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INVERSION THEOREM FOR NONCONVEX MULTIFUNCTIONS

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Abstract. Using the techniques of variational analysis, we extend the inversion theorem of Robinson for convex multifunctions to γ-paraconvex multifunctions, preinvex multifunctions and strongly convex multifunctions, respectively. The results are used to derive upper bounds on the distance from an approximate solution to the solution set.

1. Introduction

Let X and Y be normed spaces and $F : X \to 2^Y$ be a multifunction. For a given $b \in F(X)$, an inclusion problem is to find a point $\bar{x} \in X$ such that

$$b \in F(\bar{x}). \tag{1}$$

However, in general, one can only find an approximate solution of the inclusion problem (1). Therefore, it is important to have an estimate of the distance $\mathrm{d}(x, F^{-1}(b))$ which from an approximate solution x to the solution set $F^{-1}(b)$. One estimate of $\mathrm{d}(x, F^{-1}(b))$ is given by using the distance $\mathrm{d}(b, F(x))$. A positive constant τ is called a global error bound for the problem (1) if for each $x \in X$,

$$\mathrm{d}(x, F^{-1}(b)) \leqslant \tau \mathrm{d}(b, F(x)).$$

Error bounds occur in many optimization problems, and have important applications in the convergence analysis of some algorithms and in the sensitive analysis of mathematical programming [8]. One of the important results on error bounds is the inversion theorem, which is established by Robinson [Theorem 2, 7] (see also [11]) for convex multifunctions. Li and Singer [Theorem 3.1, 5] gave several equivalent characterizations of the inversion theorem of Robinson. Under the weaker conditions than Robinson's, Zheng [Theorem 2.1, 13] gave an improvement of the inversion theorem for convex multifunctions.

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Recall [10] that a multifunction F from a normed space X to a normed space Y is called γ-paraconvex ($\gamma > 0$) if there is a constant $C > 0$ such that for all $x, u \in X$ and all $\alpha \in [0,1]$,

$$\alpha F(x) + (1-\alpha)F(u) \subset F(\alpha x + (1-\alpha)u) + C\|x-u\|^{\gamma} B_Y,$$

here B_Y denotes the closed unit ball of Y. In the case $\gamma > 1$, Rolewicz [10] proved that F is γ-paraconvex if and only if there exists $C > 0$ such that for all $x, u \in X$ and all $\alpha \in [0,1]$,

$$\alpha F(x) + (1-\alpha)F(u) \subset F(\alpha x + (1-\alpha)u) + C\min(\alpha, 1-\alpha)\|x-u\|^{\gamma} B_Y.$$

Motivated by this, we adopt the following definition of γ-paraconvex multifunction throughout this paper (see also this adoption in [Remarks, 4]).

DEFINITION 1.1. A multifunction F from a normed space X to a normed space Y is called γ-paraconvex ($\gamma > 0$) if there is a constant $C > 0$ such that for all $x, u \in X$ and all $\alpha \in [0,1]$,

$$\alpha F(x) + (1-\alpha)F(u) \subset F(\alpha x + (1-\alpha)u) + C\min(\alpha, 1-\alpha)\|x-u\|^{\gamma} B_Y.$$

Jourani [4] established the following inversion theorem, which extended the result of Robinson from convex multifunction to γ-paraconvex multifunctions.

THEOREM 1.1. *Let X and Y be Banach spaces, and $F : X \to 2^Y$ be a γ-paraconvex multifunction with closed graph. Then the following are equivalent:*
(i) $b \in \operatorname{int}(F(X))$;
(ii) for all $a \in F^{-1}(b)$ there are $\tau, r > 0$ such that

$$\mathrm{d}(x, F^{-1}(y)) \leqslant \tau \mathrm{d}(y, F(x))$$

for all $x \in B(a,r)$ and $y \in B(b,r)$. Here we use $B(a,r)$ to denote the closed ball centered at a with radius r (in X and Y, this will not lead to confusion in the context).

