Mathematical
Inequalities
& Applications
Volume 13, Number 4 (2010), 803–815

A NEW METHOD FOR ESTABLISHING AND PROVING ACCURATE BOUNDS FOR THE WALLIS RATIO

CRISTINEL MORTICI

(*Communicated by N. Elezović*)

Abstract. The aim of this paper is to establish new inequalities about the Wallis ratio that improve the Gautschi-Kershaw results.

1. Introduction

In the recent past, the problem of finding new, sharp inequalities about the Euler gamma function and in particular about the Wallis ratio

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}=\frac{1}{\sqrt{\pi}}\cdot\frac{(2n)!!}{(2n-1)!!}$$

has attracted the attention of many authors. See, *e.g.,* [1-3, 6-27]. From the following result of Gautschi [9]

$$n^{1-s}\leqslant\frac{\Gamma(n+1)}{\Gamma(n+s)}\leqslant(n+1)^{1-s},\quad 0\leqslant s\leqslant 1,$$

it follows, for $s=1/2$,

$$\sqrt{n}\leqslant\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\leqslant\sqrt{n+1},$$

and Wallis [26] proved before that $\sqrt{n+1}$ can be replaced by $\sqrt{n+\frac{1}{2}}$.

D. K. Kazarinoff [11] has proved

$$\sqrt{n+\frac{1}{4}}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n+\frac{1}{2}},$$

for every positive integer n, but these estimates are not optimal because Watson [27] proposed the following inequality

$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\leqslant\sqrt{x+\frac{1}{\pi}},\quad x\geqslant 0,$$

Mathematics subject classification (2010): 33B15, 41A10, 42A16.
Keywords and phrases: Wallis ratio, gamma function, approximations, inequalities.

wherein the constants $1/4$ and $1/\pi$ are the best possible.

Moreover, using the following inequality [15, p. 322]

$$\sqrt{\frac{2\pi n(2n+1)}{4n+1}} < \frac{(2n)!!}{(2n-1)!!},$$

we get

$$\sqrt{n+\frac{1}{4}-\frac{1}{16n+4}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \tag{1}$$

and, according to a result of Gurland [10], we have

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{4}+\frac{1}{16n+2}} \tag{2}$$

Related to (1)–(2), we mention the double inequality

$$\sqrt{n+\frac{1}{4}-\frac{1}{(4n-2)^2}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{4}+\frac{1}{16n-4}},$$

which is a consequence of a result of Chu [5].

A better result is the following inequality, for every $x \geqslant 0$,

$$\sqrt{x+\frac{1}{4}+\frac{1}{32x+8+\frac{36}{4x-1}}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x+\frac{1}{4}+\frac{1}{32x+8}} \tag{3}$$

obtained by Slavić [25], who improved the earlier work of Boyd [4]:

$$\sqrt{n+\frac{1}{4}+\frac{1}{32n+32}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{4}+\frac{1}{32n-\frac{64n-148}{8n+11}}}. \tag{4}$$

We can see now that the constants $\frac{1}{4}$ and $\frac{1}{32}$ (the coefficient of n^{-1}) are sharp in the problem of approximation of the Wallis ratio.

One of the purpose of this paper is to prove the sharp approximations family

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}-\frac{5}{2048n^3}+\frac{23}{8192n^4}} \tag{5}$$

where the involved coefficients provide the best results. Furthermore, we show the following double inequality for every $x \geqslant 0$,

$$\sqrt{x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}-\frac{5}{2048x^3}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}}, \tag{6}$$

which improves much all the above results.

Looking carefully at (3)–(4), a new idea arises naturally, that is to introduce a new type of approximations in terms of continued fractions. We are speaking about the following new estimates of the form

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n+\frac{1}{4}+\cfrac{1}{32n+8+\cfrac{9}{2n+\frac{1}{2}+\cfrac{25}{32n+8+\frac{49}{2n}+\ddots}}}}. \tag{7}$$

It is to be noticed that Slavić correctly found in (3) the first coefficients 32 and 8 (from $32x+8$), but the next performant coefficient proved to be $\frac{9}{2}$ as in (7).

