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CLASSIFIED CONSTRUCTION OF GENERALIZED FURUTA TYPE OPERATOR FUNCTIONS, II

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Abstract. As a continuation of our previous work with the same title, the construction of grand Furuta inequality (GFI) is improved in order to give a constructive proof of Uchiyama's result (2003). Afterwards the generalization of Uchiyama's result is obtained by mathematical induction which implies Furuta's recent results.

1. Introduction

A capital letter (such as T) means a bounded linear operator on a Hilbert space. $T \geqslant 0$ and $T > 0$ mean a positive operator and an invertible positive operator respectively.

As an affirmative answer to a conjecture by Chan-Kwong [4] and an essential extension of the celebrated Löwner-Heinz inequality (L-H): $A \geqslant B \geqslant 0$ ensures $A^p \geqslant B^p$ for each $p \in [0,1]$, Furuta [11] showed the following operator inequality.

THEOREM 1.1. (Furuta inequality (FI), [11]) *Let* $r \geqslant 0$ *and* $p > 0$, *then* $A \geqslant B \geqslant 0$ *ensures*

$$\left(B^{r/2}A^pB^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}} \geqslant \left(B^{r/2}B^pB^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}},$$
$$\left(A^{r/2}A^pA^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}}.$$

See [5, 12, 24] for the alternate proofs of FI. Tanahashi [26] proved the optimality of the outer exponent $\min\{1,p\}+r$.

Inspired by Ando-Hiai log majorization [2], a kind of generalized Furuta type inequalities was obtained.

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THEOREM 1.2. (Grand Furuta inequality (GFI), [14, 29]) *Let $t \in [-1,0]$ and $p \geqslant 1$; then $C \geqslant A \geqslant B \geqslant 0$ with $A > 0$ ensures the function*

$$F(r,s) = C^{-r/2}\left(C^{r/2}(A^{t/2}B^pA^{t/2})^sC^{r/2}\right)^{(1+t+r)/((p+t)s+r)}C^{-r/2}$$

is decreasing for both $r \geqslant -t$ and $s \geqslant 1$. In particular, the inequality

$$C^{1+t+r} \geqslant \left(C^{r/2}(A^{t/2}B^pA^{t/2})^sC^{r/2}\right)^{(1+t+r)/((p+t)s+r)} \tag{1.1}$$

holds for $r \geqslant -t$ and $s \geqslant 1$.

Uchiyama [29] proved GFI by using the theory of operator mean founded by Kubo-Ando [25] and Furuta [16] gave an alternative proof of (1.1). The case $C = A$ is the original form of GFI [14] which interpolates FI (as extremal case $t = 0$ in (1.1)) and Ando-Hiai inequality (AH) [2] (as extremal case $t = -1$ and $r = s$ in (1.1)), and is equivalent to a kind of log majorization [17]. See [7, 15, 36] for the alternate proofs of GFI. It is interesting that the outer exponent $1+t+r$ in the inequality (1.1) is also optimal similar to that of FI [9, 28, 30].

It's well known that Furuta type inequalities have many applications. See [1, 37, 34] for some applications on Aluthge transformation and classes of operators related to hyponormal operators, [35, 3, 10, 18] for some applications on Pedersen-Takesaki operator equation, operator entropy and log majorization, and so on.

In [36], all generalized Furuta type operator functions (inequalities)

$$F_{p,t,q}(r,s) = A^{-r/2}(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2})^{(q+r)/((p+t)s+r)}A^{-r/2}$$

were divided into two classes according to the existence of Furuta inequality (see [27]):

(F) Call a function $F_{p,t,q}(r,s)$ belongs to class (F) if the Furuta type inequality

$$(A^{t/2}B^pA^{t/2})^{1/\alpha} \leqslant A^{\frac{p+t}{\alpha}}$$

is valid for some $\alpha \neq 0$, it is only need to consider the case $t < 0$, $1 \geqslant p > 0$ and the case $t \geqslant 0$, $p \geqslant 0$;

(NF) Call a function $F_{p,t,q}(r,s)$ belongs to class (NF) if the Furuta inequality

$$(A^{t/2}B^pA^{t/2})^{1/\alpha} \leqslant A^{\frac{p+t}{\alpha}}$$

is not valid for any $\alpha \neq 0$, it is only need to consider the case $t < 0$, $p > 1$.

