j-1)}(x_{k-1})\right)+I_{k,j-1}.$$

Summing last identity from $j=1$ to n we obtain

$$I_{kn}=\sum_{j=1}^{n}(-1)^j\left[w_{kj}(x_k)f^{(j-1)}(x_k)-w_{kj}(x_{k-1})f^{(j-1)}(x_{k-1})\right]+\int_{x_{k-1}}^{x_k} f(t)w(t)dt.$$

Finally, summing this identity from $k=1$ to m we get (2.11) since

$$\sum_{k=1}^{m} I_{kn}=(-1)^n\int_a^b W_{n,w}(t,\sigma)df^{(n-1)}(t). \qquad \square$$

REMARK 2. Let us suppose $w:[a,b]\to[0,\infty)$ is some integrable function such that $\int_a^b w(t)=1$, $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\in\mathbb{N}$ and let $x\in[a,b]$ be some fixed point. We consider three-point subdivision of the segment $[a,b]$: $x_0=a$, $x_1=x$ and $x_2=b$. Let us define function $W_{j,w}(t,x)$, $j\in\mathbb{N}$ in the following way

$$W_{j,w}(t,x):=\begin{cases} w_{1j}(t)=\frac{(-1)^{j-1}(b-a)^{j-1}}{j!}\int_a^b\left(B_j^*(\frac{s-t}{b-a})-B_j(\frac{x-t}{b-a})\right)w(s)ds, & t\in[a,x]\\ w_{2j}(t)=\frac{(-1)^{j-1}(b-a)^{j-1}}{j!}\int_a^b\left(B_j^*(\frac{s-t}{b-a})-B_j(1+\frac{x-t}{b-a})\right)w(s)ds, & t\in(x,b].\end{cases} \tag{2.12}$$

Obviously, function $W_{n,w}(\cdot,x)$ can be represent in the following way using periodic functions B_n^*

$$W_{n,w}(t,x)=\frac{(-1)^{n-1}(b-a)^{n-1}}{n!}\int_a^b\left(B_n^*\left(\frac{s-t}{b-a}\right)-B_n^*\left(\frac{x-t}{b-a}\right)\right)w(s)ds.$$

Then the identity (2.11) becomes (1.6). In particular, for $w(t)=\frac{1}{b-a}$, identity (2.11) becomes (1.3).

For each $k=1,\dots,m$ and $n\in\mathbf{N}$ we define functions $u_{kj}:[x_{k-1},x_k]\to\mathbb{R}$ by relation

$$u_{kj}(t):=\frac{1}{(j-1)!}\cdot\int_{x_{k-1}}^{t}(t-s)^{j-1}\cdot w(s)ds, \quad t\in[x_{k-1},x_k]. \tag{2.13}$$

It is easy to check that $u_{kn}^{(n)}(t)=w(t)$ and $u_{kn}(x_{k-1})=0$.

Further, sequences $\{u_{kj}\}_{j=0,\dots,n}$ are the examples of w-harmonic sequences of functions on $[x_{k-1},x_k]$, for $k=1,\dots,m$.

Now we can give the general representation of the w-harmonic sequences.

LEMMA 2. *Let $\{w_{kj}\}_{j=0,\ldots,n}$ be w-harmonic sequences on $[x_{k-1},x_k]$, for $k = 1,\ldots,m$. Then there exist unique sequences $\{Q_{kj}\}_{j=0,\ldots,n}$ of polynomials satisfying*

$$Q'_{kj}(t) = Q_{k,j-1}(t), \quad t \in [x_{k-1},x_k], \quad \deg Q_{kj} \leqslant j-1, \quad Q_{k0} = 0, \quad j = 1,\ldots,n \tag{2.14}$$

such that

$$w_{kj}(t) = u_{kj}(t) + Q_{kj}(t), \quad j \geqslant 0, \tag{2.15}$$

for $k = 1,\ldots,m$.

Proof. For $j \geqslant 1$ and $k = 1\ldots,m$ the $j-$th derivative of the function $w_{kj}(t) - u_{kj}(t)$ is zero by definition of the function u_{kj}. So, there must exist a polynomial $Q_{kj}(t)$ of degree at most $j-1$ such that

$$w_{kj}(t) = u_{kj}(t) + Q_{kj}(t).$$

Evidently, Q_{kj} satisfies properties (2.14), so we have proved the existence. The uniqueness of Q_{kj} is obvious. □

REMARK 3. The case $m = 1$ in Lemma 2 was obtained in [1].

REMARK 4. If we put in (2.8) $w \equiv 1$ and $w_{kj} = P_{kj}$, where $\{P_{kj}\}_{j=0,\ldots,n}$ are sequences of harmonic polynomials on $[x_{k-1},x_k]$, for $k = 1,\ldots,m$, then we recapture relation (1.2).

Now we will give the general L_p theorem.

THEOREM 3. *Let us suppose $w : [a,b] \to \mathbb{R}$ is an integrable function and $\{w_{kj}\}_{j=1,\ldots,n}$ w-harmonic sequences of functions on $[x_{k-1},x_k]$, for $k = 1,\ldots,m$. If $g : [a,b] \to \mathbb{R}$ is a function such that $g^{(n-1)}$ is absolutely continuous and $g^{(n)} \in \mathrm{L_p}$ for some $1 \leqslant p \leqslant \infty$, then the following inequality holds*

$$\begin{aligned} \Big| \int_a^b w(t)g(t)dt - \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\ + \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \Big| \\ \leqslant C(n,p,w) \cdot \|g^{(n)}\|_p, \end{aligned} \tag{2.16}$$

where

$$C(n,p,w) = \begin{cases} \left[\int_a^b |W_{n,w}(t,\sigma)|^q \, dt \right]^{\frac{1}{q}}, \ 1 < p \leqslant \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 \\ \sup_{t \in [a,b]} |W_{n,w}(t,\sigma)|, \quad p = 1. \end{cases} \tag{2.17}$$

The inequality is the best possible for $p = 1$ and sharp for $1 < p \leqslant \infty$. Equality is attained for every function g such that

$$g(t) = M \cdot g_*(t) + p_{n-1}(t),$$

where $M \in \mathbf{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n-1$ and $g_*(t)$ is function on $[a,b]$ defined by

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} W_{n,w}(\xi,\sigma) \cdot |W_{n,w}(\xi,\sigma)|^{\frac{1}{p-1}} d\xi, \tag{2.18}$$

for $1 < p < \infty$, and

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} W_{n,w}(\xi,\sigma) d\xi, \tag{2.19}$$

for $p = \infty$.