Let X and Y be normed spaces, and $F : X \to 2^Y$ be a multifunction. The following notions are needed in this paper. As usual, $\mathrm{Dom}(F) := \{x \in X : F(x) \neq \emptyset\}$ denotes the domain of F. The multifunction F is said to have closed values if $F(x)$ is a closed subset of Y for each $x \in X$. For $A \subset X$, $\mathrm{diam}(A)$ denote the diameter of A, where

$$\mathrm{diam}(A) := \sup\{\|x-y\| : x, y \in A\}.$$

Let $b \in Y$ and $r > 0$, we define

$$B_F(b,r) := \left\{ b + r\frac{b-y}{\|b-y\|} : y \in F(x) \text{ and } \mathrm{d}(b, F(x)) > 0 \right\}.$$

Clearly, $B_F(b,r) \subset B(b,r)$. Recently, Huang [Theorem 4.3, 3] obtained the following result.

THEOREM 1.2. *Let X and Y be normed spaces, and $F : X \to 2^Y$ be a convex multifunction with closed values. Let $b \in F(X)$, $a \in X$, $r > 0$, $\delta > 0$. If $B_F(b,r) \subset F(B(a,\delta))$, then*

$$\mathrm{d}(x, F^{-1}(b)) \leqslant \frac{\mathrm{d}(b,F(x))}{r + \mathrm{d}(b,F(x))}(\delta + \|x-a\|), \text{ for all } x \in \mathrm{Dom}(F).$$

In this paper, we study the inversion theorem for γ-paraconvex multifunctions, preinvex multifunctions and strongly convex multifunctions, respectively. The coefficient of error bound of these results can be easily calculated, which will bring convenience for applications.

2. Inversion Theorem for γ-Paraconvex Multifunction

THEOREM 2.1. *Let X and Y be normed spaces, and $F : X \to 2^Y$ be a multifunction with closed values. Let $b \in F(X)$, $a \in X, r > 0$ and $\delta > 0$. Suppose that F^{-1} is γ-paraconvex and $B_F(b,r) \subset F(B(a,\delta))$. Then for all $x \in \mathrm{Dom}(F)$,*

$$\mathrm{d}(x, F^{-1}(b)) \leqslant \frac{\mathrm{d}(b,F(x))}{r + \mathrm{d}(b,F(x))}\left[\|x-a\| + \delta + C(\mathrm{d}(b,F(x)) + r)^\gamma\right],$$

where C is as in the definition of γ-paraconvexity for F^{-1}.

Proof. Let $x \in \mathrm{Dom}(F)$. Without loss of generality, we assume that $x \in \mathrm{Dom}(F) \setminus F^{-1}(b)$. Let $\varepsilon > 0$ be an arbitrary positive number. Then there exists $y \in F(x)$ such that

$$\|b-y\| < d(b,F(x)) + \varepsilon.$$

Since $F(x)$ is closed in Y, $\mathrm{d}(b,F(x)) > 0$. Therefore, $z := b + r\frac{b-y}{\|b-y\|} \in B_F(b,r)$. By assumption, there exists $a_1 \in B(a,\delta)$ such that $z \in F(a_1)$. Clearly, $b = \frac{\|b-y\|}{r+\|b-y\|}z + \frac{r}{r+\|b-y\|}y$. Let $\lambda = \frac{\|b-y\|}{r+\|b-y\|}$. Then $b = \lambda z + (1-\lambda)y$. By the γ-paraconvexity of F^{-1}, we have

$$\begin{aligned}
&\lambda a_1 + (1-\lambda)x \in \lambda F^{-1}(z) + (1-\lambda)F^{-1}(y)\\
&\subset F^{-1}(\lambda z + (1-\lambda)y) + C\min(\lambda, 1-\lambda)\|y-z\|^\gamma B_Y\\
&\subset F^{-1}b + C\lambda\|y-z\|^\gamma B_Y.
\end{aligned}$$

Then there exists $e \in B_Y$ such that

$$\lambda a_1 + (1-\lambda)x - C\lambda\|y-z\|^\gamma e \in F^{-1}(b).$$

Therefore,

$$\begin{aligned} d(x,F^{-1}(b)) &\leqslant \|x-\lambda a_1-(1-\lambda)x+C\lambda\|y-z\|^{\gamma}e\| \\ &= \lambda\|x-a_1+C\|y-z\|^{\gamma}e\| \\ &\leqslant \lambda(\|x-a_1\|+C\|y-z\|^{\gamma}) \\ &\leqslant \lambda[\|x-a\|+\|a-a_1\|+C(\|y-b\|+\|b-z\|)^{\gamma}] \\ &\leqslant \lambda[\|x-a\|+\delta+C(\|y-b\|+r)^{\gamma}] \\ &= \frac{\|y-b\|}{r+\|y-b\|}[\|x-a\|+\delta+C(\|y-b\|+r)^{\gamma}] \\ &\leqslant \frac{\mathrm{d}(b,F(x))+\varepsilon}{r+\mathrm{d}(b,F(x))+\varepsilon}[\|x-a\|+\delta+C(\mathrm{d}(b,F(x))+\varepsilon+r)^{\gamma}] \end{aligned}$$

