

## APPROXIMATIONS TO EULER'S CONSTANT

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*Abstract.* We study a problem of finding good approximations to Euler's constant  $\gamma = \lim_{n \rightarrow \infty} S_n$ , where  $S_n = \sum_{k=1}^n \frac{1}{k} - \log(n+1)$ , by linear forms in logarithms and harmonic numbers. In 1995, C. Elsner showed that slow convergence of the sequence  $S_n$  can be significantly improved if  $S_n$  is replaced by linear combinations of  $S_n$  with integer coefficients. In this paper, considering more general linear transformations of the sequence  $S_n$  we establish new accelerating convergence formulae for  $\gamma$ . Our estimates sharpen and generalize recent Elsner's, Rivoal's and author's results.

### 1. Introduction

Let  $\alpha \geq 0$  be a real number and

$$\gamma_\alpha = \sum_{k=1}^{\infty} \left( \frac{1}{k+\alpha} - \log \left( \frac{k+\alpha+1}{k+\alpha} \right) \right).$$

We denote the partial sum of the above series by

$$\begin{aligned} S_n(\alpha) &= \sum_{k=1}^n \left( \frac{1}{k+\alpha} - \log \left( \frac{k+\alpha+1}{k+\alpha} \right) \right) \\ &= \sum_{k=1}^n \frac{1}{k+\alpha} - \log(\alpha+n+1) + \log(\alpha+1) \end{aligned} \tag{1}$$

and  $S_n := S_n(0)$ . It follows easily (see [12, formula (2)]) that

$$\lim_{n \rightarrow \infty} S_n(\alpha) = -\frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)} + \log(\alpha+1) = -\psi(\alpha+1) + \log(\alpha+1),$$

where  $\psi(\alpha)$  is the logarithmic derivative of the gamma function (or the digamma function) and therefore,

$$\gamma_\alpha = \log(\alpha+1) - \psi(\alpha+1).$$

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In particular,  $\gamma_0 = -\psi(1) = \gamma = 0.577215\dots$  is Euler’s constant. It is well-known that the sequence  $S_n$  converges slowly to the Euler constant  $\gamma$  (see, for details, [7])

$$\gamma = S_n + O(n^{-1}).$$

In 1995, Elsner [1] found out that  $\gamma$  can be approximated by linear combinations of partial sums (1) with integer coefficients

$$\left| \gamma - \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \binom{k+n+\tau-1}{k+\tau-1} S_{k+\tau-1} \right| \leq \frac{1}{2n\tau \binom{n+\tau}{n}}, \quad \tau, n \in \mathbb{N} \quad (2)$$

and this inequality exhibits geometric convergence if  $\tau = O(n)$ . Formula (2) for  $\tau > n$  was generalized by Rivoal in [10], where, in particular, it was shown that

$$\left| \gamma - \frac{1}{2^n} \sum_{k=0}^n (-1)^{k+n} \binom{n}{k} \binom{2k+2n}{n} S_{2k+n} \right| = O\left(\frac{1}{n2^{n/2}}\right), \quad \text{as } n \rightarrow \infty.$$

Another such kind formula

$$\left| \gamma - \sum_{k=0}^n (-1)^{k+n} \binom{n}{k} \binom{n+k}{k} S_{k+n} \right| = \frac{1}{4^{n+o(n)}}, \quad \text{as } n \rightarrow \infty$$

was proved in [4]. Recently, C. Elsner [2] presented a two-parametric series transformation of the sequence  $S_n$

$$\sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \binom{n+\tau_1+k}{n} S_{k+\tau_2-1} \quad (3)$$

converging more rapidly to  $\gamma$ , when  $\tau_2 > \tau_1 + 1$  and  $n$  increases, than in the case  $\tau_2 = \tau_1 + 1$  considered in (2).

In this paper, we consider a more general series transformation of the type

$$\frac{n_1! \dots n_m!}{N! r^N} \sum_{k=0}^N (-1)^{N+k} \binom{N}{k} \binom{rk+n_1+\tau_1}{n_1} \dots \binom{rk+n_m+\tau_m}{n_m} S_{rk+\tau_0} \quad (4)$$

with  $m, r, n_1, \dots, n_m \in \mathbb{N}$ ,  $\tau_0, \tau_1, \dots, \tau_m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , and  $N = \sum_{j=1}^m n_j$ , and we give new accelerating convergence formulae for Euler’s constant  $\gamma$ . In particular, we show (see Theorem 2 and Corollary 1 below) that if  $\tau_1, \tau_2$  are linear functions of  $n$ , then the sum (3) converges to  $\gamma$  at the least geometric rate and represents the best approximation in the set of all the sums (3) with a fixed value of  $\lim_{n \rightarrow \infty} \tau_2/n$ , provided that  $\lim_{n \rightarrow \infty} 2(\tau_2 - \tau_1)/n = 1$ .

