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MODULAR INEQUALITIES FOR THE HARDY–LITTLEWOOD AVERAGES

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Abstract. In this paper we establish general inequalities of the Hardy-Littlewood averages. We apply our results to obtained the higher-dimensional form of a strengthened Hardy-Knopp-type inequality. Furthermore, we discuss the inequalities given by Čižmešija et. al. [1], Cochran and Lee [2], Heinig [5], S. Kaijser et. al. [6], Levinson [8], Love [9 10], and Xiao [12], and show that these results are special cases of our results in this paper.

1. Introduction

Let X be a topological space and in X there is defined a continuous operation, scalar multiplication, such that to every pair (a,x) with $a\in(0,\infty)$ and $x\in X$ corresponds an element ax in X, in such a way that for $a,b\in(0,\infty)$ and $x\in X$ we have

$$1x=x,\quad a(bx)=(ab)x.$$

Let λ be a Borel probability measure on $(0,\infty)$. For a nonnegative Borel function f on X, we define the Hardy-Littlewood average Hf as

$$Hf(x)=\int_0^\infty f(tx)d\lambda(t),\qquad x\in X. \tag{1.1}$$

In the case $X=(0,\infty)$, the function Hf is the Hausdorff transform of f if $supp(\lambda)$, the support of λ, is contained in $(0,1]$. In particular, if $d\lambda(t)=\chi_{(0,1)}(t)k(1-t)^{k-1}dt$, $k>0$, then Hf is the (C,k) mean of f, and if $d\lambda(t)=\chi_{(0,1)}(t)\Gamma(k)^{-1}(-\log t)^{k-1}dt$, $k>0$, then Hf is the (H,k) mean of f. In [3, Eq.(11.18.4)], Hardy gave the following inequality for the Hausdorff transform for $1<p<\infty$:

$$\int_0^\infty (Hf(x))^p dx<\left(\int_0^1 t^{-1/p}d\lambda(t)\right)^p\int_0^\infty f(x)^p dx. \tag{1.2}$$

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In [12], Xiao considered the case $X=\mathbb{R}^n$ and proved the higher dimensional form of (1.2). The purpose of this paper is to extend these results to the inequality of the form

$$\int_E \phi(Hf(x))d\mu(x) \leqslant C\int_E \phi(f(x))d\nu(x), \tag{1.3}$$

where Hf is defined by (1.1), $\phi\in\Phi^+(I)$, μ and ν are σ-finite Borel measures on X, and E is a $\lambda-$balanced Borel set in X. Here I is an open interval contained in $(0,\infty)$ and $\Phi^+(I)$ denotes the class of all nonnegative convex functions ϕ on I such that ϕ takes its limiting values, finite or infinite, at the end of I. A Borel set $E\subseteq X$ is called $\lambda-$balanced if $tE\subseteq E$ for every $t\in supp(\lambda)$, where $tE=\{tx : x\in E\}$.

A considerable number of works are devoted to the study of inequalities of the type (1.2). We just mention the following, all of which to some extent have guided us in our research: [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], and the references given there. In particular, in paper [8], Levinson considered the case $X=E=(0,\infty)$, $d\lambda(t)=\chi_{(0,1)}(t)dt$, $d\mu(x)=d\nu(x)=dx$, and $\phi\phi''\geqslant(1-1/p)(\phi')^2$, where $1<p\leqslant\infty$. He proved (1.3) with $C=(p/(p-1))^p$ for $1<p<\infty$ and $C=e$ for $p=\infty$. This result generalized the classical Hardy's inequality (cf. [4, Theorem 327])

$$\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p\int_0^\infty f(x)^p dx, \tag{1.4}$$

and the Knopp's inequality (cf. [4, Theorem 335])

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)dt\right)dx \leqslant e\int_0^\infty f(x)dx. \tag{1.5}$$

In [5, Theorem 2.2], Heinig generalized Levinson's result to the case $d\mu(x)=u(x)dx$ and $d\nu(x)=v(x)dx$. On the other hand, in [6], Kaijser et. al. proved the Hardy-Knopp-type inequality

