Mathematical
Inequalities
& Applications
Volume 13, Number 3 (2010), 629–633

THE SHARP SOBOLEV TRACE INEQUALITY IN A LIMITING CASE

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(*Communicated by J. Pečarić*)

Abstract. A limiting case of the Sobolev trace inequalities is investigated and the best constant for the case is computed. Moreover, when $n = 1$, the same result is obtained from recognizing the Euler-Lagrange equation for the inequality as the mean curvature formula of plane curves.

1. Introduction

The Sobolev trace inequalities on the upper half-space $\mathbb{R}^{n+1}_{+}$ are given by

$$\left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{p/q} \leqslant A_{p,q}\left(\int_{\mathbb{R}^{n+1}_{+}} |\nabla u(x,y)|^p dxdy\right), \quad \frac{1}{q} = \frac{n+1}{np} - \frac{1}{n},$$

where u is an extension of f to the upper half-space that is continuous on the closed upper half-space and at least once differentiable on the open upper half-space, and $A_{p,q}$ is a positive constant independent of the function u. Recently, the importance of having the sharp form of the inequalities has been recognized. For example, the solution to the Yamabe problem turns out to depend on knowledge of the best constant of the classical Sobolev inequality [5].

The sharp form of the Sobolev trace inequality for the case $p = 2$ and $n > 1$ is

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{2n}{(n-1)}} dx\right)^{\frac{(n-1)}{n}} \leqslant \frac{1}{\sqrt{\pi}}\frac{1}{n-1}\left[\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right]^{\frac{1}{n}}\left(\int_{\mathbb{R}^{n+1}_{+}} |\nabla u(x,y)|^2 dxdy\right),$$

where u is a harmonic extension of f to the upper half-space that is continuous on the closed upper half-space and extremal functions for the inequality are given by $f(x) = (1+|x|^2)^{-(n-1)/2}$. Since this inequality is conformally invariant, the extremal function given above is unique up to a conformal automorphism. W. Beckner [2] proved this by inverting the inequality to a fractional integral on the dual space and using a special case of the sharp Hardy-Littlewood-Sobolev inequality.

In this paper, we treat the Sobolev trace inequality for the case when $p = 1$. It can be thought of as one of the limiting cases of the inequalities and is related to the

Mathematics subject classification (2010): 26D10, 35A15, 41A44, 53A04.
Keywords and phrases: Sobolev trace inequality, variational methods, best constant, curves in Euclidean space.

Paper MIA-13-44

isoperimetric inequality. The existence of the extremal function for this case is not guaranteed by the argument used in [7] for p with $1 < p < n+1$.

In section 2, we will show that the extremal function does not exist for this particular case. However, the sharp constant is computed using a rearrangement technique on the functions on $\mathbb{R}_+^{n+1}$.

In section 3, we exploit the variational equations for the Sobolev trace inequality to give a simple geometric argument to explain how to get the extremal functions for the case $n = 1$.

In the following, $\mathbb{R}_+$ denotes the set of all positive real numbers.

2. Sharp Sobolev Trace Inequality in a Limiting Case

The limiting case of the Sobolev trace inequality with $p = 1$ is given by

$$\int_{\mathbb{R}^n} |u(x,0)|dx \leqslant C \int_{\mathbb{R}_+^{n+1}} |\nabla u(x,y)|dxdy \tag{1}$$

for a positive constant C. To find the best constant for this inequality, we look at the following quotient:

$$J(u) \equiv \frac{\left(\int_{\mathbb{R}_+^{n+1}} |\nabla u(x,y)|dxdy\right)}{\left(\int_{\mathbb{R}^n} |u(x,0)|dx\right)},$$

where $u \in W^{1,1}(\mathbb{R}_+^{n+1})$ and $u \not\equiv 0$. Then the best constant $\mathbf{I}$ can be defined by

$$\mathbf{I} \equiv \inf\{J(u) : u \in W^{1,1}(\mathbb{R}_+^{n+1}), u \neq 0\}.$$