where the last inequality holds since $g(t)=\frac{t}{r+t}$ is increasing on $[0,+\infty)$. Leting $\varepsilon\to 0^{+}$, we have

$$\mathrm{d}(x,F^{-1}(b)) \leqslant \frac{\mathrm{d}(b,F(x))}{r+\mathrm{d}(b,F(x))}[\|x-a\|+\delta+C(\mathrm{d}(b,F(x))+r)^{\gamma}]. \qquad \square$$

COROLLARY 2.1. *Let X and Y be Banach spaces, and $F:X\to 2^{Y}$ be a multifunction with closed graph. Suppose that $b\in\operatorname{int}(F(X))$, $a\in F^{-1}(b)$ and F^{-1} is γ-paraconvex. Then there exist $r,\delta>0$ such that for all $x\in\operatorname{Dom}(F)$,*

$$\mathrm{d}(x,F^{-1}(b)) \leqslant \frac{\mathrm{d}(b,F(x))}{r+\mathrm{d}(b,F(x))}[\|x-a\|+\delta+C(\mathrm{d}(b,F(x))+r)^{\gamma}], \tag{2}$$

where C is as in the definition of γ-paraconvexity for F^{-1}.

Proof. Since X and Y are Banach spaces, and F^{-1} is a γ-paraconvex multifunction, by [Theorem 2.3 and Remarks, 4], there exist $r,\delta>0$ such that $B(b,r)\subset F(B(a,\delta))$. By Theorem 2.1, (2) holds. $\square$

COROLLARY 2.2. *Let X and Y be normed spaces, and $F:X\to 2^{Y}$ be a multifunction with closed values. Let $b\in F(X)$, $a\in X, r>0$ and $\delta>0$. Suppose that F^{-1} is γ-paraconvex, $B_F(b,r)\subset F(B(a,\delta))$ and $F^{-1}(b)$ is bounded. Then for all $x\in\operatorname{Dom}(F)$ with $\mathrm{d}(b,F(x))\leqslant L$,*

$$\mathrm{d}(x,F^{-1}(b)) \leqslant \frac{\mathrm{d}(b,F(x))}{r}\left[\operatorname{diam}(F^{-1}(b))+\mathrm{d}(a,F^{-1}(b))+\delta+C(L+r)^{\gamma}\right], \tag{3}$$

where C is as in the definition of γ-paraconvexity for F^{-1}.

Proof. Since

$$\|x-a\|\leqslant\|x-u\|+\|u-v\|+\|v-a\|\leqslant\|x-u\|+\operatorname{diam}(F^{-1}(b))+\|v-a\|,$$

for all $u,v \in F^{-1}(b)$, it follows that

$$\|x-a\| \leqslant \mathrm{d}(x,F^{-1}(b)) + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(a,F^{-1}(b)).$$

By Theorem 2.1, we have

$$\begin{aligned}\mathrm{d}(x,F^{-1}(b)) \leqslant \frac{\mathrm{d}(b,F(x))}{r+\mathrm{d}(b,F(x))}&\left[\mathrm{d}(x,F^{-1}(b)) + \mathrm{diam}(F^{-1}(b))\right.\\ &\left.+\mathrm{d}(a,F^{-1}(b)) + \delta + C(L+r)^{\gamma}\right],\end{aligned}$$

and hence (2) holds. $\square$

In the following, we give an example as application of Corollary 2.2.