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \leqslant \int_0^\infty \phi(f(x))\frac{dx}{x}. \tag{1.6}$$

They also pointed out that (1.4) and (1.5) can be obtained by (1.6). In [1], A. Čižmešija et. al. generalized (1.6) to the so-called strengthened Hardy-Knopp-type inequality. Our results in this paper are generalizations of the results of Čižmešija et. al. [1], Cochran and Lee [2], Heinig [5], S. Kaijser et. al. [6], Levinson [8], Love [9 10], and Xiao [12].

Throughout this paper we assume that all functions are Borel measurable on their domains. We also take $\exp(-\infty)=0$, $\log 0=-\infty$, and $0\cdot\infty=0$.

2. Main Theorems

In the following theorems, μ and ν are σ-finite Borel measures on X, λ is a Borel probability measure on $(0,\infty)$, and E is a $\lambda-$balanced Borel set in X. For each $t>0$, we also define a Borel measure μ_t by $\mu_t(D)=\mu(t^{-1}D)$ for all Borel set D in X.

THEOREM 2.1. *Suppose* $\phi \in \Phi^+(I)$ *and* $\mu_t \ll \nu$ *for each* $t \in supp(\lambda)$. *If the range of values of* f *lies in the closure of* I, *then we have*

$$\int_E \phi(Hf(x))d\mu(x) \leqslant \int_E \phi(f(x)) \left(\int_0^\infty \frac{d\mu_t}{d\nu}(x)d\lambda(t) \right) d\nu(x). \tag{2.1}$$

Moreover, if $\log\phi$ *is also convex and* ρ *is a positive function on* $(0,\infty)$ *such that* $G_\rho = \exp\int_0^\infty \log\rho(t)d\lambda(t)$ *exists and* $0 < G_\rho < \infty$, *then*

$$\int_E \phi(Hf(x))d\mu(x) \leqslant G_\rho \int_E \phi(f(x)) \left(\int_0^\infty \rho(t)^{-1} \frac{d\mu_t}{d\nu}(x)d\lambda(t) \right) d\nu(x). \tag{2.2}$$

Proof of Theorem 2.1. By Jensen's inequality and Fubini's theorem, we have

$$\int_E \phi(Hf(x))d\mu(x) \leqslant \int_E \int_0^\infty \phi(f(tx))d\lambda(t)d\mu(x) = \int_0^\infty \int_E \phi(f(tx))d\mu(x)d\lambda(t).$$

Since $tE \subseteq E$ and $\mu_t \ll \nu$ for each $t \in supp(\lambda)$,

$$\int_E \phi(f(tx))d\mu(x) = \int_{tE} \phi(f(y))d\mu_t(y) \leqslant \int_E \phi(f(y)) \frac{d\mu_t}{d\nu}(y)d\nu(y). \tag{2.3}$$

Therefore

$$\begin{aligned} \int_E \phi(Hf(x))d\mu(x) &\leqslant \int_0^\infty \int_E \phi(f(y)) \frac{d\mu_t}{d\nu}(y)d\nu(y)d\lambda(t) \\ &= \int_E \phi(f(y)) \left(\int_0^\infty \frac{d\mu_t}{d\nu}(y)d\lambda(t) \right) d\nu(y). \end{aligned}$$

If $\log\phi$ is also convex, then

$$\begin{aligned} \phi(Hf(x)) &\leqslant \exp\int_0^\infty \log\phi(f(tx))d\lambda(t) = G_\rho \left(\exp\int_0^\infty \log[\rho(t)^{-1}\phi(f(tx))]d\lambda(t) \right) \\ &\leqslant G_\rho \int_0^\infty \rho(t)^{-1}\phi(f(tx))d\lambda(t) \end{aligned}$$

for all $x \in E$. This implies

$$\begin{aligned} \int_E \phi(Hf(x))d\mu(x) &\leqslant G_\rho \int_E \int_0^\infty \rho(t)^{-1}\phi(f(tx))d\lambda(t)d\mu(x) \\ &= G_\rho \int_0^\infty \left(\int_E \phi(f(tx))d\mu(x) \right) \rho(t)^{-1}d\lambda(t). \end{aligned}$$