Define $\mathbf{B} \equiv \left\{g \in W^{1,1}(\mathbb{R}_+^{n+1}) : g \geqslant 0 \text{ on } \mathbf{R}_+^{n+1}, \text{ and } \int_{\mathbb{R}^n} g(x,0)\,dx = 1\right\}$. It is sufficient to consider functions in $\mathbf{B}$ to compute the best constant for the inequality (1), since $J(\cdot)$ is dilation invariant and $J(u) = J(|u|)$. Moreover, we use a rearrangement technique to reduce further the functions to consider to a class of functions with a special property. Namely, we take Φ_S^* to be the *Steiner rearrangement* of Φ. The proper definition of the Steiner rearrangement and its properties can be found in [6]. Here Φ_S^* is symmetric radial decreasing in $|x|$, and is decreasing in y. Then we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\Phi(x,0)|dx &= \int_{\mathbb{R}^n} |\Phi_S^*(x,0)|dx \\ &= \int_{\mathbb{R}^n}\int_0^\infty -\frac{\partial \Phi_S^*}{\partial y}(x,y)dydx \\ &\leqslant \int_{\mathbb{R}_+^{n+1}} |\nabla \Phi_S^*(x,y)|dxdy \qquad (2) \\ &\leqslant \int_{\mathbb{R}_+^{n+1}} |\nabla \Phi(x,y)|dxdy. \qquad (3)\end{aligned}$$

By the above observation, it suffices to consider functions in $\mathbf{B}$ having the following property (P):

$$g \text{ is symmetric radial decreasing in } |x|, \text{ and is decreasing in } y. \tag{P}$$

For any function g having the property (P), the inequality (3) becomes equality. It is now clear that $\inf\{J(g) \mid g \text{ has the property (P)}, g \in \mathbf{B}\} \geqslant 1$.

THEOREM 1. *The best constant* $\mathbf{I}$ *for the Sobolev trace inequality* (1) *is* 1*:*

$$\mathbf{I} \equiv \inf\{J(u) : u \in W^{1,1}(\mathbb{R}^{n+1}_+),\ u \neq 0\} = 1.$$

Proof. We will look at the inequalities above. The inequality (3) becomes equality, since we choose f with the property (P). The question is when the inequality (2) becomes equality. For that we require that f satisfy

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = |\nabla f(x,y)| \text{ for } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}_+,$$

which implies that

$$\frac{\partial f}{\partial x_j}(x,y) = 0 \text{ on } \mathbb{R}^{n+1}_+ \text{ for } j = 1,2,3,\cdots,n.$$

From this, we can see that f should be a function of y variable only. On the other hand, $f(x,0)$ is a function in $L^1(\mathbb{R}^n)$, so we need some restrictions on the decay of the function at infinity. Any function of y with appropriate decay multiplied by a characteristic function in the x variable will be an extremal function. The problem is that such functions do not belong to $W^{1,1}(\mathbb{R}^{n+1}_+)$, which means that the extremal function does not exist. However, we can use an *approximation argument* to compute the best constant. Take a function $f(x,y) = \phi(y)\chi_B(x)$, where ϕ is a positive non-increasing function of y variable and B is the unit ball centered at the origin in $\mathbb{R}^n$. Then we have

$$\int_{\mathbb{R}^{n+1}_+} |\nabla f(x,y)| dxdy = \int_{\mathbb{R}^n} |f(x,0)| dx + \sigma_n \int_0^\infty \phi(y) dy,$$

where σ_n is the surface area of the unit ball in $\mathbb{R}^n$. If we can make the second term in the right hand side go away, then we get the claim we made. This can be accomplished by choosing $\phi(y) \equiv \exp(-\frac{\pi y^2}{\varepsilon^2})$, which makes the term $\int_0^\infty \phi(y)dy = \varepsilon$ as small as we want. This completes the proof. □

3. Geometric Aspect of the Variational Equation

Assume in particular that $n = 1$, and then the Euler-Lagrange equation for (1) is given by

$$\operatorname{div}\left(\frac{\nabla u(x,y)}{|\nabla u(x,y)|}\right) = 0 \text{ on } (x,y) \in \mathbb{R} \times \mathbb{R}_+.$$

