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ON SOME INEQUALITIES INVOLVING TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS WITH EMPHASIS ON THE CUSA–HUYGENS, WILKER, AND HUYGENS INEQUALITIES

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Abstract. Recently trigonometric inequalities of N. Cusa and C. Huygens (see, e.g., [9]), J. Wilker [11], and C. Huygens [4] have been discussed extensively in mathematical literature. We shall demonstrate that Wilker's inequality, Huygens' inequality, and some other related inequalities all follow from the Cusa-Huygens inequality. A generalization of the latter result is also obtained. The hyperbolic counterparts of those inequalities are also derived.

1. Introduction

In recent years the following trigonometric inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3} \tag{1.1}$$

($0 < |x| < \pi/2$) have attracted attention of several researchers. The first one was established by D.D. Adamović and D.S. Mitrinović (see, e.g., [5, p. 238]) while the second inequality in (1.1) is due to N. Cusa and C. Huygens (see [9] for more details regarding this result). Recently A. Baricz and J. Sándor [1] have pointed out that the first inequality in (1.1) implies two other inequalities

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1.2}$$

and

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \tag{1.3}$$

which hold true provided $0 < |x| < \pi/2$. The inequality (1.2) was discovered by J. Wilker [11]. There are several published proofs of this result (see [3], [8], [10], [14], [18]). For its generalization see [15]. Inequality (1.3) is due to Huygens [4]. In what follows we will call the second inequality in (1.1) the Cusa-Huygens inequality, (1.2)

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will be called the first Wilker inequality while (1.3) will be called the Huygens inequality.

Recently S. Wu and H. Srivastava [14] have established another inequality

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \tag{1.4}$$

($0 < |x| < \pi/2$) which in the sequel will be called the second Wilker inequality.

In [14] the authors have asked the question: "Does there exist an inequality which unifies (and possibly also extends) Wilker's inequality (1.2) and Huygens' inequality (1.3) ?" A complete answer to this question is provided in Section 2. Therein we will give an answer to the similar question about a common source of inequalities involving hyperbolic counterparts of the inequalities mentioned above. Recently L. Zhu [19] has established a hyperbolic version of the first Wilker inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 \tag{1.5}$$

($x \neq 0$). We shall give generalizations and extensions of the inequalities (1.1)–(1.4) to the case of hyperbolic functions (see Theorem 2.2, Corollary 2.3, Theorem 2.4, and Theorem 2.5).

2. Main results

We begin giving a generalization of inequalities (1.1). Also, we offer a similar result for the hyperbolic functions. To facilitate presentation we recall some facts about the Schwab-Borchardt mean $SB(x,y)$ ($x \geqslant 0,\ y > 0$). This is the iterative mean, i.e.,

$$SB(x,y) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n\,,$$

where

$$x_0 = x,\ y_0 = y,\ x_{n+1} = \frac{x_n + y_n}{2},\ y_{n+1} = \sqrt{x_{n+1} y_n} \tag{2.1}$$

($n = 0, 1, \ldots$) (see, e.g., [7]), It has been shown in [7, Theorem 3.3] that for all $n \geqslant 0$ and $x \neq y$

$$(x_n y_n^2)^{1/3} < SB(x,y) < \frac{x_n + 2y_n}{3}\,. \tag{2.2}$$

Moreover, the sequence $\left\{(x_n y_n^2)^{1/3}\right\}_0^\infty$ is strictly increasing while the sequence $\left\{(x_n + 2y_n)/3\right\}_0^\infty$ is strictly decreasing. Also, it is known that [2, (2.3)]

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2 - x^2}}{\arccos(x/y)}, & 0 \leqslant x < y \\ \dfrac{\sqrt{x^2 - y^2}}{\operatorname{arccosh}(x/y)}, & y < x \\ x, & x = y. \end{cases} \tag{2.3}$$

We are in a position to prove the following

THEOREM 2.1. *Let* $0 < |x| < \pi/2$. *Then for* $n = 0, 1, \ldots$ *the following inequalities*

$$\left(\cos\frac{x}{2^n}\right)^{1/3}\prod_{k=1}^{n}\cos\frac{x}{2^k} < \frac{\sin x}{x} < \frac{\cos\frac{x}{2^n}+2}{3}\prod_{k=1}^{n}\cos\frac{x}{2^k} \tag{2.4}$$

hold true.