Moreover, the constructions of class (F) and (NF) were also provided.

Construction of class (F)

(I) Denote $A_1 = A$, $B_1 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$, then there exist a real number $\alpha \neq 0$ so that $A_1^{\alpha} \geqslant B_1^{\alpha}$.

(II) By applying the related results of Furuta type operator functions (such as [13, 33]) to $A_1^\alpha \geqslant B_1^\alpha$, we obtain the monotonicity of the variables (at least one) in $F_{p,t,q}(r,s)$.

(III) If the construction is not completed by (II). By applying Furuta's classical steps (p144, [20]), the construction is complete.

Construction of class (NF)

(I) Denote $B_1(s) = (A^{-t}\natural_s B^p)^{1/((p+t)s-t)}$, then try to find a fixed s_0 so that $A \geqslant B_1(s_0)$.

(II) By applying Theorem 1 in Ito-Yamazaki [23] and Proposition 4 in Yanagida [31] to $A \geqslant B_1(s_0)$, we obtain the monotonicity of $B_1(s)$ for $s \geqslant s_0$ (or $s \leqslant s_0$), thus $A \geqslant B_1(s)$ when $s \geqslant s_0$ (or $s \leqslant s_0$).

(III) By applying the related results of Furuta type operator functions to $A \geqslant B_1(s)$, we obtain the monotonicity of r in $F_{p,t,q}(r,s)$. Warning: the monotonicity of s in $F_{p,t,q}(r,s)$ can not be obtained because $B_1(s)$ is regarded as a fixed operator now.

(IV) By applying Furuta's classical steps (p144, [20]), the monotonicity of s is obtained. Construction is over.

According to the constructions of (F) and (NF), the monotonicity of (NF) is nontrivial and it can not be followed by the monotonicity of Furuta type operator functions easily. Therefore we pay our attention on (NF) and (GFI) is the unique one of (NF) till now. See [36, 8, 18, 19, 22] for recent developments on (GFI).

The main purpose of this paper is to give a constructive proof of Uchiyama's result (Theorem 1.2) via the improved construction of (NF) stated below. Then we show a generalization of Uchiyama's result by mathematical induction from which Furuta's recent results [19] follow.

2. Improved construction and preliminaries

Improved construction of class (NF)

(I) Denote $f(s) = (C^{-t/2}(A^{t/2}B^pA^{t/2})^sC^{-t/2})^{1/((p+t)s-t)}$, then try to find a fixed s_0 so that $C \geqslant f(s_0)$ (by Hansen-Pedersen inequality [21]).

(II) By applying Theorem 1 in Ito-Yamazaki [23] and Proposition 4 in Yanagida [31] to $C \geqslant f(s)$, we obtain the monotonicity of $f(s)$ for $s \geqslant s_0$ (or $s \leqslant s_0$), thus $C \geqslant f(s)$ when $s \geqslant s_0$ (or $s \leqslant s_0$).

(III) By applying the related results of Furuta type operator functions (such as Theorem 2.1 below) to $C \geqslant f(s)$, we obtain the monotonicity of r in $F_{p,t,q}(r,s)$. Warning: the monotonicity of s in $F_{p,t,q}(r,s)$ can not be obtained because $f(s)$ is regarded as a fixed operator now.

(IV) By applying Furuta's classical steps (p144, [20]), the monotonicity of s is obtained. Construction is over.

Preliminaries.

THEOREM 2.1. ([13, 6, 33]) *Let* $A \geqslant 0$, $B \geqslant 0$ *such that* $A^t \geqslant B^t$ *for* $t \geqslant 0$ ($A > 0$, $B > 0$ *and* $\log A \geqslant \log B$ *if* $t = 0$). *Then the following hold.*

(1) *For each* $r > 0$ *and* $s > -r$, $f_{r,s}(p) = \left(A^{r/2}B^pA^{r/2}\right)^{(s+r)/(p+r)}$ *is decreasing for* $p \geqslant \max\{s,0\}$.

(2) *For each* $p > 0$ *and* $s < p$, $f_{p,s}(r) = A^{-r}\sharp_{(s+r)/(p+r)}B^p$ *is decreasing for* $r \geqslant \max\{-s,0\}$.