Then (2.2) is followed by (2.3) and Fubini's theorem. This completes the proof. □

Consider the case $X = \mathbb{R}^n$, $d\mu(x) = |x|^{-1}\chi_E(x)u(x)dx$, and $d\nu(x) = dx$ in Theorem 2.1, where u is a nonnegative function on $\mathbb{R}^n$ and $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. Then $(d\mu_t/d\nu)(x) = t^{1-n}|x|^{-1}\chi_{tE}(x)u(t^{-1}x)$ for each $t > 0$. If

$supp(\lambda) \subseteq (0,1]$, $E = E_1 = \{x = \xi\sigma : \sigma \in A, 0 < \xi < b\}$, where A is a Borel subset of the unit sphere in $\mathbb{R}^n$ and $0 < b \leqslant \infty$, then (2.1) can be reduced to

$$\int_{E_1} \phi\left(\int_0^1 f(tx)d\lambda(t)\right) u(x)\frac{dx}{|x|} \leqslant \int_{E_1} \phi(f(x)) \left(\int_{|x|/b}^1 t^{1-n}u(t^{-1}x)d\lambda(t)\right) \frac{dx}{|x|} \quad (2.4)$$

and (2.2) can be reduced to

$$\int_{E_1} \phi\left(\int_0^1 f(tx)d\lambda(t)\right) u(x)\frac{dx}{|x|} \quad (2.5)$$

$$\leqslant G_\rho \int_{E_1} \phi(f(x)) \left(\int_{|x|/b}^1 t^{1-n}\rho(t)^{-1}u(t^{-1}x)d\lambda(t)\right) \frac{dx}{|x|}.$$

Choose $n = 1$, $E_1 = (0,b)$, and $d\lambda(t) = \chi_{(0,1)}(t)dt$. Then (2.4) can be reduced to the strengthened Hardy-Knopp-type inequality [1, Eq.(4)]. If $b = \infty$, $\rho(t) = t^{-\alpha}$, and replacing $u(x)$ by $xu(x)$, then $G_\rho = e^\alpha$ and (2.5) can be reduced to the result of [5, Theorem 2.2(ii)]. On the other hand, if $supp(\lambda) \subseteq [1,\infty)$, $E = E_2 = \{x = \xi\sigma : \sigma \in A, \xi > b\}$, where A is a Borel subset of the unit sphere in $\mathbb{R}^n$ and $0 \leqslant b < \infty$, then (2.1) can be reduced to

$$\int_{E_2} \phi\left(\int_1^\infty f(tx)d\lambda(t)\right) u(x)\frac{dx}{|x|} \leqslant \int_{E_2} \phi(f(x)) \left(\int_1^{|x|/b} t^{1-n}u(t^{-1}x)d\lambda(t)\right) \frac{dx}{|x|} \quad (2.6)$$

and (2.2) can be reduced to

$$\int_{E_2} \phi\left(\int_1^\infty f(tx)d\lambda(t)\right) u(x)\frac{dx}{|x|} \quad (2.7)$$

$$\leqslant G_\rho \int_{E_2} \phi(f(x)) \left(\int_1^{|x|/b} t^{1-n}\rho(t)^{-1}u(t^{-1}x)d\lambda(t)\right) \frac{dx}{|x|}.$$

Inequalities (2.6) and (2.7) may be seen as a dual relation to (2.4) and (2.5), respectively. If $n = 1$, $E_2 = (b,\infty)$, and $d\lambda(t) = t^{-2}\chi_{(1,\infty)}(t)dt$, then (2.6) can be reduced to [1, Eq.(6)].