### 2. Statement of the main results

As usual, we denote the Gauss hypergeometric function (see, for details, [9]) by

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = \sum_{v=0}^{\infty} \frac{(a)_v (b)_v}{v! (c)_v} z^v,$$

where  $(\lambda)_\nu$  is the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \nu = 0; \\ \lambda(\lambda + 1)\dots(\lambda + \nu - 1), & \text{if } \nu \in \mathbb{N}. \end{cases}$$

We then prove the following theorems:

**THEOREM 1.** *Let  $m, r, n_1, \dots, n_m \in \mathbb{N}$ ,  $\tau_0, \tau_1, \dots, \tau_m \in \mathbb{N}_0$ ,  $0 \leq \tau_0 - \tau_m \leq n_m$ ,  $n_m + \tau_m \geq n_j + \tau_j$ ,  $j = 1, \dots, m - 1$ , and  $N = \sum_{j=1}^m n_j$ . Then*

$$\begin{aligned} & \frac{N!(-r)^N}{n_1! \dots n_m!} \gamma - \sum_{k=0}^N (-1)^k \binom{N}{k} \binom{rk + n_1 + \tau_1}{n_1} \dots \binom{rk + n_m + \tau_m}{n_m} S_{rk + \tau_0} \\ &= \pm \prod_{j=1}^m \binom{n_m + \tau_m - \tau_j}{n_j} \int_0^1 \int_0^1 \frac{x^{n_m + \tau_m} (1 - x^r)^N t^{n_m + \tau_m - \tau_0} (1 - t)^{\tau_0 - \tau_m} \omega(t)}{(1 - t + xt)^{n_m + 1}} \omega(t) \\ & \quad \times Q_m \left( \frac{xt}{1 - t + xt} \right) dx dt, \end{aligned} \tag{5}$$

where

$$\omega(t) = \frac{1}{t(\log^2(1/t - 1) + \pi^2)} \tag{6}$$

and  $Q_m(y)$  is a polynomial of degree  $N - n_m$  given by the formula

$$Q_m(y) = \sum_{k_1=0}^{n_1} \dots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} \frac{(-n_j)_{k_j} (1 + n_m + \tau_m - \tau_{j+1})_{k_1 + \dots + k_j}}{k_j! (1 + n_m + \tau_m - n_j - \tau_j)_{k_1 + \dots + k_j}} y^{k_j} \tag{7}$$

if  $m \geq 2$ , and  $Q_1(y) \equiv 1$ .

**THEOREM 2.** *Let  $b, c, r \in \mathbb{N}$ ,  $a \in \mathbb{N}_0$ ,  $0 \leq b - a \leq c$ . Then for  $n \in \mathbb{N}$  we have*

$$\left| \gamma - \frac{1}{r^{cn}} \sum_{k=0}^{cn} (-1)^{k+cn} \binom{cn}{k} \binom{rk + (a+c)n}{cn} S_{rk+bn} \right| < \left( \frac{b^{\frac{b}{r}} (c+a-b)^{c+a-b} (b-a)^{b-a}}{(b+cr)^{c+\frac{b}{r}}} \right)^n \tag{8}$$

(Here and throughout the paper  $0^0$  is treated as 1.)

If  $b, c, r$  are fixed, then the minimum of the right-hand side of (8) is attained at  $a = b - c/2$  and in this case we have

**COROLLARY 1.** *Let  $b, c, r, n \in \mathbb{N}$  and  $b \geq c$ . Then*

$$\left| \gamma - \frac{1}{r^{2cn}} \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk + (b+c)n}{2cn} S_{rk+bn} \right| < \left( \frac{b^{\frac{b}{r}} c^{2c}}{(b+2cr)^{2c+\frac{b}{r}}} \right)^n.$$

**THEOREM 3.** *Let  $b, c, r \in \mathbb{N}$ ,  $a \in \mathbb{N}_0$  and  $0 \leq b - a \leq c$ . Then for any positive integer  $n \geq 2/c$  one has*

$$\left| \gamma - \frac{((cn)!)^2}{(2cn)! r^{2cn}} \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk + (a+c)n}{cn}^2 S_{rk+bn} \right| < cn \left( \frac{b^{\frac{b}{r}} c^c (c+a-b)^{c+a-b} (b-a)^{b-a}}{(b+2cr)^{2c+\frac{b}{r}}} \right)^n.$$

By the similar argument as above putting  $a = b - c/2$  we get a sharper bound than in Corollary 1.