Proof. Relation (2.8) implies

$$\begin{aligned}\int_a^b w(t)g(t)dt - \sum_{j=1}^{n} (-1)^{j-1} \Big[& w_{mj}(b) g^{(j-1)}(b) \\ & + \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k)\right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\ = (-1)^n \int_a^b & W_{n,w}(t,\sigma) g^{(n)}(t) dt.\end{aligned} \tag{2.20}$$

so, by Hölder inequality we get (2.16). For the proof of the sharpness, we need to find function g such that

$$\left| \int_a^b W_{n,w}(t,\sigma) g^{(n)}(t) dt \right| = C(n,p,w) \cdot \|g^{(n)}\|_p,$$

for $1 < p \leqslant \infty$ and $1 \leqslant q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The function g_* defined by (2.18) and (2.19) is n times differentiable, and its $n-$th derivative is piecewise continuous function. Further, g_* is a solution of the differential equation

$$W_{n,w}(t,\sigma) g^{(n)}(t) = |W_{n,w}(t,\sigma)|^q,$$

so the above identity holds.
For $p = 1$ we shall prove that

$$\left| \int_a^b W_{n,w}(t,\sigma) g^{(n)}(t) dt \right| \leqslant \sup_{t \in [a,b]} |W_{n,w}(t,\sigma)| \cdot \int_a^b |g^{(n)}(t)| dt \tag{2.21}$$

is the best possible inequality. Obviously, because of the continuity of the functions $w_{kn}(t)$ on $[x_{k-1}, x_k]$, for $k \in \{1,\ldots,n\}$, there exists $t_0 \in [a,b]$ and $k = 1,\ldots,n$ such that $\sup_{t\in[a,b]} |W_n(t)| = |w_{kn}(t_0)|$. First, let us assume that $w_{kn}(t_0) > 0$. For ε small enough define $g_\varepsilon^{(n-1)}(t)$ by

$$g_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leqslant t_0 - \varepsilon \\ \frac{t-t_0+\varepsilon}{\varepsilon}, & t \in [t_0-\varepsilon, t_0] \\ 1, & t \geqslant t_0, \end{cases}$$

if $t_0 \in (x_{k-1}, x_k]$. Then, for ε small enough,

$$\left| \int_a^b W_{n,w}(t,\sigma) g_\varepsilon^{(n)} dt \right| = \left| \int_{t_0-\varepsilon}^{t_0} w_{kn}(t) \frac{1}{\varepsilon} dt \right| = \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} w_{kn}(t) dt. \tag{2.22}$$

Now, relation (2.21) implies

$$\frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} w_{kn}(t) dt \leqslant w_{kn}(t_0) \int_{t_0-\varepsilon}^{t_0} \frac{1}{\varepsilon} dt = w_{kn}(t_0). \tag{2.23}$$

Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} w_{kn}(t) dt = w_{kn}(t_0),$$

the statement follows.
If $t_o = x_{k-1}$, then we define, for $\varepsilon > 0$ small enough, function $g_\varepsilon^{(n-1)}(t)$ by

$$g_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leqslant t_0 \\ \frac{t-t_0}{\varepsilon}, & t \in [t_0, t_0+\varepsilon] \\ 1, & t \geqslant t_0 + \varepsilon. \end{cases}$$

Then, for ε small enough,

$$\left| \int_a^b W_{n,w}(t,\sigma) g_\varepsilon^{(n)} dt \right| = \left| \int_{t_0}^{t_0+\varepsilon} W_{n,w}(t,\sigma)) \frac{1}{\varepsilon} dt \right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} w_{kn}(t) dt. \tag{2.24}$$

Now, relation (2.21) implies

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} w_{kn}(t) dt \leqslant w_{kn}(t_0) \int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon} dt = w_{kn}(t_0). \tag{2.25}$$

Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} w_{kn}(t) dt = w_{kn}(t_0),$$

the statement follows.

For the case $w_{kn}(t_0) < 0$ the proof is similar. □

COROLLARY 1. *For an integrable function $w : [a,b] \to \mathbf{R}$ and $\{w_k\}_{k=0,\dots,n}$ w-harmonic sequence on $[a,b]$, and $g : [a,b] \to \mathbf{R}$ such that $g^{(n)} \in L_p[a,b]$, for some $1 \leqslant p \leqslant \infty$, we have the following inequalitiy*

$$|R_n(w,g;a,b)| \leqslant ||w_n||_q ||g^{(n)}||_p, \tag{2.26}$$

if $w_n \in L_q[a,b]$ and $\frac{1}{p} + \frac{1}{q} = 1, \quad p \geqslant 1$.