We can also obtain [10, Theorem 1] by our Theorem 2.1. Let $X = (0,\infty)$, u and w are positive functions on $(0,\infty)$, w is integrable on $(0,1)$, and $W = \int_0^1 w(t)dt$. Consider the case $E = (0,b)$, $0 < b \leqslant \infty$, $d\lambda(t) = \chi_{(0,1)}(t)W^{-1}w(t)dt$, $d\mu = \chi_{(0,b)}(x)u(x)dx$, and $d\nu = u(x)dx$ in Theorem 2.1. Then $(d\mu_t/d\nu)(x) = t^{-1}u(x)^{-1}u(t^{-1}x)\chi_{(0,tb)}(x)$. If

$$\int_0^\infty \rho(t)^{-1}\frac{d\mu_t}{d\nu}(x)d\lambda(t) = \frac{1}{W}\int_{x/b}^1 \frac{u(t^{-1}x)w(t)}{\rho(t)tu(x)}dt \leqslant M \text{ for almost all } x \in (0,b),$$

then (2.2) can be reduced to

$$\int_0^b \phi\left(\frac{1}{W}\int_0^1 f(tx)w(t)dt\right) u(x)dx \leqslant MG_\rho \int_0^b \phi(f(x))u(x)dx. \quad (2.8)$$

Choosing $\phi(x) = e^x$ and replacing f by $\log f$, we have [10, Theorem 1].

If to the hypotheses of Theorem 2.1 is added that

$$\sup_{x \in E} \frac{d\mu_t}{d\nu}(x) \leqslant \alpha(t) \text{ for each } t \in supp(\lambda) \tag{2.9}$$

for some positive function α on $(0,\infty)$, then we have the following theorem.

THEOREM 2.2. *Suppose $\phi \in \Phi^+(I)$, $\mu_t \ll \nu$ for each $t \in supp(\lambda)$, condition* (2.9) *holds, and the range of values of f lies in the closure of I.*

(i) *If $\phi^{1/p} \in \Phi^+(I)$ and $0 < \int_0^\infty \alpha(t)^{1/p} d\lambda(t) < \infty$, where $1 \leqslant p < \infty$, then*

$$\int_E \phi(Hf(x))d\mu(x) \leqslant \left(\int_0^\infty \alpha(t)^{1/p} d\lambda(t)\right)^p \int_E \phi(f(x))d\nu(x). \tag{2.10}$$

(ii) *If $\log\phi$ is convex, $G_\alpha = \exp\int_0^\infty \log\alpha(t)d\lambda(t)$ exists, and $0 < G_\alpha < \infty$, then*

$$\int_E \phi(Hf(x))d\mu(x) \leqslant \left(\exp\int_0^\infty \log\alpha(t)d\lambda(t)\right) \int_E \phi(f(x))d\nu(x). \tag{2.11}$$

Proof of Theorem 2.2. We first prove case (i). Since $\phi^{1/p}$ is convex, we have

$$\phi(Hf(x)) \leqslant \left(\int_0^\infty \phi^{1/p}(f(tx))d\lambda(t)\right)^p$$

for all $x \in E$. By Minkowski inequality for integrals, we see that

$$\int_E \phi(Hf(x))d\mu(x) \leqslant \left\{\int_0^\infty \left(\int_E \phi(f(tx))d\mu(x)\right)^{1/p} d\lambda(t)\right\}^p. \tag{2.12}$$

By (2.3) and (2.9), we see that for each $t \in supp(\lambda)$,

$$\int_E \phi(f(tx))d\mu(x) \leqslant \alpha(t)\int_E \phi(f(x))d\nu(x). \tag{2.13}$$

Putting (2.12) and (2.13) together, we have (2.10). The result of case (ii) can be obtained by choosing $\rho = \alpha$ in Theorem 2.1. □

In the case $X = E = (0,\infty)$, $d\mu = d\nu = dx$, and $d\lambda(t) = \psi(t)dt$, where ψ is a nonnegative function on $(0,\infty)$ such that $\int_0^\infty \psi(t)dt = 1$, we may choose $\alpha(t) = t^{-1}$ and Theorem 2.2(i) can be reduced to [5, Corollary 2.1].