(3) *For each* $s \in R$, $f_s(p,r) = A^{-r}\sharp_{(s+r)/(p+r)}B^p$ *is decreasing for* $p \geqslant \max\{s,0\}$ *and* $r \geqslant \max\{-s,0\}$.

The notation $\sharp$ means geometric mean of operators [25].

LEMMA 2.2. ([13, 14]) *Let* $\alpha \in R$ *and* X *be invertible. Then* $(X^*X)^\alpha = X^*(XX^*)^{\alpha-1}X$, *especially if* $\alpha \geqslant 1$ *the equality holds without invertibility of* X.

THEOREM 2.3. ([23]) *Let* $A, B \geqslant 0$. *Then for each* $p, r \geqslant 0$, *the following assertions hold:*

(1) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geqslant B^r \Rightarrow (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \leqslant A^p$.

(2) $(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \leqslant A^p$ *and* $N(A) \subset N(B) \Rightarrow (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geqslant B^r$.

The following result is a generalization of Theorem 2.1 because of FI.

THEOREM 2.4. ([31, 32]) *Let* $A, B \geqslant 0$; α_0, $\beta_0 > 0$; $-\beta_0 < \delta_0 \leqslant \alpha_0$, $-\beta_0 \leqslant \overline{\delta}_0 < \alpha_0$. *Then the following assertions hold:*

(1) *If* $(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0+\delta_0}{\beta_0+\alpha_0}} \geqslant B^{\beta_0+\delta_0}$, *then* $(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\beta+\delta_0}{\beta+\alpha_0}} \geqslant B^{\beta+\delta_0}$ *for any* $\beta \geqslant \beta_0$. *Moreover, for each fixed* $\delta_0' \leqslant \alpha_0$, *the function*

$$f_{\alpha_0,\delta_0'}(\beta) = (A^{\frac{\alpha_0}{2}}B^\beta A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0-\delta_0'}{\alpha_0+\beta}}$$

is a decreasing function for $\beta \geqslant \max\{\beta_0, -\delta_0'\}$, *thus*

$$A^{\frac{\alpha_0}{2}}B^{\beta_1}A^{\frac{\alpha_0}{2}} \geqslant (A^{\frac{\alpha_0}{2}}B^{\beta_2}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta_2}} \text{ for } \beta_2 \geqslant \beta_1 \geqslant \beta_0.$$

(2) *If* $A^{\alpha_0-\overline{\delta}_0} \geqslant (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0-\overline{\delta}_0}{\alpha_0+\beta_0}}$, *then* $A^{\alpha-\overline{\delta}_0} \geqslant (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{\alpha-\overline{\delta}_0}{\alpha+\beta_0}}$ *for any* $\alpha \geqslant \alpha_0$. *Moreover, for each fixed* $\overline{\delta}_0' \geqslant -\beta_0$, *the function*

$$g_{\beta_0,\overline{\delta}_0'}(\alpha) = (B^{\frac{\beta_0}{2}}A^{\alpha}B^{\frac{\beta_0}{2}})^{\frac{\beta_0+\overline{\delta}_0'}{\beta_0+\alpha}}$$

is an increasing function for $\alpha \geqslant \max\{\alpha_0, \overline{\delta}_0'\}$, *thus*

$$B^{\frac{\beta_0}{2}}A^{\alpha_1}B^{\frac{\beta_0}{2}} \leqslant (B^{\frac{\beta_0}{2}}A^{\alpha_2}B^{\frac{\beta_0}{2}})^{\frac{\beta_0+\alpha_1}{\beta_0+\alpha_2}} \text{ for } \alpha_2 \geqslant \alpha_1 \geqslant \alpha_0.$$

The case $\delta_0 = \overline{\delta}_0 = 0$ of Theorem 2.4 appeared in [31] which is also a special case of Theorem 3.4 in [29], Theorem 2.4 itself was obtained by [32]. It plays an important role in the discussion of some nonnormal operators [35].

3. A constructive proof of Uchiyama's result

Uchiyama [29] established Theorem 1.2 via the theory of operator mean founded by Kubo-Ando [25]. Here we provide a constructive proof of Theorem 1.2 via the improved construction of class (NF) stated in Section 2 and this proof should be the second proof of Theorem 1.2.

Proof. [Proof of Theorem 1.2] We need to show $F(r,s)$ is decreasing for both $r \geqslant -t$ and $s \geqslant 1$. Assume B is invertible without loss of generality.