Proof. The assertion follows from the Theorem 3 for the case $m = 1$. This result was also obtained in [4] for the w-Appell sequences. □

3. Weighted one-point formula of Matić, Pečarić and Ujević

In this section we develop the weighted one-point formula for numerical integration. Let $g:[a,b]\to\mathbf{R}$ be some function and $x\in[a,b]$. Let $w:[a,b]\to\mathbf{R}$ be some integrable function. The approximation of the integral $\int_a^b w(t)g(t)dt$ will involve the values of the higher order derivatives of g in the node x. We consider subdivision $\sigma=\{x_0<x_1<x_2\}$ of the interval $[a,b]$, where $x_0=a,\ x_1=x$ and $x_2=b$. Further, let $\{w_{kj}\}_{j=0,1,\dots,n}$ be w-harmonic sequences on each subinterval $[x_{k-1},x_k]$, $k=1,2$, such that $w_{1j}(a)=0$ and $w_{2j}(b)=0$, for $j=1,\dots,n$.
By Lemma 2 we have that

$$w_{kj}(t)=u_{kj}(t)+Q_{kj}(t),$$

for $k=1,2$ and $j=0,1,\dots,n$, where Q_{kj} satisfy properties (2.14). No we can state the following theorem

THEOREM 4. *Let $w:[a,b]\to\mathbb{R}$ be an integrable function and $x\in[a,b]$. Further, let us suppose $\{w_{kj}\}_{j=1\dots,n}$ are w-harmonic sequences of functions on $[x_{k-1},x_k]$, for $k=1,2$ and some $n\in\mathbb{N}$, defined by the following relations:*

$$w_{1j}(t):=\frac{1}{(j-1)!}\int_a^t(t-s)^{j-1}w(s)ds,\quad t\in[a,x]$$

$$w_{2j}(t):=\frac{1}{(j-1)!}\int_b^t(t-s)^{j-1}w(s)ds,\quad t\in(x,b],$$

for $j=1,\dots,n$. If $g:[a,b]\to\mathbf{R}$ is such that $g^{(n-1)}$ is absolutely continuous function, then we have

$$\int_a^b w(t)g(t)dt=\sum_{j=1}^n A_j(x)g^{(j-1)}(x)+(-1)^n\int_a^b W_{n,w}(t,x)g^{(n)}(t)dt, \tag{3.1}$$

where for $j=1,\dots,n$

$$A_j(x)=\frac{(-1)^{j-1}}{(j-1)!}\int_a^b(x-s)^{j-1}w(s)ds \tag{3.2}$$

and

$$W_{n,w}(t,x)=\begin{cases} w_{1n}(t)=\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}w(s)ds & \text{for } t\in[a,x],\\ w_{2n}(t)=\frac{1}{(n-1)!}\int_b^t(t-s)^{n-1}w(s)ds & \text{for } t\in(x,b].\end{cases} \tag{3.3}$$

Proof. We apply identity (2.8) for $m=2$ and $x_1=x$ to get

$$\int_a^b w(t)g(t)dt=\sum_{j=1}^n(-1)^{j-1}\left[w_{1j}(x)-w_{2j}(x)\right]g^{(j-1)}(x)$$
$$+(-1)^n\int_a^b W_{n,w}(t,x)g^{(n)}(t)dt,$$

since $w_{1j}(a) = 0$ and $w_{2j}(b) = 0$, for $j = 1, \dots, n$. Further, we calculate

$$w_{1j}(x) - w_{2j}(x) = \frac{1}{(j-1)!} \int_a^b (x-s)^{j-1} w(s) ds = (-1)^{j-1} A_j(x),$$

so the assertion of the Theorem follows. □

REMARK 5. The identity in Theorem 4. was obtained in [7], so we may call it an integral formula of Matić,Pečarić and Ujević.

REMARK 6. If we want formula (3.1) to be exact for the polynomials of degree at most 1, such that approximation formula doesn't include the first derivative, the extra condition $A_2(x) = 0$ is required. From this condition we get

$$x = \frac{\int_a^b s w(s) ds}{\int_a^b w(s) ds}.$$

The solution x of this equation yields the generalization of the Gaussian quadrature formula with one node.

THEOREM 5. *Let $w : [a,b] \to [0,\infty)$ be an integrable function and $x \in [a,b]$. Further, let $\{w_{kj}\}_{j=1,\dots,2n+1}$ be w-harmonic sequences of functions for $k = 1,2$ and some $n \in \mathbb{N}$, defined by the following relations:*

$$w_{1j}(t) := \frac{1}{(j-1)!} \int_a^t (t-s)^{j-1} w(s) ds, \quad t \in [a,x]$$

$$w_{2j}(t) := \frac{1}{(j-1)!} \int_b^t (t-s)^{j-1} w(s) ds, \quad t \in (x,b],$$

for $j = 1, \dots, 2n+1$. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(2n)}$ is continuous function, then there exists $\eta \in [a,b]$ such that

$$\int_a^b w(t) g(t) dt - \sum_{j=1}^{2n} A_j(x) g^{(j-1)}(x) = A_{2n+1}(x) \cdot g^{(2n)}(\eta). \tag{3.4}$$

Proof. It is easy to check that $W_{2n,w}(t,x) \geqslant 0$, for $t \in [a,b]$, so we can apply integral mean value theorem to the $\int_a^b W_{2n,w}(t,x) g^{(2n)}(t) dt$ to obtain

$$\int_a^b w(t) g(t) dt - \sum_{j=1}^{2n} A_j(x) g^{(j-1)}(x) = g^{(2n)}(\eta) \cdot \int_a^b W_{2n,w}(t,x) dt. \tag{3.5}$$