In the following corollaries, we consider the case $X = \mathbb{R}^n$ and the inequality

$$\int_E \phi(Hf(x))\eta(|x|)\tau(|x|)dx \leqslant C\int_E \phi(f(x))\eta(|x|)\tau(|x|)dx, \tag{2.14}$$

where $\phi \in \Phi^+(I)$, $\eta : (0,\infty) \mapsto (0,\infty)$ is submultiplicative, $\tau : (0,\infty) \mapsto (0,\infty)$ is monotone, and the range of values of f lies in the closure of I. Here η is called submultiplicative if $\eta(ab) \leqslant \eta(a)\eta(b)$ for all a and b in $(0,\infty)$.

COROLLARY 2.3. *Suppose* $supp(\lambda) \subseteq (0,1]$ *and* τ *is decreasing.*

(i) *If* $\phi^{1/p} \in \Phi^+(I)$, $1 \leqslant p < \infty$ *and* $0 < \int_0^1 (\eta(t^{-1})t^{-n})^{1/p} d\lambda(t) < \infty$, *then* (2.14) *holds with* $C = (\int_0^1 (\eta(t^{-1})t^{-n})^{1/p} d\lambda(t))^p$.

(ii) *If* $\log\phi$ *is also convex and* $0 < \exp\int_0^1 \log[\eta(t^{-1})t^{-n}] d\lambda(t) < \infty$, *then* (2.14) *holds with* $C = \exp\int_0^1 \log[\eta(t^{-1})t^{-n}] d\lambda(t)$.

Proof of Corollary 2.3. Let $d\mu = d\nu = \eta(|x|)\tau(|x|)dx$ in Theorem 2.2. Then $\mu_t \ll \nu$ and $(d\mu_t/d\nu)(x) \leqslant \eta(t^{-1})t^{-n}$ for each $0 < t \leqslant 1$. Define $\alpha(t) = \eta(t^{-1})t^{-n}$. Then Corollary 2.3 can be obtained by Theorem 2.2. □

Corollary 2.3(i) was also obtained by Xiao [12, Theorem(i)] for the case $E = \mathbb{R}^n$, $\phi(x) = x^p$, $1 \leqslant p < \infty$, $\eta = \tau = 1$, and $d\lambda(t) = \psi(t)dt$, where ψ is a nonnegative function on $(0,1]$. In the case $n = 1$, $E = (0,\infty)$, $1 < p < \infty$, $d\lambda(t) = \chi_{(0,1)}(t)dt$, and $\eta = \tau = 1$, the constant given in Corollary 2.3(i) is $C = (p/(p-1))^p$ and we have [8, Theorem 1].

Let w be a nonnegative function on $(0,\infty)$ such that $0 < W = \int_0^1 w(t)dt < \infty$. Consider the case $n = 1$, $E = (0,b)$, $0 < b \leqslant \infty$, and $d\lambda(t) = \chi_{(0,1)}(t)W^{-1}w(t)dt$ in Corollary 2.3. Then the result of (i) with $\phi(x) = x^p$ implies [9, Theorem 1]. By choosing $\phi(x) = e^x$ and replacing f by $\log f$, the result of (ii) implies [9, Theorem 2] and [10, Theorem 2]. In particular, choosing $w(t) = \alpha t^{\alpha-1}$, $\alpha > 0$, $\eta(t) = t^\gamma$, and $\tau(t) = 1$, we have the inequality proved by Cochran and Lee (cf. [2]).

Analogous to Corollary 2.3 we have the following result which can be obtained by a similar proof to that given in Corollary 2.3, .

COROLLARY 2.4. *Suppose* $supp(\lambda) \subseteq [1,\infty)$ *and* τ *is increasing.*

(i) *If* $\phi^{1/p} \in \Phi^+(I)$, $1 \leqslant p < \infty$ *and* $0 < \int_1^\infty (\eta(t^{-1})t^{-n})^{1/p} d\lambda(t) < \infty$, *then* (2.14) *holds with* $C = (\int_1^\infty (\eta(t^{-1})t^{-n})^{1/p} d\lambda(t))^p$.