EXAMPLE 2.1. Let $X=Y=R$. We define $F: X \to 2^{Y}$ as

$$F(x)=\begin{cases}(-\infty,-\sqrt{-x}], & \text{if } x\leqslant 0,\\ \emptyset, & \text{if } x>0.\end{cases}$$

Clearly,

$$F^{-1}(y)=\begin{cases}[-y^2,0], & \text{if } y\leqslant 0,\\ \emptyset, & \text{if } y>0.\end{cases}$$

Then F^{-1} is a 2-paraconvex multifunction, since for all $y_1,y_2 \in (-\infty,0]$ and $\lambda \in [0,1]$,

$$\lambda F^{-1}(y_1)+(1-\lambda)F^{-1}(y_2) \subset F^{-1}(\lambda y_1+(1-\lambda)y_2)+\lambda(1-\lambda)|y_1-y_2|^2 B_Y.$$

However, F is not convex, since

$$\frac{1}{2}F(-1)+\frac{1}{2}F(0)=\left(-\infty,-\frac{1}{2}\right] \not\subset F\left(\frac{1}{2}(-1)+\frac{1}{2}0\right)=\left(-\infty,-\frac{\sqrt{2}}{2}\right].$$

Take $b=-1$. It is easy to verify that

$$B_F(-1,1) \subset [-2,0] \subset F(B(-1,1)) = F([-2,0]) = (-\infty,0],$$

and $F^{-1}(b)=[-1,0]$ is bounded. By Corollary 2.2, for all $x \in \mathrm{Dom}(F)$ wiht $\mathrm{d}(b,F(x)) \leqslant L$,

$$\begin{aligned}\mathrm{d}(x,F^{-1}(-1)) &\leqslant \frac{\mathrm{d}(-1,F(x))}{1}\left[\mathrm{diam}(F^{-1}(-1))+\mathrm{d}(-1,F^{-1}(-1))+1+(L+1)^2\right]\\ &= \mathrm{d}(-1,F(x))\left[2+(L+1)^2\right].\end{aligned}$$

REMARK 2.1. The techniques of the proof of Theorem 2.1 are based on [4, 7]. However, in compared with Theorem 1.1, the upper bound of $\mathrm{d}(x,F^{-1}(b))$ of the results in this section are more precise.

3. Inversion Theorem for Preinvex Multifunction

It is konwn that convexity plays an important role in mathematical programming and optimization theory. In order to generalize convexity, Weir and Mond [12] introduced a kind of significant generalized convex function, i.e., the preinvex function. After that, several authors [1, 9] extended this kind of function into many aspects. Especially, Bhatia and Mehra [1] introduced the following definition.

DEFINITION 3.1. Let X, Y be Banach spaces, $F : X \to 2^Y$ be a multifunction. We say that F is preinvex on X if there exists a function $\eta : X \times X \to X$ such that for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x_2 + \lambda \eta(x_1, x_2)).$$

THEOREM 3.1. *Let X and Y be normed spaces, F be a preinvex multifunction on X with the associated function η, and with closed values. Let $b \in F(X), a \in X, r > 0, \delta > 0$. Assume that $B_F(b,r) \subset F(B(a,\delta))$. Then*

$$\mathrm{d}(x, F^{-1}(b)) \leqslant \frac{\mathrm{d}(b, F(x))}{r + \mathrm{d}(b, F(x))} \sup_{x' \in B(a,\delta)} \|\eta(x', x)\| \quad \text{for all } x \in \mathrm{Dom}(F). \tag{4}$$

Proof. Let $x \in \mathrm{Dom}(F)$ with $\mathrm{d}(x, F^{-1}(b)) > 0$ (Since the conclusion holds trivially when $\mathrm{d}(x, F^{-1}(b)) = 0$). Following the proof of Theorem 2.1, $b = \lambda z + (1-\lambda)y$. By the preinvexity of F,

$$b \in \lambda F(a_1) + (1-\lambda)F(x) \subset F(x + \lambda \eta(a_1, x)),$$

which implies that $x + \lambda \eta(a_1, x) \in F^{-1}(b)$. Hence

$$\begin{aligned} \mathrm{d}(x, F^{-1}(b)) &\leqslant \|x - x - \lambda \eta(a_1, x)\| \\ &\leqslant \frac{\mathrm{d}(b, F(x)) + \varepsilon}{r + \mathrm{d}(b, F(x)) + \varepsilon} \|\eta(a_1, x)\| \\ &\leqslant \frac{\mathrm{d}(b, F(x)) + \varepsilon}{r + \mathrm{d}(b, F(x)) + \varepsilon} \sup_{x' \in B(a,\delta)} \|\eta(x', x)\|. \end{aligned}$$