We calculate

$$\int_a^b W_{2n}(t,x) dt = \int_a^x w_{1,2n}(t) dt + \int_x^b w_{2,2n}(t) dt = w_{1,2n+1}(x) - w_{2,2n+1}(x) = A_{2n+1}(x),$$

so we get the assertion. □

Now we can state the L_p-inequality for weighted one-point formula

THEOREM 6. *Let* $g : [a,b] \to \mathbb{R}$ *and* $\{w_{kj}\}_{j=1,\dots,n}$ *be as in Theorem* 4, *let* $g^{(n-1)}$ *be absolutely continuous on* (x_{k-1}, x_k), *for* $k \in \{1,2\}$ *and let* $g^{(n)} \in \mathrm{L_p}$ *for some* $1 \leqslant p \leqslant \infty$. *Then we have*

$$\Big| \int_a^b w(t)g(t)dt - \sum_{j=1}^{n} A_j(x) g^{(j-1)}(x) \Big| \leqslant C_1(n,p,x,w) \cdot \|g^{(n)}\|_p,$$

for $\frac{1}{p} + \frac{1}{q} = 1$, *where*

$$C_1(n,p,x,w) = \frac{1}{(n-1)!} \left[\int_a^x \left| \int_a^t (t-s)^{n-1} w(s)ds \right|^q dt + \int_x^b \left| \int_b^t (t-s)^{n-1} w(s)ds \right|^q dt \right]^{\frac{1}{q}}, \tag{3.6}$$

for $1 < p \leqslant \infty$, *and*

$$C_1(n,1,x,w) = \frac{1}{(n-1)!} \max \left\{ \sup_{t \in [a,x]} \left| \int_a^t (t-s)^{n-1} w(s)ds \right|, \sup_{t \in [x,b]} \left| \int_b^t (t-s)^{n-1} w(s)ds \right| \right\}. \tag{3.7}$$

The inequality is the best possible for $p = 1$ *and sharp for* $1 < p \leqslant \infty$. *Equality is attained for some function* $g(t) = Mg_*(t) + p_{n-1}(t)$ *where* $M \in \mathbf{R}$, p_{n-1} *is an arbitrary polynomial of degree at most* $n-1$ *and* g_* *is function on* $[a,b]$ *defined by*

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} W_{n,w}(\xi,x) \cdot |W_{n,w}(\xi,x)|^{\frac{1}{p-1}} d\xi, \tag{3.8}$$

for $1 < p < \infty$, *and*

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} W_{n,w}(\xi,x) d\xi, \tag{3.9}$$

for $p = \infty$.

Proof. This theorem is special case of the Theorem 3. □

4. Special cases of weight function w for one-point formulae

In this section we will give some examples of the general one-point formula by choosing different interesting weight functions w. Further, we will show that such formulae are the generalizations of the well known Gaussian quadrature formulae.

4.1. $w(t) = 1, t \in [a,b]$

For this case we have by Theorem 4

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) = \frac{(t-a)^n}{n!} & \text{for } t \in [a,x], \\ w_{2n}(t) = \frac{(t-b)^n}{n!} & \text{for } t \in (x,b]. \end{cases}$$

Further, by Theorem 4 we get

$$A_j(x) = \frac{(b-x)^j - (a-x)^j}{j!},$$

so the one-point formula states

$$\int_a^b g(t)dt = \sum_{j=1}^{n} \frac{(b-x)^j - (a-x)^j}{j!} g^{(j-1)}(x) + \frac{(-1)^n}{n!}\left[\int_a^x (t-a)^n g^{(n)}(t)dt + \int_x^b (t-b)^n g^{(n)}(t)dt\right]. \tag{4.1}$$

The L_p inequality for the general case of the one-point formula states

$$\left|\int_a^b g(t)dt - \sum_{j=1}^{n} B_j g^{(j-1)}(x)\right| \leqslant C_1(n,p,x,1)\cdot \|g^{(n)}\|_p, \tag{4.2}$$

where

$$C_1(n,p,x,1) = \begin{cases} \frac{1}{n!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}} & \text{for } 1 < p \leqslant \infty, \quad \frac{1}{p}+\frac{1}{q}=1 \\ \frac{1}{n!}\max\{(x-a)^n,(b-x)^n\} & \text{for } p=1. \end{cases} \tag{4.3}$$

If the assumptions of Theorem 5 hold,we can state the following identity

$$\int_a^b g(t)dt = \sum_{j=1}^{2n} \frac{(b-x)^j - (a-x)^j}{j!} g^{(j-1)}(x) + \frac{(b-x)^{2n+1} - (a-x)^{2n+1}}{(2n+1)!} g^{(2n)}(\eta) \quad \text{for some } \eta \in (a,b). \tag{4.4}$$

Specially, according to the Remark 6, from the condition $A_2(x) = 0$ we get $x = \frac{a+b}{2}$ which gives the generalization of the midpoint rule ([5])

$$\int_a^b g(t)dt - \sum_{j=1}^{n} \frac{(b-a)^{2j-1}}{2^{2j-2}(2j-1)!} g^{(2j-2)}\left(\frac{a+b}{2}\right) = \frac{(b-a)^{2n+1}}{2^{2n}(2n+1)!} g^{(2n)}(\eta) \tag{4.5}$$

Specially, for $n = 1$ this formula becomes the well known midpoint rule

$$\int_a^b g(t)dt = (b-a)g\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} g''(\eta). \tag{4.6}$$

4.2. $w(t) = (b-t)^{\alpha}(t-a)^{\beta}, \quad t \in (a,b), \quad \alpha,\beta > -1$