Proof. It follows from (2.3) that $SB(\cos x, 1) = \sin x/x$. This in conjunction with (2.2) and (2.1), when $n = 0$, gives the inequalities (1.1). To obtain the inequality (2.4) for $n = 1$, we use (1.1) with x replaced by $x/2$. Next, multiplying each member of the resulting inequality by $\cos(x/2)$ we obtain

$$\left(\cos\frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} < \frac{\cos^2\frac{x}{2}+2\cos\frac{x}{2}}{3}\,. \tag{2.5}$$

Easy induction completes the proof of (2.4). □

COROLLARY 2.1. (Euler) *For* $|x| < \pi/2$

$$\sin x = x\prod_{k=1}^{\infty}\cos\frac{x}{2^k}\,.$$

This follows immediately from (2.4).

More bounds for the function $\sin x/x$ are obtained in [12], [13], [16] and [17].

The hyperbolic version of (2.4) is contained in the following.

THEOREM 2.2. *Let* $x \neq 0$. *Then*

$$\left(\cosh\frac{x}{2^n}\right)^{1/3}\prod_{k=1}^{n}\cosh\frac{x}{2^k} < \frac{\sinh x}{x} < \frac{\cosh\frac{x}{2^n}+2}{3}\prod_{k=1}^{n}\cosh\frac{x}{2^k} \tag{2.6}$$

the following inequalities are valid for all $n \geqslant 0$.

Proof. We follow the lines introduced in the proof of Theorem 2.1. It follows from (2.3) that $SB(\cosh x, 1) = \sinh x/x$. Then using (2.1) and (2.2) we obtain for $n = 0$

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3} \tag{2.7}$$

and

$$\left(\cosh\frac{x}{2}\right)^{4/3} < \frac{\sinh x}{x} < \frac{\cosh^2\frac{x}{2}+2\cosh\frac{x}{2}}{3} \tag{2.8}$$

for $n = 1$. To complete the proof of (2.6) we use mathematical induction. □

COROLLARY 2.2. *For* $x \in \mathbb{R}$

$$\sinh x = x\prod_{k=1}^{\infty}\cosh\frac{x}{2^k}\,.$$

The first inequality in (2.7) was obtained by I. Lazarević (see, e.g., [5, p. 270]). We will refer to the second inequality in (2.7) as the hyperbolic Cusa-Huygens inequality.

COROLLARY 2.3. *For* $x \neq 0$

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3. \tag{2.9}$$

Proof. The first inequality in (2.7) can be written as

$$1 < \left(\frac{\sinh x}{x}\right)^{2/3}\left(\frac{\tanh x}{x}\right)^{1/3}.$$

Application of the inequality of weighted arithmetic and geometric means, with weights $2/3$ and $1/3$, gives the desired result. □

In what follows inequality (2.9) will be called the hyperbolic Huygens inequality.

We shall now prove that the trigonometric and hyperbolic Cusa-Huygens inequalities (see the second inequalities in (1.1) and (2.7)) imply the second trigonometric and the second hyperbolic Wilker inequalities (1.4) and (2.13).

Let us note that (1.4) can be written as

$$\frac{\sin x}{x} < \frac{1}{2}\left(\frac{x}{\sin x} + \cos x\right). \tag{2.10}$$

In order to obtain the desired implication for the trigonometric functions we shall prove the following.

THEOREM 2.3. *Let* $0 < |x| < \pi/2$. *Then*

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3} < \frac{1}{2}\left(\frac{x}{\sin x} + \cos x\right). \tag{2.11}$$

Proof. We have to establish the second inequality in (2.11) because the first one is the Cusa-Huygens inequality. It is easy to see that the second inequality in (2.11) can be written as

$$3\frac{x}{\sin x} + \cos x > 4. \tag{2.12}$$

Let $f(x)$ denote the left-hand side of (2.12). Then $f'(x) = l(x)/\sin^2 x$, where $l(x) = 3\sin x - 3x\cos x - \sin^3 x$. Differentiation of $l(x)$ gives

$$l'(x) = 3\sin x(x - \sin x\cos x) > 0,$$

where the last inequality follows from $x > \sin x > \sin x\cos x$ provided $0 < x < \pi/2$. Since $l(0) = 0$, $l(x) > 0$. This in turn implies that $f(x)$ is an increasing function on $(0, \pi/2)$. Taking into account that $f(0) = 4$ we obtain the desired result (2.12) when $0 < x < \pi/2$. Since $f(x)$ is an even function, (2.12) is satisfied for all values of x which satisfy $0 < |x| < \pi/2$. □