(I) Put $f(s) = \left(C^{-t/2}(A^{t/2}B^pA^{t/2})^sC^{-t/2}\right)^{1/((p+t)s-t)}$. Since $C^t \leqslant A^t$ for $-1 \leqslant t < 0$, $A^{-t/2}C^tA^{-t/2} \leqslant I$ holds. By $p \geqslant 1$ and Hansen-Pedersen inequality [21],

$$\begin{aligned} f(1) &= (C^{-t/2}A^{t/2}B^pA^{t/2}C^{-t/2})^{1/p} \\ &= \left((C^{t/2}A^{-t/2}B^{-p}A^{-t/2}C^{t/2})^{1/p}\right)^{-1} \\ &\leqslant (C^{t/2}A^{-t/2}B^{-1}A^{-t/2}C^{t/2})^{-1} \\ &= C^{-t/2}A^{t/2}BA^{t/2}C^{-t/2} \\ &\leqslant C^{-t/2}A^{1+t}C^{-t/2} \\ &\leqslant C. \end{aligned}$$

(II) Put $D = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$. By (I),

$$(C^{-t/2}D^{p+t}C^{-t/2})^{(-t)/(p+t-t)} = \left(f(1)\right)^{-t} \leqslant C^{-t}$$

for $-t \in [0,1]$. Thus, by Theorem 2.3 (2),

$$(D^{(p+t)/2}C^{-t}D^{(p+t)/2})^{(p+t)/(p+t-t)} \geqslant D^{p+t}.$$

Therefore, by Theorem 2.4 (1), $f(s) = (C^{-t/2}D^{(p+t)s}C^{-t/2})^{(1+t-t)/((p+t)s-t)}$ is decreasing for $s \geqslant 1$, so that $f(s) \leqslant f(1) \leqslant C$.

(III) Put $r_1 = r+t \geqslant 0$, $q_1 = 1 \geqslant 0$. By Theorem 2.1 (2) or (3), for each $q_1 \geqslant 0$ and $p_1 = (p+t)s - t \geqslant \max\{p, q_1\} > 0$, the function

$$C^{-r_1} \natural_{\frac{q_1+r_1}{p_1+r_1}} (f(s))^{p_1}$$

is decreasing for $r_1 \geqslant \max\{0, -q_1\}$, that is, the function $F(r,s)$ is decreasing for $r \geqslant -t$.

(IV) Let $r \geqslant -t$ and $s \geqslant 1$, by (III) and (II), $F(r,s) \leqslant F(-t,s) \leqslant C^{1+t}$. Thus

$$C^{1+r+t} \geqslant \left(C^{r/2} D^{(p+t)s} C^{r/2}\right)^{(1+r+t)/((p+t)s+r)}.$$

By L-H and Theorem 2.3 (2),

$$\begin{aligned} C^r &\geqslant \left(C^{r/2} D^{(p+t)s} C^{r/2}\right)^{r/((p+t)s+r)}, \\ D^{(p+t)s} &\leqslant \left(D^{(p+t)s/2} C^r D^{(p+t)s/2}\right)^{(p+t)s/((p+t)s+r)}. \end{aligned} \tag{3.1}$$

Then for $0 < w \leqslant s$, by Lemma 2.2,

$$\begin{aligned} F(r,s) &= C^{-r} \natural_{(1+t+r)/((p+t)s+r)} D^{(p+t)s} \\ &= C^{-r/2} \{ \left(C^{r/2} D^{(p+t)s} C^{r/2}\right)^{\frac{(p+t)(s+w)+r}{(p+t)s+r}} \}^{\frac{1+t+r}{(p+t)(s+w)+r}} C^{-r/2} \\ &= C^{\frac{-r}{2}} \{ C^{\frac{r}{2}} D^{\frac{(p+t)s}{2}} \left(D^{\frac{(p+t)s}{2}} C^r D^{\frac{(p+t)s}{2}}\right)^{\frac{(p+t)w}{(p+t)s+r}} D^{\frac{(p+t)s}{2}} C^{\frac{r}{2}} \}^{\frac{1+t+r}{(p+t)(s+w)+r}} C^{\frac{-r}{2}} \\ &\geqslant C^{-r/2} \{ C^{\frac{r}{2}} D^{\frac{(p+t)s}{2}} D^{(p+t)w} D^{\frac{(p+t)s}{2}} C^{\frac{r}{2}} \}^{\frac{1+t+r}{(p+t)(s+w)+r}} \quad \text{by (3.1)} \\ &= C^{-r/2} \{ C^{r/2} D^{(p+t)(s+w)} C^{r/2} \}^{(1+t+r)/((p+t)(s+w)+r)} C^{-r/2} \\ &= F(r, s+w). \end{aligned}$$

Therefore the function $F(r,s)$ is decreasing for $s \geqslant 1$ and the proof is complete.