The general one-point formula for this case is

$$\int_a^b (b-t)^{\alpha}(t-a)^{\beta} g(t)dt = \sum_{j=1}^{n} A_j(x) g^{(j-1)}(x) + (-1)^n \int_a^b W_{n,w}(t,x) g^{(n)}(t)dt, \tag{4.7}$$

where, according to the Theorem 4, we have

$$A_j(x)=\begin{cases}\frac{(a-x)^{j-1}(b-a)^{\alpha+\beta+1}}{(j-1)!}B(\alpha+1,\beta+1)F(1-j,\beta+1,\alpha+\beta+2;\frac{b-a}{x-a}), & x\neq a,\\ \frac{(b-a)^{\alpha+\beta+j}}{(j-1)!}B(\alpha+1,\beta+j), & x=a\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases}\frac{(b-a)^{\alpha}(t-a)^{n+\beta}}{(n-1)!}B(\beta+1,n)F(-\alpha,\beta+1,\beta+n+1;\frac{t-a}{b-a}) & \text{for } t\in[a,x],\\ (-1)^n\frac{(b-a)^{\beta}(b-t)^{n+\alpha}}{(n-1)!}B(\alpha+1,n)F(-\beta,\alpha+1,\alpha+n+1;\frac{b-t}{b-a}) & \text{for } t\in(x,b],\end{cases}$$

where

$$B(u,v)=\int_0^1 s^{u-1}(1-s)^{v-1}ds$$

is the Beta function, and

$$F(\alpha,\beta,\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,$$

for $\gamma>\beta>0$ and $z<1$ is the hypergeometric function. We also use notation of the hypergeometric function when integral $\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$ converges. Specially, when $\alpha<0$, then any $z\in\mathbf{R}$ is allowed.

Further, if the assumptions of the Theorem 5 hold, we get

$$\int_a^b (b-t)^{\alpha}(t-a)^{\beta}g(t)dt-\sum_{j=1}^{2n}A_j(x)g^{(j-1)}(x)=A_{2n+1}(x)g^{(2n)}(\eta),\ \text{ for some } \eta\in(a,b). \tag{4.8}$$

Specially, for $[a,b]=[-1,1]$ and from the condition $A_2(x)=0$, we get $x=\frac{\beta-\alpha}{\alpha+\beta+2}$, so we have for $n=1$

$$\begin{aligned}\int_{-1}^1(1-t)^{\alpha}(t+1)^{\beta}g(t)dt &= 2^{\alpha+\beta+1}B(\alpha+1,\beta+1)\cdot g(x)\\ &\quad+\frac{2^{\alpha+\beta+2}(\beta+1)B(\alpha+2,\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)}\cdot g''(\eta),\end{aligned} \tag{4.9}$$

which is the Jacobi-Gaussian quadrature formula with 1 node (see [6]).

Now we consider special cases for α and β:

4.2.1. $\alpha=\beta=-\frac{1}{2}$, $\quad w(t)=\frac{1}{\sqrt{1-t^2}}$, $\quad t\in(-1,1)$

For this case we have

$$A_j(x)=\begin{cases}\frac{(-1-x)^{j-1}\pi}{(j-1)!}F(1-j,\frac{1}{2},1;\frac{2}{x+1}), & x\neq -1,\\ \frac{2^{j-1}}{(j-1)!}B(\frac{1}{2},j-\frac{1}{2}), & x=-1\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases}\frac{2^{-\frac{1}{2}}(t+1)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{t+1}{2}) & \text{for } t\in[-1,x],\\ (-1)^n\frac{2^{-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{1-t}{2}) & \text{for } t\in(x,1],\end{cases}$$

so we have

$$\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt=\sum_{j=1}^{n}A_j(x)g^{(j-1)}(x)+(-1)^n\int_{-1}^{1}W_{n,w}(t,x)g^{(n)}(t)dt. \tag{4.10}$$

If the assumptions of Theorem 5 hold, then we have

$$\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt-\sum_{j=1}^{2n}A_j(x)g^{(j-1)}(x)=A_{2n+1}(x)g^{(2n)}(\eta),\quad \text{for some } \eta\in(-1,1). \tag{4.11}$$

Specially, from the condition $A_2(x)=0$ we get $x=0$, so we have for $n=1$

$$\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt=\pi g(0)+\frac{\pi}{4}g''(\eta),\quad \text{for some } \eta\in(a,b), \tag{4.12}$$

which coincides with the Chebyshev-Gaussian quadrature formula with 1 node ([6]). If $g^{(n)}\in L_p$, for $n\in\{1,2\}$, then Theorem 6 implies

$$\left|\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt-\pi g(0)\right|\leqslant C_1(1,p,0,w)\|g'\|_p. \tag{4.13}$$

Specially,

$$C_1(1,\infty,0,w)=2,\quad C_1(1,1,0,w)=\frac{\pi}{2}\approx 1.570796$$
$$C_2(2,\infty,0,w)=\frac{\pi}{4}\approx 0.785398,\quad C_1(2,1,0,w)=1.$$

4.2.2. $\alpha=\beta=\frac{1}{2},\quad w(t)=\sqrt{1-t^2},\quad t\in[-1,1]$

For this case we have

$$A_j(x)=\begin{cases}\frac{4(-1-x)^{j-1}}{(j-1)!}B(\frac{3}{2},\frac{3}{2})F(1-j,\frac{3}{2},3;\frac{2}{x+1}), & x\neq-1,\\ \frac{2^{j+1}}{(j-1)!}B(\frac{3}{2},j+\frac{1}{2}), & x=-1\end{cases}$$

and

$$W_{n,w}(t,\sigma)=\begin{cases}\frac{2^{\frac{1}{2}}(t+1)^{n+\frac{1}{2}}}{(n-1)!}B(\frac{3}{2},n)F(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{t+1}{2}) & \text{for } t\in[-1,x],\\ (-1)^n\frac{2^{\frac{1}{2}}(1-t)^{n+\frac{1}{2}}}{(n-1)!}B(\frac{3}{2},n)F(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{1-t}{2}) & \text{for } t\in(x,1],\end{cases}$$

so we have

$$\int_{-1}^{1} g(t)\sqrt{1-t^2}dt = \sum_{j=1}^{n} A_j(x)g^{(j-1)}(x) + (-1)^n \int_{-1}^{1} W_{n,w}(t,x)g^{(n)}(t)dt. \tag{4.14}$$