(ii) *If* $\log\phi$ *is also convex and* $0 < \exp\int_1^\infty \log[\eta(t^{-1})t^{-n}] d\lambda(t) < \infty$, *then* (2.14) *holds with* $C = \exp\int_1^\infty \log[\eta(t^{-1})t^{-n}] d\lambda(t)$.

Let $E = \mathbb{R}^n$, $\phi(x) = x^p$, $1 \leqslant p < \infty$, and $\eta = \tau = 1$ in Corollary 2.4(i). If $d\lambda(t) = \chi_{[1,\infty)}(t)\psi(t^{-1})t^{n-2}dt$, where ψ is a nonnegative function on $(0,1]$, then we have [12, Corollary(i)].

Let w be a nonnegative function on $(0,\infty)$ such that $0 < W = \int_1^\infty w(t)dt < \infty$. Consider the case $n = 1$, $E = (b,\infty)$, $0 \leqslant b < \infty$, and $d\lambda(t) = \chi_{(1,\infty)}(t)W^{-1}w(t)dt$ in Corollary 2.4. Then the result of (i) with $\phi(x) = x^p$ implies [9, Theorem 7] and [9, Theorem 6] can be obtained by choosing $\phi(x) = e^x$ and replacing f by $\log f$ in the result of (ii). In particular, choosing $w(t) = -\alpha t^{\alpha-1}$, $\alpha < 0$, $\eta(t) = t^\gamma$, and $\tau(t) = 1$, we have the inequality proved by Love (cf. [9, Corollary 6]).

3. Best possible constants

In this section, we discuss the best possible constants in our theorems. Consider the case that $(X,\|\cdot\|)$ is a real norm linear space and $\mu=\nu$. Let $\phi(x)=x^p$, where $1\leqslant p<\infty$, in Theorem 2.2. Then (2.10) can be reduced to

$$\int_E (Hf(x))^p d\mu(x)\leqslant\left(\int_0^\infty \alpha(t)^{1/p}d\lambda(t)\right)^p\int_E f(x)^p d\mu(x). \tag{3.1}$$

If we choose $\phi(x)=e^x$ and replace f by $\log f$, inequality (2.11) then can be reduced to

$$\int_E \exp\left(\int_0^\infty \log f(tx)d\lambda(t)\right)d\mu(x)\leqslant G_\alpha\int_E f(x)d\mu(x). \tag{3.2}$$

The following Theorem 3.1 is concerned with the best possible constants in (3.1) and (3.2).

THEOREM 3.1. *Let the hypotheses of Theorem 2.2 hold with $\mu=\nu$ and α be multiplicative. Suppose that there exist a $\lambda-$balanced Borel set D in E and a sequence of positive multiplicative functions $\{\alpha_m\}_{m\in\mathbb{N}}$ defined on $(0,\infty)$ such that $0<\int_D\alpha_m(\|x\|)d\mu(x)<\infty$ for all sufficiently large m.*

(i) *If $\liminf_{m\to\infty}\alpha_m(t)\geqslant\alpha(t)$ for each $t\in supp(\lambda)$, then $(\int_0^\infty\alpha(t)^{1/p}d\lambda(t))^p$ in (3.1) is the best possible constant.*

(ii) *Suppose that G_{α_m} exists and $0<G_{\alpha_m}<\infty$ for all sufficiently large m. If $\liminf_{m\to\infty}G_{\alpha_m}\geqslant G_\alpha$, then the constant G_α in (3.2) is the best possible.*