THEOREM 2.4. *For $x \neq 0$ the second inequality in* (2.7) *implies the inequality*

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2. \tag{2.13}$$

Proof. We will follow the lines introduced in the proof of the last theorem. First, let us rewrite (2.13) as

$$\frac{\sinh x}{x} < \frac{1}{2}\left(\frac{x}{\sinh x} + \cosh x\right).$$

We shall establish the inequality in question by showing that the second inequality in

$$\frac{\sinh x}{x} < \frac{\cosh x + 2}{3} < \frac{1}{2}\left(\frac{x}{\sinh x} + \cosh x\right)$$

holds true for each $x \neq 0$ or what is the same that

$$3\frac{x}{\sinh x} + \cosh x > 4. \tag{2.14}$$

Denote the left-hand side of (2.14) by $f(x)$. Since this function is an even function it suffices to prove (2.14) for $x > 0$. We have

$$f'(x) = \frac{l(x)}{\sinh^2 x} \quad \text{where} \quad l(x) = 3\sinh x - 3x\cosh x + \sinh^3 x.$$

Hence

$$\begin{aligned} l'(x) &= 3x\sinh x\left(\frac{\sinh x}{x}\cosh x - 1\right) \\ &> 3x\sinh x(\cosh x - 1) > 0 \end{aligned}$$

because $\sinh x/x > 1$ and $\cosh x > 1$. This, in conjunction with $l(0) = 0$, gives $l(x) > 0$. Thus $f(x)$ is an increasing function for $x > 0$. Taking into account that $f(0) = 4$ we obtain $f(x) > 0$ for $x > 0$. The proof is complete. □

Our next result reads as follows.

THEOREM 2.5. *If $t > 0$, then the following inequalities*

$$\left(\frac{\sin x}{x}\right)^{2t} + \left(\frac{\tan x}{x}\right)^{t} > \left(\frac{x}{\sin x}\right)^{2t} + \left(\frac{x}{\tan x}\right)^{t} \tag{2.15}$$

($0 < |x| < \pi/2$) and

$$\left(\frac{\sinh x}{x}\right)^{2t} + \left(\frac{\tanh x}{x}\right)^{t} > \left(\frac{x}{\sinh x}\right)^{2t} + \left(\frac{x}{\tanh x}\right)^{t} \tag{2.16}$$

($x \neq 0$) hold true. Inequalities (2.15) *and* (2.16) *are reversed if $t < 0$.*

Proof. We shall establish that the inequalities in question are valid using the following elementary result. Let a and b $(a \neq b)$ be positive numbers. If $ab > 1$, then

$$a+b > \frac{1}{a} + \frac{1}{b} \tag{2.17}$$

with the inequality reversed if $ab < 1$. This is an immediate consequence of the identity $AH = G^2$, where A, H and G are the arithmetic, harmonic and geometric means, with equal weights, of a and b. For the proof of (2.15) we let

$$a = \left(\frac{\sin x}{x}\right)^{2t} \text{ and } b = \left(\frac{\tan x}{x}\right)^{t}.$$

Using the first inequality in (1.1) we obtain for $t > 0$

$$ab = \left[\left(\frac{\sin x}{x}\right)^3 \frac{1}{\cos x}\right]^t > 1.$$

The last inequality is reversed if $t < 0$. Application of (2.17) gives the desired result. Inequality (2.16) can be established in a similar way. We omit further details. □

Letting in (2.15) and (2.16) $t = 1$ we obtain the following.

COROLLARY 2.4. *The second Wilker inequality* (1.4) *implies the first Wilker inequality* (1.2)*. Similarly, inequality* (2.13) *implies Zhu's inequality* (1.5)*.*

Inequality (2.15) is the limiting case of an inequality (3.15) in [6] which has been established for the Jacobian elliptic functions.

In [1] the authors have shown that the first Wilker inequality implies the second Wilker inequality. This also follows from (2.15) when $t = -1$. Similarly, inequality (1.5) implies inequality (2.13).