If the assumptions of Theorem 5 hold, then we have

$$\int_{-1}^{1} \sqrt{1-t^2}g(t)dt - \sum_{j=1}^{2n} A_j(x)g^{(j-1)}(x) = A_{2n+1}(x)g^{(2n)}(\eta), \quad \text{for some } \eta \in (-1,1). \tag{4.15}$$

Specially, from the condition $A_2(x) = 0$, we get $x = 0$, so we have for $n = 1$

$$\int_{-1}^{1} \sqrt{1-t^2}g(t)dt = \frac{\pi}{2}g(0) + \frac{\pi}{16}g''(\eta), \quad \text{for some } \eta \in (-1,1), \tag{4.16}$$

which coincides with the Chebyshev-Gaussian quadrature formula of the second kind with one node ([6]). If $g^{(n)} \in \mathrm{L_p}$, for $n \in \{1,2\}$, then Theorem 6 implies

$$\left| \int_{-1}^{1} g(t)\sqrt{1-t^2}dt - \frac{\pi}{2}g(0) \right| \leqslant C_1(n,p,0,w)\|g'\|_p. \tag{4.17}$$

Specially,

$$C_1(1,\infty,0,w) = \frac{2}{3} \approx 0.66667, \quad C_1(1,1,0,w) = \frac{\pi}{4} \approx 0.78538$$
$$C_1(2,\infty,0,w) = \frac{\pi}{16} \approx 0.19635, \quad C_1(2,1,0,w) = \frac{1}{3} \approx 0.33333.$$

4.2.3. $\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2} \quad w(t) = \sqrt{\frac{1-t}{1+t}}, \quad t \in (-1,1)$

For this case we have

$$A_j(x) = \begin{cases} \frac{2(-1-x)^{j-1}}{(j-1)!}B(\frac{3}{2},\frac{1}{2})F(1-j,\frac{1}{2},2,\frac{2}{x+1}), & x \neq -1 \\ \frac{2^j}{(j-1)!}B(\frac{3}{2},j-\frac{1}{2}), & x = -1, \end{cases}$$

and

$$W_{n,w}(t,\sigma) = \begin{cases} \frac{2^{\frac{1}{2}}(t+1)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(-\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{t+1}{2}) & \text{for } t \in [-1,x], \\ (-1)^n\frac{2^{-\frac{1}{2}}(1-t)^{n+\frac{1}{2}}}{(n-1)!}B(\frac{3}{2},n)F(\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{1-t}{2}) & \text{for } t \in (x,1], \end{cases}$$

so we have

$$\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt = \sum_{j=1}^{n} A_j(x)g^{(j-1)}(x) + (-1)^n \int_{-1}^{1} W_{n,w}(t,\sigma)g^{(n)}(t)dt. \tag{4.18}$$

If the assumptions of Theorem 5 hold, then we have

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}g(t)dt-\sum_{j=1}^{2n}A_j(x)g^{(j-1)}(x)=A_{2n+1}(x)g^{(2n)}(\eta),\quad \text{for some } \eta\in(-1,1). \tag{4.19}$$

Specially, from the condition $A_2(x)=0$ we get $x=-\frac{1}{2}$, so we have for $n=1$

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}g(t)dt=\pi g\left(-\frac{1}{2}\right)+\frac{\pi}{8}g''(\eta),\quad \text{for some } \eta\in(-1,1), \tag{4.20}$$

which coincides with the well known quadrature formula with n nodes, with the highest degree of precision $2n-1$, for the special case $n=1$ ([6]). If $g^{(n)}\in L_p$, for $n\in\{1,2\}$, then Theorem 6 implies

$$\left|\int_{-1}^{1}g(t)\sqrt{\frac{1-t}{1+t}}dt-\pi g\left(-\frac{1}{2}\right)\right|\leqslant C_1(n,p,-\frac{1}{2},w)\|g'\|_p. \tag{4.21}$$

Specially,

$$C_1\left(1,\infty,-\frac{1}{2},w\right)=\frac{3\sqrt{3}}{4}\approx 1.29904,\quad C_1\left(1,1,-\frac{1}{2},w\right)=\frac{\pi}{3}+\frac{\sqrt{3}}{2}\approx 1.91322$$
$$C_1\left(2,\infty,-\frac{1}{2},w\right)=\frac{\pi}{8}\approx 0.392699,\quad C_1\left(2,1,-\frac{1}{2},w\right)=\frac{3\sqrt{3}}{8}\approx 0.649519.$$

4.2.4. $\alpha=0,\quad \beta=\frac{1}{2}\quad w(t)=\sqrt{t},\quad t\in[0,1]$

For this case we have

$$A_j(x)=\begin{cases}\frac{(-x)^j}{(j-1)!}B(1,\frac{3}{2})F(1-j,\frac{3}{2},\frac{5}{2},\frac{1}{x}), & x\neq 0\\ \frac{1}{(j-1)!}B(1,j+\frac{1}{2}), & x=0\end{cases}$$

$$W_{n,w}(t,x)=\begin{cases}\frac{t^{n+1/2}}{(n-1)!}B(n,\frac{3}{2}) & \text{for } t\in[0,x],\\ \frac{(t-1)^n}{n!}F(-\frac{1}{2},1,n+1;1-t) & \text{for } t\in(x,1],\end{cases}$$

so we have

$$\int_{0}^{1}\sqrt{t}g(t)dt=\sum_{j=1}^{n}A_j(x)g^{(j-1)}(x)+(-1)^n\int_{-1}^{1}W_{n,w}(t,x)g^{(n)}(t)dt. \tag{4.22}$$