Proof of Theorem 3.1. For large $m\in\mathbb{N}$, choose $f(x)=\chi_D(x)\alpha_m(\|x\|)^{1/p}$ in (3.1) and (3.2). Inequality (3.1) then gives rise to

$$\begin{aligned}\left(\int_0^\infty \alpha(t)^{1/p}d\lambda(t)\right)^p\int_D\alpha_m(\|x\|)d\mu(x)&\geqslant\int_E\left(\int_0^\infty\chi_D(tx)\alpha_m(t\|x\|)^{1/p}d\lambda(t)\right)^p d\mu(x)\\&\geqslant\left(\int_0^\infty\alpha_m(t)^{1/p}d\lambda(t)\right)^p\int_D\alpha_m(\|x\|)d\mu(x).\end{aligned}$$

Therefore $(\int_0^\infty\alpha(t)^{1/p}d\lambda(t))^p\geqslant(\int_0^\infty\alpha_m(t)^{1/p}d\lambda(t))^p$. By the Fatou's lemma and the condition in (i), we readily deduce that $(\int_0^\infty\alpha(t)^{1/p}d\lambda(t))^p$ in (3.1) must be the best possible. On the other hand, by (3.2) we have

$$\begin{aligned}G_\alpha\int_D\alpha_m(\|x\|)d\mu(x)&\geqslant\int_E\exp\left(\int_0^\infty\log[\chi_D(tx)\alpha_m(t\|x\|)]d\lambda(t)\right)d\mu(x)\\&\geqslant G_{\alpha_m}\int_D\alpha_m(\|x\|)d\mu(x).\end{aligned}$$

Therefore $G_\alpha\geqslant G_{\alpha_m}$. By (ii) we see that G_α in (3.2) must be the best possible. □

Consider the case $(X,\|\cdot\|)=(\mathbb{R}^n,|\cdot|)$ and $d\mu(x)=\eta(|x|)\tau(|x|)dx$ in $(3.1)-(3.2)$. The following Corollary 3.2 and Corollary 3.3 are concerned with the best possible constants in Corollary 2.3 and Corollary 2.4, respectively.

COROLLARY 3.2. *If to the hypotheses of Corollary 2.3 is added that η is multiplicative and $\int_0^1 \xi^{\varepsilon-1}\tau(\xi)d\xi$ is finite for all sufficiently small $\varepsilon > 0$, then for $\phi(x) = x^p$, $1 \leqslant p < \infty$, the constant C given in Corollary 2.3(i) is the best possible. Moreover, if $\int_0^1 \log[\eta(t^{-1})t^{\varepsilon-n}]d\lambda(t)$ is also finite for all sufficiently small $\varepsilon > 0$, then for $\phi(x) = e^x$ and replacing f by $\log f$, the constant C given in Corollary 2.3(ii) is the best possible.*

Proof of Corollary 3.2. Since η is multiplicative, the function α given in the proof of Corollary 2.3 is also multiplicative. Let $D = \{x \in E : |x| < 1\}$ and $\alpha_m(t) = \eta(t^{-1})t^{1/m-n}$, $m \in \mathbb{N}$. Then conditions given in Theorem 3.1 are satisfied and hence the constants given in Corollary 2.3 are the best possible. $\square$

COROLLARY 3.3. *If to the hypotheses of Corollary 2.4 is added that η is multiplicative and $\int_1^\infty \xi^{-\varepsilon-1}\tau(\xi)d\xi$ is finite for all sufficiently small $\varepsilon > 0$, then for $\phi(x) = x^p$, $1 \leqslant p < \infty$, the constant C given in Corollary 2.4(i) is the best possible. Moreover, if $\int_1^\infty \log[\eta(t^{-1})t^{-\varepsilon-n}]d\lambda(t)$ is also finite for all sufficiently small $\varepsilon > 0$, then for $\phi(x) = e^x$ and replacing f by $\log f$, the constant C given in Corollary 2.4(ii) is the best possible.*

Corollary 3.3 can be proved by choosing $D = \{x \in E : |x| > 1\}$ and $\alpha_m(t) = \eta(t^{-1})t^{-1/m-n}$, $m \in \mathbb{N}$ in Theorem 3.1.

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