**COROLLARY 2.** *Let  $b, c, r, n \in \mathbb{N}$ ,  $2b \geq c$ , and  $c$  is even. Then*

$$\left| \gamma - \frac{((cn)!)^2}{(2cn)! r^{2cn}} \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk + (b + \frac{c}{2})n}{cn}^2 S_{rk+bn} \right| < cn \left( \frac{b^{\frac{b}{r}} c^{2c}}{2^c (b+2cr)^{2c+\frac{b}{r}}} \right)^n.$$

For example, setting  $b = c = 4, r = 1$  we get the following estimate:

**COROLLARY 3.** *For any positive integer  $n$  one has*

$$\left| \gamma - \frac{(4n)!^2}{(8n)!} \sum_{k=0}^{8n} (-1)^k \binom{8n}{k} \binom{k+6n}{4n}^2 S_{k+4n} \right| < \frac{4n}{(2^4 \cdot 3^{12})^n} < 4n(0.00000012)^n.$$

**THEOREM 4.** *Let  $m, r, n_1, \dots, n_m \in \mathbb{N}$ ,  $\tau_0, \tau_1, \dots, \tau_m \in \mathbb{N}_0$ ,  $0 \leq \tau_0 - \tau_m \leq n_m$ ,  $n_m + \tau_m > \tau_{j+1} > n_j + \tau_j$ ,  $j = 1, \dots, m-1$ , and  $N = \sum_{j=1}^m n_j$ . Then*

$$\left| \frac{N! (-r)^N}{n_1! \dots n_m!} \gamma - \sum_{k=0}^N (-1)^k \binom{N}{k} \binom{rk + n_1 + \tau_1}{n_1} \dots \binom{rk + n_m + \tau_m}{n_m} S_{rk+\tau_0} \right| \leq \prod_{j=1}^m \binom{n_m + \tau_m - \tau_j}{n_j} \int_0^1 \int_0^1 \frac{x^{n_m + \tau_m} (1-x^r)^N t^{n_m + \tau_m - \tau_0} (1-t)^{\tau_0 - \tau_m} \omega(t)}{(1-t+xt)^{n_m+1}} dx dt.$$

Setting  $\tau_{j+1} = n_j + \tau_j + 1$ ,  $j = 1, \dots, m-1$ , in Theorem 4 we get

**COROLLARY 4.** *Let  $m, r, n_1, \dots, n_m \in \mathbb{N}$ ,  $\tau_0, \tau_1 \in \mathbb{N}_0$ ,  $N = \sum_{j=1}^m n_j$ , and  $N - n_m + \tau_1 + (m-1) \leq \tau_0 \leq N + \tau_1 + (m-1)$ . Then*

$$\left| \gamma - \frac{n_1! \dots n_m!}{N! (-r)^N} \sum_{k=0}^N (-1)^k \binom{N}{k} \prod_{j=1}^m \binom{rk + n_1 + \dots + n_j + \tau_1 + j - 1}{n_j} S_{rk+\tau_0} \right| \leq \prod_{j=1}^{m-1} \frac{N+j}{n_{j+1} + \dots + n_m + m - j} \times \int_0^1 \int_0^1 \frac{x^{N+\tau_1+m-1} (1-x^r)^N t^{N+\tau_1+m-1-\tau_0} (1-t)^{\tau_0+n_m-N-\tau_1-m+1} \omega(t)}{r^N (1-t+xt)^{n_m+1}} dx dt$$

**THEOREM 5.** Let  $m, c_1, \dots, c_m, r, b, n \in \mathbb{N}$ ,  $a \in \mathbb{N}_0$ ,  $C = \sum_{j=1}^m c_j$ , and  $a - c_m \leq b - C \leq a$ . Then

$$\left| \gamma - \frac{(c_1 n)! \dots (c_m n)!}{(Cn)! (-r)^{Cn}} \sum_{k=0}^{Cn} (-1)^k \binom{Cn}{k} \prod_{j=1}^m \binom{rk + (a + c_1 + \dots + c_j)n + j}{c_j n} S_{rk+bn+m} \right| < M(\bar{c}) \left( \frac{b^{\frac{b}{r}} C^C (C+a-b)^{C+a-b} (c_m + b - a - C)^{c_m+b-a-C}}{c_m^{c_m} (b+Cr)^{C+\frac{b}{r}}} \right)^n,$$

where  $M(\bar{c}) < C^{m-1}$  is some constant depending only on  $c_1, \dots, c_m$ .