Letting $\varepsilon \to 0^+$, then (4) holds. □

COROLLARY 3.1. *Let X and Y be normed spaces, F be a preinvex multifunction on X with the associated function η, and with closed values. Let $b \in F(X), a \in X, r > 0, \delta > 0$. Suppose that for all $x', x \in X$, $\|\eta(x', x)\| \leqslant \|x' - x\|$. Assume that $B_F(b,r) \subset F(B(a,\delta))$. Then*

$$\mathrm{d}(x, F^{-1}(b)) \leqslant \frac{\mathrm{d}(b, F(x))}{r + \mathrm{d}(b, F(x))} (\delta + \|x - a\|) \text{ for all } x \in \mathrm{Dom}(F).$$

Proof. The conclusion follows from Theorem 3.1 immediately by noting that

$$\sup_{x' \in B(a,\delta)} \|\eta(x', x)\| \leqslant \sup_{x' \in B(a,\delta)} \|x' - x\| \leqslant \sup_{x' \in B(a,\delta)} (\|x' - a\| + \|a - x\|) = \delta + \|x - a\|. \quad \square$$

In the following, we give an example as an application of Corollary 3.1.

EXAMPLE 3.1. Let $X = Y = R$. We define $F : X \to 2^Y$ as

$$F(x) = [-|x|, +\infty) \quad \text{for all } x \in X.$$

Then F is a preinvex multifunction on X. In fact, for all $\lambda \in [0,1]$ and all $x_1, x_2 \in X$, we have

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x_2 + \lambda\eta(x_1,x_2)),$$

where

$$\eta(x_1,x_2) = \begin{cases} |x_1| - |x_2|, & \text{if either } x_1 \geqslant 0,\ x_2 \geqslant 0 \text{ or } x_1 \leqslant 0,\ x_2 \leqslant 0, \\ |x_2| - |x_1|, & \text{if either } x_1 > 0,\ x_2 < 0 \text{ or } x_1 < 0,\ x_2 > 0. \end{cases}$$

However, F is not convex, since

$$\frac{1}{2}F(-1) + \frac{1}{2}F(1) = [-1, +\infty) \not\subset F\left(\frac{1}{2}(-1) + \frac{1}{2}\right) = F(0) = [0, +\infty).$$

Take $b = -3$. It is easy to verify that $|\eta(x_1,x_2)| \leqslant |x_1 - x_2|$,

$$B_F(-3,1) \subset [-4,2] \subset F(B(-3,1)) = F([-4,2]) = [-4,+\infty)$$

and $F^{-1}(-3) = (-\infty,-3] \cup [3,+\infty)$. By Corollary 3.1, we have

$$\mathrm{d}(x, F^{-1}(-3)) \leqslant \mathrm{d}(-3, F(x))\frac{1+|x+3|}{1} = \mathrm{d}(-3, F(x))(1 + |x+3|), \text{ for all } x \in X.$$

Similar to the proof of Corollary 2.2 but using Corollary 3.1 in place of Theorem 2.1, we immediately obtain the following result.

COROLLARY 3.2. *Let X and Y be normed spaces, F be a preinvex multifunction on X with the associated function η, and with closed values. Let $b \in F(X), a \in X, r > 0, \delta > 0$. Suppose that for all $x', x \in X$, $\|\eta(x',x)\| \leqslant \|x' - x\|$. Assume that $B_F(b,r) \subset F(B(a,\delta))$, and $F^{-1}(b)$ is bounded. Then*

$$\mathrm{d}(x, F^{-1}(b)) \leqslant \frac{\delta + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(a, F^{-1}(b))}{r}\mathrm{d}(b, Fx), \text{ for all } x \in X,$$

where $\mathrm{d}(b, F(x))$ is understood as ∞ if $F(x) = \emptyset$.

REMARK 3.1. In compared with Theorem 1.2, the objective multifunction in this section are not necessarily convex.

4. Inversion Theorem for Strongly Convex Multifunction

Let X be a normed space. Recall [6] that a function $f : X \to R$ is said to be strongly convex of order γ $(\gamma > 0)$ if there exists a contant $C > 0$ such that for all $x, y \in X$ and all $t \in [0,1]$,

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - Ct(1-t)\|x-y\|^{\gamma}.$$

Motivated by this, Huang [3] extended this definition to multifunctions.