We prove that for every $x \geqslant 5$,

$$\sqrt{x+\frac{1}{4}+\cfrac{1}{32x+8+\frac{9}{2x+\frac{1}{2}}}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x+\frac{1}{4}+\cfrac{1}{32x+8+\cfrac{9}{2x+\frac{1}{2}+\frac{25}{32x}}}}. \tag{8}$$

Fihtenholt [8, p. 371] proved the formula

$$\frac{(2n)!!}{(2n-1)!!} = \sqrt{\pi n}\exp\frac{4\theta-\theta'}{4n},$$

for some $\theta, \theta' \in (0,1)$, which can be used to get the inequality

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n}\exp\frac{1}{6n}. \tag{9}$$

We establish the following new approximation formula

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n-\frac{1}{12}+\frac{1}{288n}-\frac{109}{10368n^2}+\frac{433}{497\,664n^3}+\frac{92\,231}{29\,859\,840n^4}}\exp\frac{1}{6n} \tag{10}$$

and we prove that for every $x \geqslant 2$,

$$\sqrt{x-\frac{1}{12}+\frac{1}{288x}-\frac{109}{10368x^2}}\exp\frac{1}{6x} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x-\frac{1}{12}+\frac{1}{288x}}\exp\frac{1}{6x}. \tag{11}$$

We show that better results can be obtained if in (9)–(11) we replace $\frac{1}{6n}$ by $\frac{1}{8n}$. More precisely, we establish the approximation formula

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n}\exp\left(\frac{1}{8n}-\frac{1}{192n^3}+\frac{1}{640n^5}-\frac{17}{14\,336n^7}+\frac{31}{18\,432n^9}\right) \tag{12}$$

and we prove the following inequality for every $x \geqslant 2$,

$$\sqrt{x}\exp\left(\frac{1}{8x}-\frac{1}{192x^3}\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\exp\left(\frac{1}{8x}-\frac{1}{192x^3}+\frac{1}{640x^5}\right), \tag{13}$$

that refines (10)–(11). Moreover, numerical computations show that (11) improves much the following double inequalities involving the psi, or digamma function ψ :

$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x-\frac{1}{2}+\sqrt{\frac{3}{4}}}$$

$$\exp\left(\frac{1}{2}\mathrm{Psi}\left(x+\sqrt{\frac{1}{2}}\right)\right)<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\exp\left(\frac{1}{2}\mathrm{Psi}\left(x+\frac{3}{4}\right)\right),$$

called the first and the second Kershaw's double inequality, see [12].

2. A series under the radical

In order to illustrate our new method, we begin this section by searching the best approximation of the form

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\approx\sqrt{n+\frac{1}{4}+\frac{a}{n}+\frac{b}{n^2}}, \tag{14}$$

where a,b are real parameters. A method to compare two approximations (14) is to define the sequence $(w_n)_{n\geqslant 1}$ by the relations

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}=\sqrt{n+\frac{1}{4}+\frac{a}{n}+\frac{b}{n^2}}\cdot\exp w_n \tag{15}$$

and to a consider an estimate (14) as better as $(w_n)_{n\geqslant 1}$ faster converges to zero. Note that a powerful way to measure the rate of convergence of a sequence is the following result first used by Mortici [16]-[23] to construct asymptotic expansions or to accelerate some convergences.

LEMMA 2.1. *If* $(x_n)_{n\geqslant 1}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(x_n-x_{n+1})=l\in\mathbb{R}, \tag{16}$$

with $k>1$, *then there exists the limit:*

$$\lim_{n\to\infty} n^{k-1}x_n=\frac{l}{k-1}.$$

For complete proof and other details, see, e.g., [18].

Now we can see from this Lemma 2.1 that the speed of convergence of the sequence $(x_n)_{n\geqslant 1}$ is even higher as the value k satisfying (16) is greater.