We shall demonstrate now that the Cusa-Huygens inequality, which can be written as,

$$2\frac{x}{\sin x} + \frac{x}{\tan x} > 3$$

implies the Huygens inequality (1.3). We have

THEOREM 2.6. *Let* $0 < |x| < \pi/2$. *Then*

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3. \tag{2.18}$$

Proof. With

$$a = \frac{\sin x}{x} \quad \text{and} \quad c = \cos x$$

the first inequality in (2.18) can be written as

$$a^2 > \frac{c(2+c)}{2c+1}. \tag{2.19}$$

On the other hand, the first inequality in (2.5) can be written in terms of a and c as

$$a^2 > \left(\frac{1+c}{2}\right)^{4/3} \equiv t^4,$$

where

$$t = \left(\frac{1+c}{2}\right)^{1/3}. \tag{2.20}$$

In order to prove (2.19) it suffices to show that

$$t^4 > \frac{c(2+c)}{2c+1}. \tag{2.21}$$

It follows from (2.20) that $c = 2t^3 - 1$. Substituting this into (2.21) we obtain

$$t^4 > \frac{4t^6-1}{4t^3-1}$$

which is equivalent to

$$(t-1)^2(4t^5+4t^4+4t^3+3t^2+2t+1) > 0.$$

This completes the proof of (2.19) which is the same as the first inequality in (2.18). The proof is complete. □

The hyperbolic version of the last result can be obtained in the same way. We omit further details.

Our next result reads as follows.

THEOREM 2.7. *Huygens' inequality* (1.3) *implies the first Wilker inequality* (1.2). *Similarly, the hyperbolic Huygens inequality* (2.9) *implies Zhu's inequality* (1.5).

Proof. A simple algebra shows that (1.3) can be written as follows

$$\frac{\sin x}{x} > \frac{3\cos x}{1+2\cos x}.$$

Thus

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \left(\frac{3\cos x}{1+2\cos x}\right)^2 + \frac{3\cos x}{1+2\cos x}\cdot\frac{1}{\cos x}$$
$$= \frac{9\cos^2 x + 6\cos x + 3}{(1+2\cos x)^2} = 2 + \left(\frac{1-\cos x}{1+2\cos x}\right)^2 > 2.$$

The second part of the thesis can be established in a similar way. □

In the proof of the last result of this paper we shall utilize the following.

LEMMA 2.1. *Let the numbers a and c be such that $0<a<1$, $1<a^3c$, and*

$$\frac{1+c}{2}<a^{3/2}c. \tag{2.22}$$

Then

$$2<a^2+ac<1+c<a+(ac)^2 \tag{2.23}$$

Proof. The first inequality in (2.23) follows from the inequality of the arithmetic and geometric means applied to numbers a^2 and ac. Using the assumption that $1<a^3c$ we obtain

$$1<\sqrt{a^3c}<\frac{a^2+ac}{2}.$$

Since $0<a<1$, $a^2+ac<1+c$. The last inequality (2.23) follows from (2.22) and the inequality of arithmetic and geometric means. We have

$$\frac{1+c}{2}<\sqrt{a(ac)^2}<\frac{1}{2}\left[a+(ac)^2\right].$$

This completes the proof of (2.23). □

We close this section with the following.

THEOREM 2.8. *Let $0<|x|<\pi/2$. Then*

$$2<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<1+\sec x<\frac{\sin x}{x}+\left(\frac{\tan x}{2}\right)^2. \tag{2.24}$$

Proof. We shall employ Lemma 2.1 with $a=\sin x/x$ and $c=\sec x$. Clearly $0<a<1$ and $1<a^3c$ which is the first inequality in (1.1). To verify that a and c satisfy the assumption (2.22) we use the first inequality in (2.5), which can also be written as

$$\left(\frac{1+\cos x}{2}\right)^2<\left(\frac{\sin x}{x}\right)^3.$$

Multiplying both sides by $1/\cos^2 x$ we obtain

$$\left(\frac{1+\cos x}{2\cos x}\right)^2<\frac{\sin x}{x}\left(\frac{\tan x}{x}\right)^2.$$

The last inequality can be written in terms of a and c as

$$\left(\frac{1+c}{2}\right)^2<a(ac)^2.$$

This completes the proof. □

It is easy to see that with $a=\sinh x/x$ and $c=\operatorname{sech} x$ the last inequality in (2.23) takes the form

$$1+\operatorname{sech} x<\frac{\sinh x}{x}+\left(\frac{\tanh x}{x}\right)^2 \tag{2.25}$$

$(x\neq 0)$.

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