If the assumptions of Theorem 5 hold, then we have

$$\int_{0}^{1}\sqrt{t}g(t)dt-\sum_{j=1}^{2n}A_j(x)g^{(j-1)}(x)=A_{2n+1}(x)g^{(2n)}(\eta),\quad \text{for some } \eta\in(0,1). \tag{4.23}$$

Specially, from the condition $A_2(x)=0$ we get $x=\frac{3}{5}$, so we have for $n=1$

$$\int_0^1 \sqrt{t}g(t)dt = \frac{2}{3}g\left(\frac{3}{5}\right)+\frac{4}{175}g''(\eta), \quad \text{for some } \eta\in(0,1). \tag{4.24}$$

If $g^{(n)}\in \mathrm{L_p}$, for $n\in\{1,2\}$, then Theorem 6 implies

$$\left|\int_0^1 g(t)\sqrt{t}dt-\frac{2}{3}g(\frac{3}{5})\right| \leqslant C_1(n,p,\frac{3}{5},w)\|g'\|_p. \tag{4.25}$$

Specially,

$$C_1\left(1,\infty,\frac{3}{5},w\right)=\frac{24}{125}\sqrt{\frac{3}{5}}\approx 0.148722, \quad C_1\left(1,1,\frac{3}{5},w\right)=\frac{2}{3}-\frac{2}{5}\sqrt{\frac{3}{5}}\approx 0.35682$$
$$C_1\left(2,\infty,\frac{3}{5},w\right)=\frac{4}{175}\approx 0.02285, \quad C_1\left(2,1,\frac{3}{5},w\right)=\frac{12}{125}\sqrt{\frac{3}{5}}\approx 0.07436$$

4.2.5. $\alpha=0, \quad \beta=-\frac{1}{2} \quad w(t)=\frac{1}{\sqrt{t}}, \quad t\in(0,1]$

For this case we have

$$A_j(x)=\begin{cases} \frac{2(-x)^j}{(j-1)!}F(1-j,\frac{1}{2},\frac{3}{2},\frac{1}{x}), & x\neq 0\\ \frac{2}{(2j-1)(j-1)!}, & x=0\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases} \frac{t^{n-1/2}}{(n-1)!}B(n,\frac{1}{2}) & \text{for } t\in[0,x],\\ \frac{(t-1)^n}{n!}F(\frac{1}{2},1,n+1;1-t) & \text{for } t\in(x,1],\end{cases}$$

so we have

$$\int_0^1 \frac{g(t)}{\sqrt{t}}dt=\sum_{j=1}^n A_j(x)g^{(j-1)}(x)+(-1)^n\int_{-1}^1 W_{n,w}(t,x)g^{(n)}(t)dt. \tag{4.26}$$

If the assumptions of Theorem 5 hold, then we have

$$\int_0^1 \frac{g(t)}{\sqrt{t}}dt-\sum_{j=1}^{2n}A_j(x)g^{(j-1)}(x)=A_{2n+1}(x)g^{(2n)}(\eta), \quad \text{for some } \eta\in(0,1). \tag{4.27}$$

Specially, from the condition $A_2(x)=0$ we get $x=\frac{1}{3}$, so we have for $n=1$

$$\int_0^1 \frac{g(t)}{\sqrt{t}}dt=2g\left(\frac{1}{3}\right)+\frac{4}{45}g''(\eta), \quad \text{for some } \eta\in(0,1). \tag{4.28}$$

If $g^{(n)} \in \mathrm{L_p}$, for $n \in \{1,2\}$, then Theorem 6 implies

$$\left|\int_0^1 \frac{g(t)}{\sqrt{t}}dt - 2g\left(\frac{1}{3}\right)\right| \leqslant C_1(n,p,\frac{1}{3},w)\|g'\|_p. \tag{4.29}$$

Specially,

$$C_1\left(1,\infty,\frac{1}{3},w\right) = \frac{8}{9\sqrt{3}} \approx 0.5132 \quad C_1\left(1,1,\frac{1}{3},w\right) = \frac{2}{\sqrt{3}} \approx 1.1547$$
$$C_1\left(2,\infty,\frac{1}{3},w\right) = \frac{4}{45} \approx 0.08889 \quad C_1\left(2,1,\frac{1}{3},w\right) = \frac{4}{9\sqrt{3}} \approx 0.25666.$$

4.3. $w(t) = e^{-t^2}, \quad t \in \mathbf{R}$

For this case we have

$$A_j(x) = \frac{(-1)^{j-1}}{(j-1)!}\int_{-\infty}^{\infty}(x-s)^{j-1}e^{-s^2}ds \tag{4.30}$$

and

$$W_{n,w}(t,x) = \begin{cases} \frac{1}{(n-1)!}\int_{-\infty}^{t}(t-s)^{n-1}e^{-s^2}ds & \text{for } t \in (-\infty,x], \\ \frac{1}{(n-1)!}\int_{\infty}^{t}(t-s)^{n-1}e^{-s^2}ds & \text{for } t \in (x,\infty), \end{cases}$$

so we have

$$\int_{-\infty}^{\infty} g(t)e^{-t^2}dt = \sum_{j=1}^{n} A_j(x)g^{(j-1)}(x) + (-1)^n\int_{-1}^{1} W_{n,w}(t,x)g^{(n)}(t)dt. \tag{4.31}$$

If the assumptions of Theorem 5 hold, then we have

$$\int_{-\infty}^{\infty} e^{-t^2}g(t)dt - \sum_{j=1}^{2n} A_j(x)g^{(j-1)}(x) = A_{2n+1}(x)g^{(2n)}(\eta), \quad \text{for some } \eta \in \mathbf{R}. \tag{4.32}$$