By (15), we have

$$w_n=\ln\Gamma(n+1)-\ln\Gamma\left(n+\frac{1}{2}\right)-\frac{1}{2}\ln\left(n+\frac{1}{4}+\frac{a}{n}+\frac{b}{n^2}\right)$$

and we develop the difference $w_n - w_{n+1}$ as power series in n^{-1},

$$w_n - w_{n+1} = \left(\frac{1}{32} - a\right)\frac{1}{n^3} + \left(\frac{15}{8}a - \frac{3}{2}b - \frac{9}{128}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{17}$$

Having in mind Lemma 2.1, we can see that the fastest sequence $(w_n)_{n\geqslant 1}$ is obtained when the first two coefficients of the expansion (17) vanish. The corresponding values $a = \frac{1}{32}$, $b = -\frac{1}{128}$ give the best approximation of the form (14):

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n + \frac{1}{4} + \frac{1}{32n} - \frac{1}{128n^2}}.$$

Using the same idea, the approximation (5) was obtained.

In fact, using (17) and Lemma 2.1, we can state the following theoretical result:

THEOREM 2.1. *(i) If* $a \neq \frac{1}{32}$, *then the sequence* $(w_n)_{n\geqslant 1}$ *behaves as* n^{-2}, *for* $n \to \infty$, *since* $\lim_{n\to\infty} n^2 w_n = \frac{1}{2}\left(\frac{1}{32} - a\right) \neq 0$.

(ii) If $a = \frac{1}{32}$ *and* $b \neq -\frac{1}{128}$, *then the sequence* $(w_n)_{n\geqslant 1}$ *behaves as* n^{-3}, *for* $n \to \infty$, *since* $\lim_{n\to\infty} n^3 w_n = -\frac{1}{256}(128b+1) \neq 0$.

(iii) If $a = \frac{1}{32}$ *and* $b = -\frac{1}{128}$, *then the sequence* $(w_n)_{n\geqslant 1}$ *behaves as* n^{-4}, *for* $n \to \infty$, *since* $\lim_{n\to\infty} n^4 w_n = -\frac{5}{4096}$.

The same method presented above can be used to obtain other new estimates (5) of increasing accuracy. This represents a systematically way to proceed, unlike the methods used to obtained (1)–(4). Inductively, if assume that the coefficients a_1, a_2, ..., a_k are already known in the estimate

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n + \frac{1}{4} + \frac{a_1}{n} + \ldots + \frac{a_k}{n^k} + \frac{a_{k+1}}{n^{k+1}}},$$

then the new coefficient a_{k+1} is the value which vanishes the first term of the expansion in power series of $w_n - w_{n+1}$, where the sequence $(w_n)_{n\geqslant 1}$ is defined by

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} = \sqrt{n + \frac{1}{4} + \frac{a_1}{n} + \ldots + \frac{a_k}{n^k} + \frac{a_{k+1}}{n^{k+1}}} \cdot \exp w_n, \quad n \geqslant 1.$$

In this way, we obtained (5), but we omit the proofs for sake of simplicity.

In order to prove the announced inequalities, we use a result of Alzer, who proved in [2, Theorem 8] that for every integer $n \geqslant 0$, the functions

$$F_n(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln 2\pi - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \tag{18}$$

and

$$G_n(x) = -\ln\Gamma(x) + \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \tag{19}$$

are strictly completely monotonic on $(0,\infty)$ (B_j is the jth Bernoulli number). In particular, by taking $n=1,2$ in (18)–(19), we obtain that for every $x>0$,

$$\exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}\right)<$$
$$<\frac{\Gamma(x+1)}{\sqrt{2\pi}\cdot x^{x+\frac{1}{2}}e^{-x}}<\exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}\right). \tag{20}$$

Designating by

$$f(x)=\sqrt{2\pi}\cdot x^{x+\frac{1}{2}}e^{-x}\exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}\right)$$

and

$$g(x)=\sqrt{2\pi}\cdot x^{x+\frac{1}{2}}e^{-x}\exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}\right),$$

(20) can be written as $f(x)<\Gamma(x+1)<g(x)$. The following bounds for the Wallis ratio can be stated now:

LEMMA 2.2. *For every $x>\frac{1}{2}$, it holds:*

$$\sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^x\exp\left(-\frac{P(x)}{x^7(2x-1)^5}\right)<$$
$$<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^x\exp\left(-\frac{Q(x)}{x^5(2x-1)^7}\right), \tag{21}$$

where

$$\begin{aligned}5040P(x)=\;&6720x^{10}-13440x^9+9408x^8-2352x^7+180x^6\\&-84x^5-94x^4+200x^3-116x^2+30x-3\end{aligned}$$

and

$$\begin{aligned}2520Q(x)=\;&13440x^{10}-40320x^9+49056x^8-30240x^7+\\&+9768x^6-2088x^5+742x^4-462x^3+161x^2-28x+2\end{aligned}$$

Proof. Using (20) in terms of f and g, we have

$$\frac{f(x)}{g\left(x-\frac{1}{2}\right)}<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\frac{g(x)}{f\left(x-\frac{1}{2}\right)},$$

and the conclusion follows by direct computations. □

THEOREM 2.2. *For every $x \geqslant 3$, it holds:*

$$\sqrt{x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}-\frac{5}{2048x^3}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}}.$$

Proof. By combining the requested inequalities with (21), it suffices to show that

$$\sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^x \exp\left(-\frac{P(x)}{x^7(2x-1)^5}\right) > \sqrt{x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}-\frac{5}{2048x^3}}$$

and

$$\sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^x \exp\left(-\frac{Q(x)}{x^5(2x-1)^7}\right) < \sqrt{x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}},$$

or equivalently $u(x) > 0$ and $v(x) < 0$, where

$$u(x) = \frac{1}{2}\ln x - \frac{1}{2} + x\ln\frac{x}{x-\frac{1}{2}} - \frac{P(x)}{x^7(2x-1)^5} - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}-\frac{5}{2048x^3}\right)$$

and

$$v(x) = \frac{1}{2}\ln x - \frac{1}{2} + x\ln\frac{x}{x-\frac{1}{2}} - \frac{Q(x)}{x^5(2x-1)^7} - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}\right).$$

We have

$$u''(x) = \frac{S(x)}{210x^9(2x-1)^7(2048x^4+512x^3+64x^2-16x-5)^2}$$

and

$$v''(x) = -\frac{R(x)}{210x^7(2x-1)^9(128x^3+32x^2+4x-1)^2},$$

where

$$\begin{aligned} S(x) = {} & 175 - 1330x - 3793x^2 + 33\,486x^3 + 34\,491x^4 + 234\,150x^5 \\ & -3754\,991x^6 + 1577\,142x^7 + 38\,668\,410x^8 - 74\,954\,668x^9 + 287\,074\,624x^{10} \\ & -2384\,044\,544x^{11} + 8141\,418\,656x^{12} - 12\,731\,165\,888x^{13} + 7387\,989\,504x^{14} \\ & +5192\,728\,576x^{15} - 13\,991\,362\,560x^{16} + 4748\,083\,200x^{17} \end{aligned}$$

and

$$\begin{aligned} R(x) = {} & 5 - 130x + 1193x^2 - 4618x^3 + 10\,995x^4 - 77\,385x^5 + 522\,624x^6 - 1720\,276x^7 \\ & +3485\,128x^8 - 12\,092\,528x^9 + 65\,962\,400x^{10} - 243\,328\,960x^{11} + 507\,941\,504x^{12} \\ & -839\,207\,936x^{13} + 655\,818\,240x^{14} - 262\,348\,800x^{15} + 43\,008\,000x^{16}. \end{aligned}$$

As all the coefficients of the polynomials $S(x+3)$ and $R(x+3)$ are positive, it results that $S(x) > 0$ and $R(x) > 0$, for every $x \geqslant 3$. In consequence, $u'' > 0$ and $v'' < 0$ on $[3,\infty)$. But $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = 0$, so $u > 0$ and $v < 0$, and the conclusion is proved. $\square$

3. A continued fraction under the radical

In order to improve (3)–(4), we introduce the estimates family

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n+\frac{1}{4}+\frac{1}{32n+a+\frac{b}{n}}}, \quad a,b \in \mathbb{R}, \tag{22}$$

which is in fact the first approximation of a future continued fraction. As before, we define the sequence $(z_n)_{n \geqslant 1}$ by the relations