4. A generalization of the results by Furuta and Uchiyama

Very recently, Furuta [19] showed new developments on GFI below.

THEOREM 4.1. (Extension of (GFI), [19]) *Let $t \in [-1, 0]$, $p_i \geqslant 1$, $r \geqslant -t$ and $s_i \geqslant 1$ for natural numbers i and n with $i \leqslant n$; then $A \geqslant B \geqslant 0$ with $A > 0$ ensures*

$$A^{1+t+r} \geqslant \{ A^{\frac{r}{2}} \left(A^{\frac{t}{2}} \{ \cdots \{ A^{\frac{-t}{2}} \left(A^{\frac{t}{2}} B^{p_1} A^{\frac{t}{2}}\right)^{s_1} A^{\frac{-t}{2}} \}^{p_2} \cdots \}^{p_n} A^{\frac{t}{2}}\right)^{s_n} A^{\frac{r}{2}} \}^{\frac{1+t+r}{q(n)+r+t}} \tag{4.1}$$

where $q(0) = 1$ and $q(i) = (q(i-1)p_i + t)s_i - t$.

Here a generalization of Theorems 1.2 and 4.1 is obtained.

THEOREM 4.2. *Let i and n be positive integral numbers with $i \leqslant n$, $t_i \in [-1,0]$, $r_i \geqslant -t_i$, $s_i \geqslant 1$, and $p_i \geqslant \frac{1}{q(i-1,r_{i-1})}$ where $q(0,r_0)=1$ and $q(i,r_i)=(q(i-1,r_{i-1})p_i+t_i)s_i+r_i$. Then $C_n \geqslant A_n \geqslant C_{n-1} \geqslant A_{n-1} \geqslant \cdots \geqslant C_i \geqslant A_i \geqslant \cdots \geqslant C_1 \geqslant A_1 \geqslant B \geqslant 0$ with $A_1 > 0$ ensures the function*

$$F(r_n,s_n) = C_n^{\frac{-r_n}{2}} \{C_n^{\frac{r_n}{2}} (A_n^{\frac{t_n}{2}} \{\cdots \{C_1^{\frac{r_1}{2}} (A_1^{\frac{t_1}{2}} B^{p_1} A_1^{\frac{t_1}{2}})^{s_1} C^{\frac{r_1}{2}}\}^{p_2} \cdots\}^{p_n} A_n^{\frac{t_n}{2}})^{s_n} C_n^{\frac{r_n}{2}}\}^{\frac{1+t_n+r_n}{q(n,r_n)}} C_n^{\frac{-r_n}{2}}$$

is decreasing for both $r_n \geqslant -t_n$ and $s_n \geqslant 1$. In particular, the inequality

$$C_n^{1+t_n+r_n} \geqslant \{C_n^{\frac{r_n}{2}} (A_n^{\frac{t_n}{2}} \{\cdots \{C_1^{\frac{r_1}{2}} (A_1^{\frac{t_1}{2}} B^{p_1} A_1^{\frac{t_1}{2}})^{s_1} C^{\frac{r_1}{2}}\}^{p_2} \cdots\}^{p_n} A_n^{\frac{t_n}{2}})^{s_n} C_n^{\frac{r_n}{2}}\}^{\frac{1+t_n+r_n}{q(n,r_n)}} \tag{4.2}$$

holds for $r_n \geqslant -t_n$ and $s_n \geqslant 1$.