Consider several illustrative examples of Theorem 5. Taking  $c_1 = \dots = c_m = 2c$ ,  $C = 2mc$ ,  $b = 2mc$ ,  $a = c$ ,  $c \in \mathbb{N}$ , we get

**COROLLARY 5.** Let  $c, m, r \in \mathbb{N}$ . Then for any positive integer  $n$  one has

$$\left| \gamma - \frac{((2cn)!)^m}{(2mcn)! r^{2mcn}} \sum_{k=0}^{2mcn} (-1)^k \binom{2mcn}{k} \prod_{j=1}^m \binom{rk + (2j+1)cn + j}{2cn} S_{rk+2mcn+m} \right| < \frac{m^{m-1}}{(m-1)!} \left( \frac{1}{4^c (r+1)^{2mc+\frac{2mc}{r}}} \right)^n$$

Setting  $c_1 = \dots = c_m = 2c$ ,  $C = 2mc$ ,  $b = (2m-1)c$ ,  $a = 0$ ,  $c \in \mathbb{N}$ , we get

**COROLLARY 6.** Let  $c, m, r \in \mathbb{N}$ . Then for any positive integer  $n$  one has

$$\left| \gamma - \frac{((2cn)!)^m}{(2mcn)! r^{2mcn}} \sum_{k=0}^{2mcn} (-1)^k \binom{2mcn}{k} \prod_{j=1}^m \binom{rk + 2jcn + j}{2cn} S_{rk+(2m-1)cn+m} \right| < \frac{m^{m-1}}{(m-1)!} \left( \frac{4^{-c} \left(1 - \frac{1}{2m}\right)^{\frac{(2m-1)c}{r}}}{\left(r + 1 - \frac{1}{2m}\right)^{2mc + \frac{(2m-1)c}{r}}} \right)^n.$$

**3. Analytical construction**

We define the generalized Legendre polynomial by  $A(x) = \sum_{k=0}^N A_k x^{rk}$  with

$$A_k = (-1)^{k+N} \binom{N}{k} \binom{rk + n_1 + \tau_1}{n_1} \dots \binom{rk + n_m + \tau_m}{n_m}.$$

**LEMMA 1.** There holds

$$A(1) = \sum_{k=0}^N A_k = \frac{N! r^N}{n_1! \dots n_m!}.$$

*Proof.* For the proof, let

$$R(t) = \frac{N!}{n_1! \dots n_m!} \frac{(rt - n_1 - \tau_1)_{n_1} (rt - n_2 - \tau_2)_{n_2} \dots (rt - n_m - \tau_m)_{n_m}}{t(t+1) \dots (t+N)}.$$

Such rational functions were considered early by the authors [5], [6] to derive explicit Padé approximations of the first and second kinds for polylogarithmic functions. As it is easily seen the rational function  $R(t)$  has the following partial-fraction expansion:

$$R(t) = \sum_{k=0}^N \frac{A_k}{t+k},$$

from which it follows that

$$\sum_{k=0}^N A_k = \sum_{k=0}^N \operatorname{res}_{t=-k} R(t) = -\operatorname{res}_{t=\infty} R(t) = \frac{N! r^N}{n_1! \dots n_m!}. \quad \square$$

Put

$$I(\alpha) := \int_0^1 x^{\tau_0+\alpha} A(x) \left( \frac{1}{1-x} + \frac{1}{\log x} \right) dx$$

LEMMA 2. *There holds the equality*

$$I(\alpha) = \frac{N! r^N}{n_1! \dots n_m!} \gamma_\alpha - \sum_{k=0}^N A_k S_{rk+\tau_0}(\alpha).$$

*Proof.* Substituting

$$\frac{1}{1-x} + \frac{1}{\log x} = \int_0^1 \frac{1-x^t}{1-x} dt,$$

we get

$$I(\alpha) = \int_0^1 \int_0^1 x^{\tau_0+\alpha} A(x) \frac{1-x^t}{1-x} dt dx = \sum_{k=0}^N A_k \int_0^1 \int_0^1 \frac{x^{rk+\tau_0+\alpha} (1-x^t)}{1-x} dx dt.$$

Expanding  $(1-x)^{-1}$  in a geometric series and applying Lemma 1 we find

$$\begin{aligned} I(\alpha) &= \sum_{k=0}^N A_k \sum_{l=0}^{\infty} \int_0^1 \int_0^1 x^{rk+\tau_0+l+\alpha} (1-x^t) dx dt \\ &= \sum_{k=0}^N A_k \sum_{l=0}^{\infty} \int_0^1 \left( \frac{1}{rk+\tau_0+l+\alpha+1} - \frac{1}{rk+\tau_0+t+l+\alpha+1} \right) dt \\ &= \sum_{k=0}^N A_k \sum_{l=1}^{\infty} \left( \frac{1}{rk+\tau_0+l+\alpha} - \log \left( \frac{rk+\tau_0+l+\alpha+1}{rk+\tau_0+l+\alpha} \right) \right) \\ &= \sum_{k=0}^N A_k (\gamma_\alpha - S_{rk+\tau_0}(\alpha)) = \frac{N! r^N}{n_1! \dots n_m!} \gamma_\alpha - \sum_{k=0}^N A_k S_{rk+\tau_0}(\alpha). \quad \square \end{aligned}$$

Next, we consider two differential operators

$$S_{\tau,n}(f(x)) = \frac{(-1)^n}{n!} x^{-\tau} (x^{n+\tau} f(x))^{(n)},$$

$$T_{\tau,n}(f(x)) = \frac{1}{n!} x^{n+\tau} (x^{-\tau} f(x))^{(n)},$$

where  $\tau$  is a real number and  $n$  is a non-negative integer. We show that  $S_{\tau,n}$  and  $T_{\tau,n}$  are adjoint operators in some sense.