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} = \sqrt{n+\frac{1}{4}+\frac{1}{32n+a+\frac{b}{n}}} \cdot \exp z_n\ , \quad n \geqslant 1,$$

for which

$$z_n - z_{n+1} = \frac{3(a-8)}{2048n^4} + \frac{32b-56a-a^2+368}{16\,384n^5} + O\left(\frac{1}{n^6}\right).$$

The solution $a=8,\ b=\frac{9}{2}$ of the system

$$\begin{cases} a-8=0 \\ 32b-56a-a^2+368=0 \end{cases},$$

provides the best approximation of the form (22), namely

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n+\frac{1}{4}+\frac{1}{32n+8+\frac{9}{2n}}}.$$

Using the same method, approximations (7) were obtained.

THEOREM 3.1. *For every $x \geqslant 5$, it holds*

$$\sqrt{x+\frac{1}{4}+\frac{1}{32x+8+\frac{9}{2x+\frac{1}{2}}}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x+\frac{1}{4}+\frac{1}{32x+8+\frac{9}{2x+\frac{1}{2}+\frac{25}{32x}}}}. \tag{23}$$

Proof. For the left-hand side inequality of (23), using (21), it suffices to show that

$$\sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^x \exp\left(-\frac{P(x)}{x^7(2x-1)^5}\right) > \sqrt{x+\frac{1}{4}+\frac{1}{32x+8+\frac{9}{2x+\frac{1}{2}}}},$$

or $h(x) > 0$, where

$$h(x) = \frac{1}{2}\ln x - \frac{1}{2} + x\ln\frac{x}{x-\frac{1}{2}} - \frac{P(x)}{x^7(2x-1)^5} - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x+8+\frac{9}{2x+\frac{1}{2}}}\right).$$

We have

$$h''(x) = \frac{T(x)}{210x^9(4x+1)^2(2x-1)^7(64x^2+32x+13)^2(64x^2+32x+15)^2},$$

where

$$\begin{aligned} T(x) = {} & 266175 + 849030x - 4753313x^2 - 18580378x^3 + 4378923x^4 \\ & + 194938270x^5 + 349569953x^6 - 952687221x^7 - 2583975732x^8 \\ & - 1993959336x^9 + 16439848808x^{10} + 5908730096x^{11} \\ & - 19460247040x^{12} - 166020044288x^{13} - 109494392448x^{14} \\ & - 397316964352x^{15} - 407518806016x^{16} - 1020855320576x^{17} \\ & + 260112384000x^{18}. \end{aligned}$$

As all the coefficients of the polynomial $T(x+5)$ are positive, it results that $T(x) > 0$, for every $x > 0$. Now, $h > 0$ on $[5,\infty)$, since $h'' > 0$ on $[5,\infty)$ and $\lim_{x\to\infty} h(x) = 0$.

The bounds (21) prove to be weak for showing the right-hand side inequality (23). We are forced to use the following stronger double inequality

$$\exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7}\right) <$$

$$< \frac{\Gamma(x+1)}{\sqrt{2\pi}\cdot x^{x+\frac{1}{2}}e^{-x}} < \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9}\right),$$

which is also a particular case of (18)–(19). Then

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^x \frac{\exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9}\right)}{\exp\left(\frac{1}{12\left(x-\frac{1}{2}\right)} - \frac{1}{360\left(x-\frac{1}{2}\right)^3} + \frac{1}{1260\left(x-\frac{1}{2}\right)^5} - \frac{1}{1680\left(x-\frac{1}{2}\right)^7}\right)}$$

$$< \sqrt{x + \frac{1}{4} + \frac{1}{32x + 8 + \frac{9}{2x+\frac{1}{2}+\frac{25}{32x}}}}.$$

The last inequality can be equivalently written as $q(x) < 0$, where

$$q(x) = \frac{1}{2}\ln x - \frac{1}{2} + x\ln\frac{x}{x-\frac{1}{2}} + \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9}\right)$$

$$- \left(\frac{1}{12\left(x-\frac{1}{2}\right)} - \frac{1}{360\left(x-\frac{1}{2}\right)^3} + \frac{1}{1260\left(x-\frac{1}{2}\right)^5} - \frac{1}{1680\left(x-\frac{1}{2}\right)^7}\right).$$