Proof. This result is proved by induction. Denote $B_1 = B$ and

$$B_i = \{C_{i-1}^{\frac{r_{i-1}}{2}} (A_{i-1}^{\frac{t_{i-1}}{2}} B_{i-1}^{q(i-2,r_{i-2})p_{i-1}} A_{i-1}^{\frac{t_{i-1}}{2}})^{s_{i-1}} C_{i-1}^{\frac{r_{i-1}}{2}}\}^{\frac{1}{q(i-1,r_{i-1})}}$$

when $i \geqslant 2$. Then

$$F(r_n,s_n) = C_n^{\frac{-r_n}{2}} \{C_n^{\frac{r_n}{2}} (A_n^{\frac{t_n}{2}} B_n^{q(n-1,r_{n-1})p_n} A_n^{\frac{t_n}{2}})^{s_n} C_n^{\frac{r_n}{2}}\}^{\frac{1+t_n+r_n}{q(n,r_n)}} C_n^{\frac{-r_n}{2}}.$$

If $n=1$, it is sufficient to prove that the function

$$F(r_1,s_1) = C_1^{\frac{-r_1}{2}} \{C_1^{\frac{r_1}{2}} (A_1^{\frac{t_1}{2}} B^{p_1} A_1^{\frac{t_1}{2}})^{s_1} C_1^{\frac{r_1}{2}}\}^{\frac{1+t_1+r_1}{q(1,r_1)}} C_1^{\frac{-r_1}{2}}$$

is decreasing for both $r_1 \geqslant -t_1$ and $s_1 \geqslant 1$, and this is just Theorem 1.2.

Suppose Theorem 4.2 holds for n, that is, the function $F(r_n,s_n)$ is decreasing for both $r_n \geqslant -t_n$ and $s_n \geqslant 1$. In order to complete the proof, it is sufficient to prove Theorem 4.2 holds for $n+1$.

In fact, by the case n of Theorem 4.2, (4.2) holds, that is,

$$C_n^{1+t_n+r_n} \geqslant \{C_n^{\frac{r_n}{2}} (A_n^{\frac{t_n}{2}} B_n^{q(n-1,r_{n-1})p_n} A_n^{\frac{t_n}{2}})^{s_n} C_n^{\frac{r_n}{2}}\}^{\frac{1+t_n+r_n}{q(n,r_n)}}. \tag{4.3}$$

Hence $C_{n+1} \geqslant A_{n+1} \geqslant C_n \geqslant B_{n+1}$, thus by the case $n=1$ of Theorem 2.1 and $q(n,r_n)p_{n+1} \geqslant 1$, the function

$$F(r_{n+1},s_{n+1}) = C_{n+1}^{\frac{-r_{n+1}}{2}} \{C_{n+1}^{\frac{r_{n+1}}{2}} (A_{n+1}^{\frac{t_{n+1}}{2}} B_{n+1}^{q(n,r_n)p_{n+1}} A_{n+1}^{\frac{t_{n+1}}{2}})^{s_{n+1}} C_{n+1}^{\frac{r_{n+1}}{2}}\}^{\frac{1+t_{n+1}+r_{n+1}}{q(n+1,r_{n+1})}} C_n^{\frac{-r_{n+1}}{2}}$$

is decreasing for both $r_{n+1} \geqslant -t_{n+1}$ and $s_{n+1} \geqslant 1$.

COROLLARY 4.3. *Let i and n be positive integral numbers with $i \leqslant n$, $t \in [-1,0]$, $r_i \geqslant -t$, $s_i \geqslant 1$, and $p_i \geqslant \frac{1}{q(i-1,r_{i-1})}$ where $q(0,r_0)=1$ and $q(i,r_i)=(q(i-1,r_{i-1})p_i+t)s_i+r_i$. Then $C \geqslant A \geqslant B \geqslant 0$ with $A > 0$ ensures the function*

$$F(r_n,s_n) = C^{\frac{-r_n}{2}} \{C^{\frac{r_n}{2}} (A^{\frac{t}{2}} \{\cdots \{C^{\frac{r_1}{2}} (A^{\frac{t}{2}} B^{p_1} A^{\frac{t}{2}})^{s_1} C^{\frac{r_1}{2}}\}^{p_2} \cdots\}^{p_n} A^{\frac{t}{2}})^{s_n} C^{\frac{r_n}{2}}\}^{\frac{1+t+r_n}{q(n,r_n)}} C^{\frac{-r_n}{2}}$$

is decreasing for both $r_n \geqslant -t$ *and* $s_n \geqslant 1$. *In particular, the inequality*

$$C^{1+t+r_n} \geqslant \left\{C^{\frac{r_n}{2}}\left(A^{\frac{t}{2}}\left\{\cdots\left\{C^{\frac{r_1}{2}}\left(A^{\frac{t}{2}}B^{p_1}A^{\frac{t}{2}}\right)^{s_1}C^{\frac{r_1}{2}}\right\}^{p_2}\cdots\right\}^{p_n}A^{\frac{t}{2}}\right)^{s_n}C^{\frac{r_n}{2}}\right\}^{\frac{1+t+r_n}{q(n,r_n)}}$$

holds for $r_n \geqslant -t$ *and* $s_n \geqslant 1$.