LEMMA 3. *Suppose that  $f(x)$  is a polynomial vanishing at  $x = 1$  with order at least  $n$  and  $g(x) \in C^\infty(0, 1) \cap L^1(0, 1)$  satisfies the following boundary conditions:*

$$\lim_{x \rightarrow 0^+} x^l g^{(l-1)}(x) = \lim_{x \rightarrow 1^-} (1-x)^l g^{(l-1)}(x) = 0$$

for all  $1 \leq l \leq n$ . Then we have

$$\int_0^1 S_{\tau,n}(f(x)) \cdot g(x) dx = \int_0^1 f(x) \cdot T_{\tau,n}(g(x)) dx.$$

*Proof.* The proof is analogous to the proof of Lemma 3.1 [3].  $\square$

LEMMA 4. *There holds*

$$I(\alpha) = \int_0^1 \int_0^1 (1-x^r)^N \omega(t) T_{\tau_{m-1}, n_{m-1}} \circ \dots \circ T_{\tau_1, n_1} \circ T_{\tau_m, n_m} \left( \frac{x^{\tau_0 + \alpha}}{1 - (1-x)t} \right) dx dt$$

with the weight function  $\omega(t)$  defined in (6).

*Proof.* Applying the following representation introduced by Prévost [8]:

$$\frac{1}{1-x} + \frac{1}{\log x} = \int_0^1 \frac{\omega(t)}{1 - (1-x)t} dt,$$

we have

$$I(\alpha) = \int_0^1 \int_0^1 \frac{x^{\tau_0 + \alpha} \omega(t)}{1 - (1-x)t} A(x) dt dx.$$

It follows easily that the polynomial  $A(x)$  can be written in the form

$$A(x) = S_{\tau_1, n_1} \circ S_{\tau_2, n_2} \circ \dots \circ S_{\tau_m, n_m} ((1-x^r)^N).$$

Since  $A(x)$  is symmetric in pairs  $(\tau_j, n_j)$  and does not depend on order of differential operators  $S_{\tau_j, n_j}$ , it is convenient for the sequel to write it as

$$A(x) = S_{\tau_m, n_m} \circ S_{\tau_1, n_1} \circ \dots \circ S_{\tau_{m-1}, n_{m-1}} ((1-x^r)^N).$$

Now by Fubini's theorem and Lemma 3, we get the desired equality.  $\square$

We also need the following simple lemma, which will be used for estimation purposes.

LEMMA 5. Let  $a, b, c, d, r, s \in \mathbb{R}$ ,  $b, c, d, r, s > 0$ , and  $b + d \geq a + c \geq \max\{b, d\}$ . Then the function

$$f(x, t) = \frac{x^{a+c}(1-x^r)^{sc}t^{c+a-b}(1-t)^{b+d-c-a}}{(1-t+xt)^d}$$

attains its maximum on  $[0, 1] \times [0, 1]$  at the unique point  $(x_0, t_0)$ , where

$$x_0 = \left(\frac{b}{b+scr}\right)^{\frac{1}{r}}, \quad t_0 = \frac{c+a-b}{c+a-b+x_0(b+d-a-c)} \tag{9}$$

and

$$\max_{0 \leq x, t \leq 1} f(x, t) = f(x_0, t_0) = \frac{b^{\frac{b}{r}}(scr)^{sc}(c+a-b)^{c+a-b}(b+d-a-c)^{b+d-a-c}}{d^d(b+scr)^{sc+\frac{b}{r}}}.$$

*Proof.* First note that the function  $f(x, t)$  is bounded on  $[0, 1] \times [0, 1]$ , since  $f(0, t) = 0$  ( $0 \leq t < 1$ ),

$$f(x, 1) = \begin{cases} 0, & \text{if } b + d > a + c; \\ x^b(1-x^r)^{sc}, & \text{if } b + d = a + c; \end{cases}$$

and

$$\begin{aligned} f(\varepsilon_x, 1 - \varepsilon_t) &= \frac{\varepsilon_x^{a+c}(1 - \varepsilon_x^r)^{sc}(1 - \varepsilon_t)^{a+c-b}\varepsilon_t^{b+d-c-a}}{(\varepsilon_t + \varepsilon_x - \varepsilon_x\varepsilon_t)^d} \\ &\leq \varepsilon_x^{a+c-d}(1 - \varepsilon_x^r)^{sc} \left(\frac{\varepsilon_t}{1 - \varepsilon_t}\right)^{b+d-a-c} \\ &\leq \varepsilon_x^{a+c-d} \left(\frac{\varepsilon_t}{1 - \varepsilon_t}\right)^{b+d-a-c} \rightarrow 0, \quad \text{as } \varepsilon_x, \varepsilon_t \rightarrow 0^+. \end{aligned}$$