We have

$$q''(x) = -\frac{W(x)}{2310x^{11}(2x-1)^9},$$

where

$$\begin{aligned} W(x) = {} & 175-3150x+25\,123x^2-116\,214x^3+341\,767x^4-654\,846x^5 \\ & +793\,411x^6-531\,510x^7+89\,425x^8+87\,115x^9+41\,580x^{10} \\ & -392\,700x^{11}+1413\,720x^{12}-4047\,120x^{13}+7428\,960x^{14} \\ & -8796\,480x^{15}+6652\,800x^{16}-2956\,800x^{17}+591\,360x^{18}. \end{aligned}$$

All the coefficients of the polynomial $W(x+2)$ are positive, so $W(x)>0,$ for every $x\in[2,\infty)$. As a consequence, $q<0$ on $[2,\infty)$, since $q''(x)<0,$ for every $x\in[2,\infty)$ and $\lim_{x\to\infty}q(x)=0$. $\square$

4. Involving the exponential

In order to improve (9), we introduce the approximations family

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\approx\sqrt{n+a+\frac{b}{n}+\frac{c}{n^2}}\exp\frac{1}{6n},\quad a,b,c\in\mathbb{R}, \tag{24}$$

As before, we define the sequence $(t_n)_{n\geqslant 1}$ by the relations

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}=\sqrt{n+a+\frac{b}{n}+\frac{c}{n^2}}\exp\frac{1}{6n}\cdot\exp t_n\,,\quad n\geqslant 1,$$

for which

$$\begin{aligned} t_n-t_{n+1} = {} & \left(-\frac{1}{2}a-\frac{1}{24}\right)\frac{1}{n^2}+\left(\frac{1}{2}a-b+\frac{1}{2}a^2+\frac{1}{24}\right)\frac{1}{n^3} \\ & +\left(\frac{3}{2}b-\frac{1}{2}a-\frac{3}{2}c+\frac{3}{2}ab-\frac{3}{4}a^2-\frac{1}{2}a^3-\frac{11}{192}\right)\frac{1}{n^4}+O\left(\frac{1}{n^5}\right). \end{aligned} \tag{25}$$

According to Lemma 2.1, the best approximation (24) is obtained for the values $a=-\frac{1}{12}$, $b=\frac{1}{288}$, $c=-\frac{109}{10368}$, which vanish the first three coefficients of (25). Continuing this method, other more accurate approximations of the form (9) can be obtained.

THEOREM 4.1. *For every $x\geqslant 2$, it holds*

$$\sqrt{x-\frac{1}{12}+\frac{1}{288x}-\frac{109}{10368x^2}}\exp\frac{1}{6x}<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x-\frac{1}{12}+\frac{1}{288x}}\exp\frac{1}{6x}.$$

Proof. Using (21), it suffices to prove that

$$\sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^x\exp\left(-\frac{P(x)}{x^7(2x-1)^5}\right)>\sqrt{x-\frac{1}{12}+\frac{1}{288x}-\frac{109}{10368x^2}}\exp\frac{1}{6x}$$

and

$$\sqrt{\frac{x}{e}}\left(\frac{x}{x-\frac{1}{2}}\right)^{x}\exp\left(-\frac{Q(x)}{x^{5}(2x-1)^{7}}\right)<\sqrt{x-\frac{1}{12}+\frac{1}{288x}}\exp\frac{1}{6x},$$

or $j(x)>0$ and $m(x)<0$, where

$$j(x)=\frac{1}{2}\ln x-\frac{1}{2}+x\ln\frac{x}{x-\frac{1}{2}}-\frac{P(x)}{x^{7}(2x-1)^{5}}-\frac{1}{2}\ln\left(x-\frac{1}{12}+\frac{1}{288x}-\frac{109}{10368x^{2}}\right)-\frac{1}{6x},$$

respective

$$m(x)=\frac{1}{2}\ln x-\frac{1}{2}+x\ln\frac{x}{x-\frac{1}{2}}-\frac{Q(x)}{x^{5}(2x-1)^{7}}-\frac{1}{2}\ln\left(x-\frac{1}{12}+\frac{1}{288x}\right)-\frac{1}{6x}.$$