Proof. By putting $t_i = t$, $C_i = C$ and $A_i = A$ in Theorem 4.2, Corollary 4.3 follows.

COROLLARY 4.4. *Let* i *and* n *be positive integral numbers with* $i \leqslant n$, $t \in [-1,0]$, $r \geqslant -t$, $s_i \geqslant 1$ *and* $p_i \geqslant \frac{1}{q(i-1)}$ *where* $q(0) = 1$ *and* $q(i) = (q(i-1)p_i + t)s_i - t$. *Then* $A \geqslant B \geqslant 0$ *with* $A > 0$ *ensures the function*

$$F(r,s_n) = A^{\frac{-r}{2}}\left\{A^{\frac{r}{2}}\left(A^{\frac{t}{2}}\left\{\cdots\left\{A^{\frac{-t}{2}}\left(A^{\frac{t}{2}}B^{p_1}A^{\frac{t}{2}}\right)^{s_1}A^{\frac{-t}{2}}\right\}^{p_2}\cdots\right\}^{p_n}A^{\frac{t}{2}}\right)^{s_n}A^{\frac{r}{2}}\right\}^{\frac{1+t+r}{q(n)+t+r}}A^{\frac{-r}{2}}$$

is decreasing for both $r \geqslant -t$ *and* $s_n \geqslant 1$. *In particular, the inequality*

$$A^{1+t+r} \geqslant \left\{A^{\frac{r}{2}}\left(A^{\frac{t}{2}}\left\{\cdots\left\{A^{\frac{-t}{2}}\left(A^{\frac{t}{2}}B^{p_1}A^{\frac{t}{2}}\right)^{s_1}A^{\frac{-t}{2}}\right\}^{p_2}\cdots\right\}^{p_n}A^{\frac{t}{2}}\right)^{s_n}A^{\frac{r}{2}}\right\}^{\frac{1+t+r}{q(n)+t+r}}$$

holds for $r \geqslant -t$ *and* $s_n \geqslant 1$.

Proof. By taking $C = A$, $r_1 = \cdots = r_{n-1} = -t$, $r_n = r$ and $q(0,r_0) = q(0)$ in Corollary 4.3, then $q(0,r_0) = 1$, $q(i,r_i) = q(i,-t) = q(i)$ when $1 \leqslant i \leqslant n-1$ and $q(n,r_n) = q(n,r) = q(n)+t+r$, so that Corollary 4.4 follows.

Corollary 4.4 implies that the condition $p_i \geqslant 1$ can be replaced with $p_i \geqslant \frac{1}{q(i-1)} (\leqslant 1)$ and released for $\frac{1}{q(i-1)} \leqslant 1$.

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REFERENCES

[1] A. ALUTHGE, *On* p*-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13** (1990), 307–315.

[2] T. ANDO AND F. HIAI, *Log majorization and complementary Golded-Thompson type inequality*, Linear Algebra Appl., **197** (1994), 113–131.

[3] N. BEBIANO, R. LEMOS AND J. PROVIDÊNCIA, *Inequalities for quantum relative entropy*, Linear Algebra Appl., **401** (2005), 159–172.

[4] N. N. CHAN AND M. K. KWONG, *Hermitian matrix inequalities and a conjecture*, Amer. Math. Monthly, **92** (1985), 533–541.

[5] M. FUJII, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.

[6] M. FUJII, J. F. JIANG AND E. KAMEI, *A geometrical structure in the Furuta's inequality, II*, Nihonkai Math. J., **8** (1997), 37–46.

[7] M. FUJII AND E. KAMEI, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 2751–2756.

[8] M. FUJII, E. KAMEI AND R. NAKAMOTO, *Grand Furuta inequality and its variant*, J. Math. Inequal., **1**, 3 (2007), 437–441.