Specially, from the condition $A_2(x) = 0$ we get $x = 0$, so we have for $n = 1$

$$\int_{-\infty}^{\infty} e^{-t^2}g(t)dt = \sqrt{\pi}g(0) + \frac{\sqrt{\pi}}{4}g''(\eta), \quad \text{for some } \eta \in \mathbf{R}, \tag{4.33}$$

which coincides with the Hermite-Gaussian quadrature formula with 1 node ([6]). If $g^{(n)} \in \mathrm{L_p}$, for $n \in \{1,2\}$, then Theorem 6 implies

$$\left|\int_{-\infty}^{\infty} g(t)e^{-t^2}dt - \sqrt{\pi}g(0)\right| \leqslant C_1(n,p,0,w)\|g'\|_p. \tag{4.34}$$

Specially,

$$C_1(1,\infty,0,w) = 1, \quad C_1(1,1,0,w) = \frac{\sqrt{\pi}}{2} \approx 0.88622$$
$$C_1(1,\infty,0,w) = \frac{\sqrt{\pi}}{4} \approx 0.44311, \quad C_1(1,1,0,w) = \frac{1}{2}.$$

4.4. $w(t) = t^{\alpha} e^{-t}, \quad \alpha > -1, \quad t \in (0, \infty)$

For this case we have

$$A_j(x) = \sum_{k=0}^{j-1} \frac{(-x)^k \Gamma(j-k+\alpha)}{k!(j-k-1)!} \tag{4.35}$$

and

$$W_{n,w}(t,x) = \begin{cases} \sum_{k=0}^{n-1} \frac{t^k \gamma(n-k+\alpha,t)}{k!(n-k-1)!} & \text{for } t \in (0,x], \\ -\sum_{k=0}^{n-1} \frac{t^k \Gamma(n-k+\alpha,t)}{k!(n-k-1)!} & \text{for } t \in (x,\infty), \end{cases}$$

where

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad z > 0$$

is Gamma function,

$$\gamma(z,a) = \int_0^{a} t^{z-1} e^{-t} dt, \quad z > 0$$

is the "lower" incomplete Gamma function and

$$\Gamma(z,a) = \int_a^{\infty} t^{z-1} e^{-t} dt, \quad z > 0$$

is "upper" incomplete Gamma function, so we have

$$\int_0^{\infty} g(t) t^{\alpha} e^{-t} dt = \sum_{j=1}^{n} A_j(x) g^{(j-1)}(x) + (-1)^n \int_{-1}^{1} W_{n,w}(t,x) g^{(n)}(t) dt. \tag{4.36}$$

If the assumptions of Theorem 5 hold, then we have

$$\int_0^{\infty} t^{\alpha} e^{-t} g(t) dt - \sum_{j=1}^{2n} A_j(x) g^{(j-1)}(x) = A_{2n+1}(x) g^{(2n)}(\eta), \quad \text{for some } \eta \in (0,\infty). \tag{4.37}$$

Specially, from the condition $A_2(x) = 0$ we get $x = \alpha + 1$, so we have for $n = 1$

$$\int_0^{\infty} t^{\alpha} e^{-t} g(t) dt = \Gamma(\alpha+1) g(\alpha+1) + \frac{\Gamma(\alpha+2)}{2} g''(\eta), \quad \text{for some } \eta \in (0,\infty), \tag{4.38}$$

which coincides with the Laguerre-Gaussian quadrature formula with 1 node ([6]). If $g^{(n)} \in \mathrm{L_p}$, for $n \in \{1,2\}$, then Theorem 6 implies

$$\left| \int_0^{\infty} g(t) t e^{-t} dt - g(2) \right| \leqslant C_1(n,p,2,w) \|g'\|_p. \tag{4.39}$$

Specially,

$$C_1(1,\infty,2,w) = \frac{8}{e^2} \approx 1.08268, \quad C_1(1,1,2,w) = 1 - \frac{3}{e^2} \approx 0.59399$$

$$C_1(2,\infty,2,w) = 1, \quad C_1(2,1,2,w) = \frac{4}{e^2} \approx 0.54134.$$

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ, *On weighted Euler harmonic identities with applications* , Math. Inequal. Appl., **8**, 2 (2005), 237–257.

[2] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, *The weighted Euler identity*, Math. Inequal. Appl., **8**, 2 (2005), 207–221.

[3] LJ.DEDIĆ, M. MATIĆ, J. PEČARIĆ, *On generalizations of Ostrowski inequality via some Euler-type identities*, Math. Inequal. Appl., **3**, 3 (2000), 337–353.

[4] S.S.DRAGOMIR, *The generalized integration by parts formula for Appell sequences and related results*, Communications of the Korean Mathematical Society, **19** (2004), 75–92.

[5] S. KOVAČ, J. PEČARIĆ, A. VUKELIĆ, *A generalization of general two-point formula with applications in numerical integration*, Nonlinear Analysis Series A: Theory, Methods and Applications, **68** (2008), 2445–2463.

[6] V. I. KRYLOV, *Approximate calculation of integrals*, Macmillan, New York-London, 1962.

[7] M. MATIĆ, J. PEČARIĆ, N. UJEVIĆ, *Generalizations of weighted version of Ostrowski's inequality and some related results*, J. of Inequal. Appl., **5** (2000), 639–666.

[8] J. PEČARIĆ, S. VAROŠANEC, *Harmonic Polynomials and Generalization of Ostrowski Inequality with Applications in Numerical Integration*, Nonlinear Analysis Series A: Theory, Methods and Applications, **47** (2001), 2365–2374.

(Received May 4, 2007)

S. Kovač
Faculty of Geotechnical Engineering, Varaždin
University of Zagreb
Hallerova aleja 7
42000 Varaždin
Croatia
e-mail: skovac@gfv.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: pecaric@hazu.hr