We have

$$j''(x)=\frac{Z(x)}{210x^{9}(2x-1)^{7}(10\,368x^{3}-864x^{2}+36x-109)^{2}},$$

and

$$m''(x)=-\frac{T(x)}{210x^{7}(2x-1)^{9}(288x^{2}-24x+1)^{2}},$$

where

$$\begin{aligned}Z(x)={}&83\,167-1219\,274x+9023\,263x^{2}-61\,873\,002x^{3}+405\,113\,451x^{4}\\&-2052\,079\,218x^{5}+8471\,225\,687x^{6}-32\,594\,847\,367x^{7}\\&+109\,847\,017\,800x^{8}-273\,055\,428\,688x^{9}+436\,652\,163\,416x^{10}\\&-365\,860\,041\,456x^{11}-11\,573\,535\,680x^{12}+367\,665\,187\,456x^{13}\\&-352\,043\,798\,400x^{14}+44\,836\,485\,120x^{15}+25\,140\,326\,400x^{16}\end{aligned}$$

and

$$\begin{aligned}T(x)={}&5-330x+10\,793x^{2}-210\,258x^{3}+2644\,525x^{4}-21\,495\,852x^{5}\\&+114\,989\,958x^{6}-411\,059\,684x^{7}+969\,881\,640x^{8}-1377\,152\,784x^{9}\\&+632\,956\,128x^{10}+1693\,333\,824x^{11}-4830\,053\,760x^{12}\\&+5072\,103\,680x^{13}-2624\,670\,720x^{14}+562\,544\,640x^{15}.\end{aligned}$$

We have $T(x)>0$ and $Z(x)>0$, for every $x\geqslant 2$. In consequence, $j>0$ and $m<0$ on $[2,\infty)$, since $j''>0$ and $m''<0$, with $j(\infty)=m(\infty)=0$. □

The ideas presented in the previous sections can be used to establish further improvements, when also some surpringsly results can be obtained. We refer to the fact that the best results in (10)–(11) are obtained when we replace $\frac{1}{6n}$ by $\frac{1}{8n}$. In this sense, let us introduce the following extended family of approximations:

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\approx\sqrt{n}\exp\left(\frac{a}{n}+\frac{b}{n^{3}}+\frac{c}{n^{5}}\right),\quad a,b,c\in\mathbb{R}.\tag{26}$$

The associated sequence $(y_n)_{n\geqslant 1}$, defined by

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}=\sqrt{n}\exp\left(\frac{a}{n}+\frac{b}{n^3}+\frac{c}{n^5}\right)\cdot\exp y_n$$

satisfies

$$y_n-y_{n+1}=\left(-a+\frac{1}{8}\right)\frac{1}{n^2}+\left(a-\frac{1}{8}\right)\frac{1}{n^3}+\left(-a-3b+\frac{7}{64}\right)\frac{1}{n^4}$$
$$+\left(a+6b-\frac{3}{32}\right)\frac{1}{n^5}+\left(-a-10b-5c+\frac{31}{384}\right)\frac{1}{n^6}+O\left(\frac{1}{n^7}\right).$$

The solution $a=\frac{1}{8}, b=-\frac{1}{192}, c=\frac{1}{640}$ of the system

$$\begin{cases} a=\frac{1}{8} \\ -a-3b+\frac{7}{64}=0 \\ -a-10b-5c+\frac{31}{384}=0 \end{cases}$$

provides the best approximation (26). The same method permited us to establish (12). Using (21), the bounds (13) can be similarly proved. Being similar with the previous proofs, we omit the details.

Finally, it is to be noticed that other more accurate formulas (5), (7), (10), (12), or stronger bounds (6), (11), (13) can be obtained, considering more terms in (18)–(19).

REFERENCES

[1] H. ALZER, *On some inequalities for the incomplete gamma function*, Math. Comp., **66** (1997), 771–778.

[2] H. ALZER, *On some inequalities for the gamma and psi functions*, Math. Comp., **66**, 3 (1997), 373–389.