If  $a + c = b$ , then  $f(x, t) = x^b(1-x^r)^{sc} \left(\frac{1-t}{1-t+xt}\right)^d$  and the function  $f$  takes its maximum on  $[0, 1] \times [0, 1]$  at the boundary point  $(x_0, t_0) = ((b/(b+scr))^{1/r}, 0)$ . If  $a + c = b + d$ , then  $f(x, t) = x^b(1-x^r)^{sc} \left(\frac{xt}{xt+1-t}\right)^d$  and the function  $f$  attains its maximum on  $[0, 1] \times [0, 1]$  at the boundary point  $(x_0, t_0) = ((b/(b+scr))^{1/r}, 1)$ . If  $b < a + c < b + d$ , then  $f(x, t) = 0$  on the boundary of  $[0, 1] \times [0, 1]$ , and  $f(x, t)$  attains its maximum at the unique interior point  $(x_0, t_0)$  defined in (9).  $\square$

### 4. Proof of Theorem 1

LEMMA 6. Let  $x, t \in (0, 1)$ ,  $\tau_0, n_m, \tau_m \in \mathbb{N}_0$ , and  $\tau_m \leq \tau_0 \leq n_m + \tau_m$ . Then

$$T_{\tau_m, n_m} \left(\frac{x^{\tau_0}}{1 - (1-x)t}\right) = (-1)^{n_m} \frac{x^{n_m+\tau_m}t^{n_m+\tau_m-\tau_0}(t-1)^{\tau_0-\tau_m}}{(1 - (1-x)t)^{n_m+1}}.$$



*Proof.* Clearly,

$$T_{\tau_m, n_m} \left( \frac{x^{\tau_0}}{1-t+xt} \right) = \frac{x^{n_m+\tau_m}}{n_m!} \left( \frac{x^{\tau_0-\tau_m}}{1-t+xt} \right)^{(n_m)}.$$

Decomposing the fraction  $\frac{x^{\tau_0-\tau_m}}{1-t+xt}$  into the sum

$$\frac{x^{\tau_0-\tau_m}}{1-t+xt} = p(x) + \left( \frac{t-1}{t} \right)^{\tau_0-\tau_m} \frac{1}{1-t+xt},$$

where  $p(x)$  is a polynomial of degree not exceeding  $\tau_0 - \tau_m - 1$ , and differentiating it  $n_m$  times, we get the required statement.  $\square$

LEMMA 7. *Under the hypothesis of Theorem 1 one has*

$$\begin{aligned} & T_{\tau_{m-1}, n_{m-1}} \circ \dots \circ T_{\tau_1, n_1} \circ T_{\tau_m, n_m} \left( \frac{x^{\tau_0}}{1-(1-x)t} \right) \\ &= (-1)^{n_m} \prod_{j=1}^m \binom{n_m + \tau_m - \tau_j}{n_j} \frac{x^{n_m+\tau_m} t^{n_m+\tau_m-\tau_0} (t-1)^{\tau_0-\tau_m}}{(1-t+xt)^{n_m+1}} Q_m \left( \frac{xt}{1-t+xt} \right), \end{aligned} \tag{10}$$

where the polynomial  $Q_m(y)$  is defined in (7).

*Proof.* If  $m = 1$ , then (10) follows easily by Lemma 6. Suppose  $m \geq 2$ . Then consecutive calculation of the  $n_j$ th derivatives with respect to  $x$  by Leibniz' rule for  $j = 1, 2, \dots, m-1$

$$\begin{aligned} & \frac{x^{\tau_j+n_j}}{n_j!} \left( \frac{t^k x^{n_m+\tau_m+k-\tau_j}}{(1-t+xt)^{n_m+1+k}} \right)^{(n_j)} = \binom{n_m + \tau_m - \tau_j}{n_j} \frac{x^{n_m+\tau_m}}{(1-t+xt)^{n_m+1}} \\ & \times \sum_{k_j=0}^{n_j} \frac{(-n_j)_{k_j} (n_m+1)_{k+k_j} (1+n_m+\tau_m-\tau_j)_{k}}{k_j! (n_m+1)_k (1+n_m+\tau_m-\tau_j-n_j)_{k+k_j}} \left( \frac{xt}{1-t+xt} \right)^{k+k_j} \end{aligned}$$

readily leads to the formula (10).  $\square$

Now Theorem 1 follows easily from Lemmas 2, 4, 7.