DEFINITION 4.1. Let X and Y be normed spaces. We say that $F : X \to 2^Y$ is strongly convex of order γ $(\gamma > 0)$ and rank C $(C > 0)$ if for all $x, y \in X$ and all $t \in [0,1]$,

$$tF(x) + (1-t)F(y) + Ct(1-t)\|x-y\|^{\gamma} B_Y \subset F(tx + (1-t)y). \tag{5}$$

The following lemma can be founded in [Proposition 2.4.1, p.50, 2].

LEMMA 4.1. *Let X be a normed space and $A \subset X$ be a nonempty subset of X. Then*

$$|\mathrm{d}(x,A) - \mathrm{d}(y,A)| \leqslant \|x-y\|.$$

THEOREM 4.1. *Let X and Y be Banach spaces. Let $F : X \to 2^Y$ be a strongly convex of order γ and rank C multifunction. Suppose that $b \in \mathrm{int}(F(X))$. Then for all $x \in X$,*

$$C[\mathrm{d}(x, F^{-1}(b))]^{\gamma} \leqslant \mathrm{d}(b, F(x)).$$

Proof. Let $x \in X$. Without loss of generality, we assume that $x \in \mathrm{Dom}(F) \setminus F^{-1}(b)$. Let $t \in (0,1)$. Take $e \in F^{-1}(b)$ such that

$$\|x-e\| < \frac{\mathrm{d}(x, F^{-1}(b))}{1-t}. \tag{6}$$

By Lemma 4.1 and (6),

$$\mathrm{d}(tx + (1-t)e, F^{-1}(b)) \geqslant \mathrm{d}(x, F^{-1}(b)) - \|tx + (1-t)e - x\| > 0. \tag{7}$$

Since F is a strongly convex multifunction, F is a convex multifunction. Since $b \in \mathrm{int}(F(X))$ and X, Y are Banach spaces, by [Lemma 1, 7], there exists $\delta, r > 0$ such that $B(b,r) \subset F(B(a,\delta))$. By Corollary 3.1, we have

$$\mathrm{d}(tx + (1-t)e, F^{-1}(b)) \leqslant \frac{\mathrm{d}(b, F(tx + (1-t)e))}{r}(\delta + \|tx + (1-t)e - a\|). \tag{8}$$

It follows from (7) and (8) that

$$\mathrm{d}(b, F(tx + (1-t)e)) > 0. \tag{9}$$

Since F is a strongly convex of order γ and rank C multifunction, it follows from (5) and (9) that

$$\begin{aligned} 0 &< \mathrm{d}(b, F(tx + (1-t)e)) \\ &\leqslant \mathrm{d}(b, tF(x) + (1-t)F(e) + Ct(1-t)\|x-e\|^{\gamma} B_Y) \\ &\leqslant \mathrm{d}(b, tF(x) + (1-t)b + Ct(1-t)\|x-e\|^{\gamma} B_Y) \\ &= t[\mathrm{d}(b, F(x)) - C(1-t)\|x-e\|^{\gamma}] \end{aligned}$$

that is,

$$C(1-t)[\mathrm{d}(x, F^{-1}(b))]^{\gamma} \leqslant C(1-t)\|x-e\|^{\gamma} \leqslant \mathrm{d}(b, F(x)).$$

Letting $t \to 0^+$, we have

$$C[\mathrm{d}(x, F^{-1}(b))]^\gamma \leqslant \mathrm{d}(b, F(x)). \quad \square$$

In the following, we give an example as application of Theorem 4.1.

EXAMPLE 4.1. Let $X = Y = R$. We define $F : X \to 2^Y$ as

$$F(x) = [x^2 - 1, +\infty) \quad \text{for all } x \in X.$$

Then F is a strongly convex of order 2 and rank 1 multifunction. Let $b = 0$. It is easy to verify that $b \in \mathrm{int}(F(X))$. By Theorem 4.1, we have

$$[\mathrm{d}(x, F^{-1}(0))]^2 \leqslant \mathrm{d}(0, F(x)) \quad \text{for all } x \in X.$$

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