[3] J. M. BORWEIN, AND O.-Y. CHAN, *Uniform bounds for the complementary incomplete gamma function*, Math. Inequal. Appl., **12**, 1 (2009), 115–121.

[4] A. V. BOYD, *Note on a paper by Uppuluri*, Pacific J. Math., **22** (1967), 9–10.

[5] J. T. CHU, *A modified Wallis product and some application*, Amer. Math Monthly, **69** (1962), 402–404.

[6] N. ELEZOVIĆ, C. GIORDANO, AND J. PEČARIĆ, *The best bounds in Gautschi's inequality*, Math. Inequl. App., **3**, 2 (2000), 239–252.

[7] N. ELEZOVIĆ, AND J. PEČARIĆ, *Differential and integral $f-$means and applications to digamma function*, Math. Inequal. Appl., **3**, 2 (2000), 189–196.

[8] G. M. FIHTENHOLT, *Course on differential and integral calculus*, vol. 2, Moscow, 1969.

[9] W. GAUTSCHI, *Inequalities for gamma and incomplete gamma function*, J. Math. Phys., **39** (1959), 77–81.

[10] J. GURLAND, *On Wallis' formula*, Amer. Math. Monthly, **63** (1956), 643–645.

[11] D. K. KAZARINOFF, *On Wallis' formula*, Edindurgh Math. Notes, **40** (1956), 19–21.

[12] D. KERSHAW, *Some extentions of W. Gautschi's inequalities for the gamma function*, Math. Comp., **41**, 164 (1983), 607–611.

[13] S. KOUMANDOS, *Monotonicity of some functions involving the gamma and psi functions*, Math. Comp., **77** (2008), 2261–2275.

[14] A.-J. LI, J. YUAN, AND C.-P. CHEN, *Monotonicity results for the polygamma functions*, Math. Inequal. Appl., **11**, 3 (2008), 307–316.

[15] D. S. MITRINOVIĆ, *Elementary Inequalities*, Gröningen, 1964.

[16] C. MORTICI, *Product approximations via asymptotic integration*, Amer. Math. Monthly, **117**, 5 (2010).

[17] C. MORTICI, *An ultimate extremely accurate formula for approximation of the factorial function*, Arch. Math. (Basel), **93**, 1 (2009), 37–45.

[18] C. MORTICI, *New approximations of the gamma function in terms of the digamma function*, Appl. Math. Lett., **23**, 3 (2010), 97–100.

[19] C. MORTICI, *New sharp bounds for gamma and digamma functions*, An. Ştiinţ. Univ. A. I. Cuza Iaşi Ser. N. Matem., **56** 2 (2010), in press.

[20] C. MORTICI, *Complete monotonic functions associated with gamma function and applications*, Carpathian J. Math., **25**, 3 (2009), 186–191.

[21] C. MORTICI, *Optimizing the rate of convergence in some new classes of sequences convergent to Euler's constant*, Anal. Appl. (Singap.), **8**, 1 (2010), 99–107.

[22] C. MORTICI, *Improved convergence towards the generalized Euler-Mascheroni constant*, Appl. Math. Comput., **215** (2010), 3443–3448.

[23] C. MORTICI, *Best estimates of the generalized Stirling formula*, Appl. Math. Comput., **215**, 11 (2010), 4044–4048.

[24] P. NATALINI, AND B. PALUMBO, *Inequalities for the incomplete gamma function*, Math. Inequal. Appl., **3**, 1 (2000), 69–77.

[25] D. V. SLAVIĆ, *On inequalities for* $\Gamma(x+1)/\Gamma(x+1/2)$, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz. 498–541 (1975), 17–20.

[26] J. WALLIS, *Arithmetica Infinitorum*, Oxford, 1656.

[27] G. N. WATSON, *A note on gamma function*, Proc. Edinburgh Math. Soc., **2**, 11 (1958/59) and Edinburgh Math. Notes 42 (1959), 7–9.

(Received December 15, 2009)

Cristinel Mortici
Valahia University of Târgovişte
Department of Mathematics
Bd. Unirii 18
130082 Târgovişte
Romania
e-mail: cmortici@valahia.ro