### 5. Proof of Theorem 2

If we put  $m = 1, n_1 = cn, \tau_1 = an, \tau_0 = bn, n \in \mathbb{N}$ , in Theorem 1, we get

$$\begin{aligned} & \left| \gamma - \frac{1}{r^{cn}} \sum_{k=0}^{cn} (-1)^{k+cn} \binom{cn}{k} \binom{rk+(a+c)n}{cn} S_{rk+bn} \right| \\ & \leq \frac{1}{r^{cn}} \int_0^1 \int_0^1 \frac{(1-x^r)^{cn} x^{(a+c)n} t^{(c+a-b)n} (1-t)^{(b-a)n} \omega(t)}{(1-t+xt)^{cn+1}} dx dt \\ & \leq \frac{1}{r^{cn}} \left( \max_{0 \leq x, t \leq 1} f(x, t) \right)^n \int_0^1 \int_0^1 \frac{\omega(t)}{1-t+xt} dt dx = \frac{\gamma}{r^{cn}} \left( \max_{0 \leq x, t \leq 1} f(x, t) \right)^n \end{aligned}$$

with

$$f(x,t) = \frac{x^{a+c}(1-x^r)^c t^{c+a-b}(1-t)^{b-a}}{(1-t+xt)^c}.$$

Here we used the fact (see [8, formula 2.6]) that

$$\gamma = \int_0^1 \left( \frac{1}{\log x} + \frac{1}{1-x} \right) dx.$$

Now, since  $\gamma < 1$ , by Lemma 5 with  $s = 1, d = c$ , the theorem follows.  $\square$

### 6. Proofs of Theorems 3, 4

To estimate the rate of convergence of quantities (4) to  $\gamma$  as  $N \rightarrow \infty$ , we need an upper bound for the polynomial  $Q_m(y)$ . In some situations it is possible to get suitable estimations.

First, we consider the case  $m = 2, n_1 = n_2, \tau_1 = \tau_2$ . Then by Theorem 1, we get

$$\begin{aligned} I &:= \left| \frac{(2n_1)! r^{2n_1}}{(n_1!)^2} \gamma - \sum_{k=0}^{2n_1} (-1)^k \binom{2n_1}{k} \binom{rk+n_1+\tau_1}{n_1}^2 S_{rk+\tau_0} \right| \\ &\leq \int_0^1 \int_0^1 \frac{x^{n_1+\tau_1} (1-x^r)^{2n_1} t^{n_1+\tau_1-\tau_0} (1-t)^{\tau_0-\tau_1} \omega(t)}{(1-t+xt)^{n_1+1}} |Q_2(y)| dx dt \end{aligned}$$

with  $y = xt/(1-t+xt)$ . The polynomial

$$Q_2(y) = {}_2F_1 \left( \begin{matrix} -n_1, n_1+1 \\ 1 \end{matrix} \middle| y \right) = \frac{1}{n_1!} \left( \frac{d}{dy} \right)^{n_1} \left( y^{n_1} (1-y)^{n_1} \right)$$

is a shifted Legendre polynomial  $P_{n_1}(u)$  formally identified as follows:

$$Q_2(y) = P_{n_1}(1-2y).$$

By the well-known inequality (see [11, p.162])

$$|P_{n_1}(u)| \leq 1, \quad -1 \leq u \leq 1,$$

it follows that

$$I \leq \int_0^1 \int_0^1 \frac{x^{n_1+\tau_1} (1-x^r)^{2n_1} t^{n_1+\tau_1-\tau_0} (1-t)^{\tau_0-\tau_1} \omega(t)}{(1-t+xt)^{n_1+1}} dx dt.$$

Now, setting  $n_1 = cn, \tau_1 = an, \tau_0 = bn$  with  $c, b \in \mathbb{N}, a \in \mathbb{N}_0$ , and  $0 \leq b-a \leq c$ , we get

$$\left| \frac{(2cn)! r^{2cn}}{((cn)!)^2} \gamma - \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk+(a+c)n}{cn}^2 S_{rk+bn} \right| \leq \gamma \left( \max_{0 \leq x,t \leq 1} f(x,t) \right)^n,$$

where

$$f(x,t) = \frac{x^{c+a}(1-x^r)^{2c}t^{a+c-b}(1-t)^{b-a}}{(1-t+xt)^c}.$$

By Lemma 5, the function  $f(x,t)$  takes its maximum on  $[0,1] \times [0,1]$  at the unique point  $(x_0,t_0)$ , at which

$$f(x_0,t_0) = \frac{b^{\frac{b}{r}}(4cr^2)^c(c+a-b)^{c+a-b}(b-a)^{b-a}}{(b+2cr)^{2c+\frac{b}{r}}}.$$

Since for any positive integer  $n \geq 2$

$$\gamma \frac{(n!)^2}{(2n)!} \leq \frac{n}{4^n},$$

Theorem 3 follows.  $\square$

Another interesting case is described by the following lemma.