[9] M. FUJII, A. MATSUMOTO AND R. NAKAMOTO, *A short proof of the best possibility for the grand Furuta inequality*, J. Inequal. Appl., **4** (1999), 339–344.

[10] M. FUJII, R. NAKAMOTO AND M. TOMINAGA, *Genaralized Bebiano-Lemos-Providência inequalities and their reverses*, Linear Algebra Appl., **426** (2007), 33–39.

[11] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{\frac{p+2r}{q}}$ for $r \geqslant 0, p \geqslant 0, q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.

[12] T. FURUTA, *Elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci., **65** (1989), 126.

[13] T. FURUTA, *Two operator functions with monotone property*, Proc. Amer. Math. Soc., **111** (1991), 511–516.

[14] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai* log*-majorization*, Linear Algebra Appl., **219** (1995), 139–155.

[15] T. FURUTA, *Simplified proof of an order preserving operator inequality*, Proc. Japan Acad. Ser. A Math. Sci., **74** (1998), 114.

[16] T. FURUTA, *A proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci., **78** (2002), 26.

[17] T. FURUTA, *Operator inequality implying generalized Bebiano-Lemos-Providência one*, Linear Algebra Appl., **426** (2007), 342–348.

[18] T. FURUTA, *Monotonicity of order preserving operator functions*, Linear Algebra Appl., **428** (2008), 1072–1082.

[19] T. FURUTA, *Further extension of an order preserving operator inequality*, J. Math. Inequal., **2**, 4 (2008), 465–472.

[20] T. FURUTA, *Invitation to Linear Operators -From Matrices to Bounded Linear Operators on a Hilbert Space*, Taylor & Francis, London, 2001.

[21] F. HANSEN AND G. K. PEDERSEN, *Jensen's inequality for operators and Löwner's theorem*, Math. Ann., **258** (1982), 229–241.

[22] M. ITO AND E. KAMEI, *A complement to monotonicity of generalized Furuta-type operator functions*, Linear Algebra Appl., **430** (2009), 544–546.

[23] M. ITO AND T. YAMAZAKI, *Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geqslant B^r$ and $(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \leqslant A^p$ and its applications*, Integral Equations Operator Theory, **44** (2002), 442–450.

[24] E. KAMEI, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883–886.

[25] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

[26] K. TANAHASHI, *Best possibility of Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146.

[27] K. TANAHASHI, *The Furuta inequality with negative powers*, Proc. Amer. Math. Soc., **127** (1999), 1683–1692.

[28] K. TANAHASHI, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc., **128** (2000), 511–519.

[29] M. UCHIYAMA, *Criteria for monotonicity of operator mean*, J. Math. Soc. Japan, **55**, 1 (2003), 197–207.

[30] T. YAMAZAKI, *Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality*, Math. Inequal. Appl., **2** (1999), 473–477.

[31] M. YANAGIDA, *Powers of class wA(s, t) operators associated with generalized Aluthge transformation*, J. Inequal. Appl., **7**, 2 (2002), 143–168.

[32] C. YANG AND J. YUAN, *On class $wF(p,r,q)$ operators*, Acta Math. Sci. Ser. A Chin. Ed., **27**, 5 (2007), 769–780.

[33] J. YUAN AND Z. GAO, *The Furuta inequality and Furuta type operator functions under chaotic order*, Acta Sci. Math. (Szeged), **73** (2007), 669–681.

[34] J. YUAN AND Z. GAO, *Structure on powers of p-hyponormal and* log*-hyponormal operators*, Integral Equations Operator Theory, **59** (2007), 437–448.

[35] J. YUAN AND Z. GAO, *The operator equation $K^p = H^{\frac{\delta}{2}}T^{\frac{1}{2}}(T^{\frac{1}{2}}H^{\delta+r}T^{\frac{1}{2}})^{\frac{p-\delta}{\delta+r}}T^{\frac{1}{2}}H^{\frac{\delta}{2}}$ and its applications*, J. Math. Anal. Appl., **341** (2008), 870–875.

[36] J. YUAN AND Z. GAO, *Classified construction of generalized Furuta type operator functions*, Math. Inequal. Appl., **11**, 2 (2008), 189–202.

[37] J. YUAN AND Z. GAO, *Complete form of Furuta inequality*, Proc. Amer. Math. Soc., **136**, 8 (2008), 2859–2867.

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