We take a different route to look at the nature of extremal functions for this case. Since we may assume by the rearrangement that this function $u(x,y)$ is even in x variable, we will consider functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$. Now consider any twice differentiable nonnegative function u on $\mathbb{R}_+ \times \mathbb{R}_+$. Sard theorem tells us that almost all the numbers in $\mathbb{R}_+$ are regular values of u. Take any nonnegative regular value c to have $u^{-1}(c)$ a submanifold of $\mathbb{R}_+ \times \mathbb{R}_+$ - it is actually the *level curve* of u at c. Now we want to discuss the *curvature* of this submanifold. Since this submanifold is 1-dimensional and sits in $\mathbb{R}_+ \times \mathbb{R}_+$, it is, in fact, a plane curve. The curvature $K(x,y)$ of this curve at $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ can be computed as follows:

$$K(x,y) = (-1)\frac{\det\begin{pmatrix} \nabla_{v(x,y)}N(x,y) \\ N(x,y) \end{pmatrix}}{\det\begin{pmatrix} v(x,y) \\ N(x,y) \end{pmatrix}},$$

where $N(x,y)$ is the unit normal vector field along $u^{-1}(c)$, $v(x,y)$ is a basis for the tangent space of $u^{-1}(c)$ at (x,y) and $\nabla_{v(x,y)}N(x,y)$ is the derivative of the vector field $N(x,y)$ with respect to the vector $v(x,y)$ (see [8]). Since $\nabla u(x,y)$ is a normal vector field to the level curves, we can take

$$N(x,y) = \frac{\nabla u(x,y)}{|\nabla u(x,y)|} \text{ and } v(x,y) = \left(\frac{\partial u}{\partial y}(x,y), -\frac{\partial u}{\partial x}(x,y)\right).$$

Using the fact that

$$\nabla_{v(x,y)}N(x,y) = \operatorname{div}\left(\frac{\nabla u(x,y)}{|\nabla u(x,y)|}\right)\left(\frac{\partial u}{\partial y}(x,y), -\frac{\partial u}{\partial x}(x,y)\right),$$

we can see

$$K(x,y) = -\operatorname{div}\left(\frac{\nabla u(x,y)}{|\nabla u(x,y)|}\right).$$

By the above observation, we see that the curvatures of all the level curves of an extremal function $u(x,y)$ are equal to 0, which implies that all the level curves of u are *straight lines*. By changes of the coordinates, if necessary, we may assume that the level curves are parallel to either x or y axis. Then, we can get the same kind of functions as we had in the previous section for the inequality.

Acknowledgements. The author would like to thank Dr. Beckner for his encouragement and support. She also wants to express her gratitude to the referee whose comments made it possible to improve this paper much. After completing this paper, it was pointed out that the best constant for the Sobolev trace inequality with a different approach could be found in [1].

REFERENCES

[1] T. AUBIN, *Nonlinear analysis on manifolds: Monge-Ampére equations*, Springer-Verlag, 1982.
[2] W. BECKNER, *Geometric inequalities in Fourier analysis*, Essays on Fourier Analysis in honor of Elias M. Stein, Princeton University Press, 36–68, 1995.
[3] W. BECKNER, *Inequalities in Fourier analysis*, Ann. Math., **102** (1975), 159–182.
[4] W. BECKNER, *Sharp inequalities and geometric manifolds*, J. Fourier Anal. Appl., **3** (1997), 825–836.
[5] J. F. ESCOBAR, *Sharp constant in a Sobolev trace inequality*, Indiana Math. J., **37** (1988), 687–698.
[6] B. KAWOHL, *Rearrangements and convexity of level sets in PDE*, Springer-Verlag, 1985.
[7] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The limit case, part 1 and 2*, Revista Mathematica Iberoamericana, **1** (1985), 145–201, 45–121.
[8] J. THROPE, *Elementary topics in differential geometry*, Springer-Verlag, 1979.

(Received August 13, 2007)

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