LEMMA 8. Let  $n_1, \dots, n_m \in \mathbb{N}$ ,  $\tau_0, \tau_1, \dots, \tau_m \in \mathbb{N}_0$ , and  $n_m + \tau_m > \tau_{j+1} > n_j + \tau_j$ ,  $j = 1, \dots, m-1$ . Then

$$\begin{aligned} Q_m(y) &= \prod_{j=1}^{m-1} \frac{(n_m + \tau_m - n_j - \tau_j)!}{(n_m + \tau_m - \tau_{j+1})!(\tau_{j+1} - n_j - \tau_j - 1)!} \\ &\quad \times \int_0^1 \dots \int_0^1 \prod_{j=1}^{m-1} (1 - yu_j \dots u_{m-1})^{n_j} u_j^{n_m + \tau_m - \tau_{j+1}} (1 - u_j)^{\tau_{j+1} - n_j - \tau_j - 1} du_1 \dots du_{m-1}. \end{aligned} \tag{11}$$

Moreover,  $0 \leq Q_m(y) \leq 1$  for  $y \in [0, 1]$ .

*Proof.* Denoting the integral on the right-hand side of (11) by  $J$  and substituting

$$\prod_{j=1}^{m-1} (1 - yu_j u_{j+1} \dots u_{m-1})^{n_j} = \sum_{k_1=0}^{n_1} \dots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} \frac{(-n_j)_{k_j} y^{k_j} u_j^{k_1 + \dots + k_j}}{k_j!},$$

we get

$$\begin{aligned} J &= \sum_{k_1=0}^{n_1} \dots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} \frac{(-n_j)_{k_j} y^{k_j}}{k_j!} \int_0^1 u_j^{k_1 + \dots + k_j + n_m + \tau_m - \tau_{j+1}} \\ &\quad \times (1 - u_j)^{\tau_{j+1} - n_j - \tau_j - 1} du_j = \sum_{k_1=0}^{n_1} \dots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} \frac{(-n_j)_{k_j} y^{k_j}}{k_j!} \\ &\quad \times \frac{\Gamma(k_1 + \dots + k_j + n_m + \tau_m + 1 - \tau_{j+1}) \Gamma(\tau_{j+1} - n_j - \tau_j)}{\Gamma(k_1 + \dots + k_j + n_m + \tau_m + 1 - n_j - \tau_j)} \end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^{m-1} \frac{\Gamma(1+n_m+\tau_m-\tau_{j+1})\Gamma(\tau_{j+1}-n_j-\tau_j)}{\Gamma(1+n_m+\tau_m-n_j-\tau_j)} \\
&\quad \times \sum_{k_1=0}^{n_1} \dots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} \frac{(-n_j)_{k_j} (1+n_m+\tau_m-\tau_{j+1})_{k_1+\dots+k_j}}{k_j! (1+n_m+\tau_m-n_j-\tau_j)_{k_1+\dots+k_j}} y^{k_j} \\
&= \prod_{j=1}^{m-1} \frac{(n_m+\tau_m-\tau_{j+1})!(\tau_{j+1}-n_j-\tau_j-1)!}{(n_m+\tau_m-n_j-\tau_j)!} Q_m(y).
\end{aligned}$$

The inequality  $0 \leq Q_m(y) \leq 1$  for  $y \in [0, 1]$  follows easily from the integral representation (11).  $\square$

Now, Theorem 4 is a consequence of Theorem 1 and Lemma 8.

## 7. Proof of Theorem 5

Setting  $n_j = c_j n$ ,  $j = 1, \dots, m$ ,  $C = \sum_{j=1}^m c_j$ ,  $\tau_1 = an + 1$ ,  $\tau_0 = bn + m$  in Corollary 4 we get that the absolute value of the remainder is less than

$$\frac{M(\bar{c})}{r^{Cn}} \int_0^1 \int_0^1 \frac{x^{(C+a)n+m} (1-x^r)^{Cn} t^{(C+a-b)n} (1-t)^{(b+c_m-C-a)n} \omega(t)}{(1-t+xt)^{c_m n+1}} dx dt$$

with some constant  $M(\bar{c}) < C^{m-1}$ , since

$$\prod_{j=1}^{m-1} \frac{Cn+j}{(c_{j+1}+\dots+c_m)n+m-j} < C^{m-1}.$$

Denoting

$$f(x, t) = \frac{x^{C+a} (1-x^r)^{Cn} t^{C+a-b} (1-t)^{b+c_m-C-a}}{(1-t+xt)^{c_m}}$$

and applying Lemma 5 with  $s = 1, d = c_m$ , we conclude the theorem.  $\square$

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