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A NOTE ON ONE–SIDED MAXIMAL OPERATOR IN $L^{p(\cdot)}(\mathbb{R})$

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Abstract. Consider one-sided Hardy-Littlewood maximal operator on the general Lebesgue space with variable exponent. It is known a local sufficient condition to the function $p(.)$ for the boundedness of the one-sided maximal operator on $L^{p(\cdot)}(\mathbb{R})$ provided $p(.)$ is a constant function in a neighborhood of infinity. Our main aim is to find a weaker condition to $p(.)$ at infinity to guarantee the boundedness of the one-sided maximal operator on $L^{p(\cdot)}(\mathbb{R})$. We will show two different sufficient conditions to the behavior of $p(.)$ at infinity under which the one-sided maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R})$.

1. Introduction

The study of Lebesgue spaces with variable exponent and function spaces derived from them attracts an interest of many mathematicians more and more. One of the fundamental questions in this theory is a problem of the boundedness of Hardy-Littlewood maximal operator. The basic result concerning the bounded domain $\Omega \subset \mathbb{R}^n$ was done by L. Diening (see [3]). This result was later extended to $\mathbb{R}^n$ by D. Cruiz-Uribe, A. Fiorenza and C. J. Neugebauer (see [1] and [2]) and independently by A. Nekvinda (see [11]). Further results on maximal operator can be found for instance in [4], [5], [6], [7], [8], [12] and [13].

In connection with the maximal operator there appears a problem on boundedness of one-sided maximal operators on $L^{p(\cdot)}(\mathbb{R})$. This paper generalizes results from [10] given by D. E. Edmunds, V. Kokilashvili and A. Meskhi where a sufficient local condition is given for boundedness of the one-sided maximal opeartor on $\mathbb{R}$. In fact, the condition in [10] consists of two parts. The first one controls a local behavior of $p(.)$ and the second one requires a constancy of the function $p(.)$ near the infinity.

Our main aim is to generalize of the condition at the infinity. We will find two different conditions to $p(.)$ at the infinity each of them commonly with the local control preserves the boundedness of the one-sided maximal operators. Both these conditions are more general then the constancy of $p(.)$ near infinity assumed in [10].

Recall basic definitions of maximal functions and variable Lebesgue spaces. Denote by $\mathfrak{M}$ a set of all measurable functions defined on $\mathbb{R}$ and $\mathfrak{B}$ a set of all functions $p(.) \in \mathfrak{M}$ such that

$$1 < \text{ess inf}\{p(x); x \in \mathbb{R}\} \leqslant \text{ess sup}\{p(x); x \in \mathbb{R}\} < \infty.$$

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DEFINITION 1.1. Let $f \in L^1_{loc}(\mathbb{R})$. Define the one-sided Hardy-Littlewood maximal functions M^+f, M^-f and the Hardy-Littlewood maximal function Mf by

$$(M^+f)(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| \mathrm{d}t, \qquad (M^-f)(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \mathrm{d}t,$$

$$(Mf)(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)| \mathrm{d}t.$$

It is not difficult to see the sublinearity of all operators M, M^+, M^-, i. e.

$$M^+(f+g)(x) \leqslant M^+f(x) + M^+g(x) \tag{1.1}$$

and analogously for M, M^-.

DEFINITION 1.2. Let $p(.) \in \mathfrak{B}$. Define the variable Lebesgue space $L^{p(.)}(\mathbb{R})$ as a set of all functions with a finite norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0; \int_{\mathbb{R}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}.$$

The paper is organized into several parts. In the second part we introduce main results of this paper (Theorem 1 and Theorem 2). The third one contains basic definitions and known assertions which we need in proofs of Theorem 1 and Theorem 2. These theorems are proved in further parts.

2. Main results

Recall that the following well-known class $\mathscr{L}$ of functions plays an important role in boundedness of maximal operator on $L^{p(.)}(\mathbb{R})$.

DEFINITION 2.1. Let $p(.) \in \mathfrak{B}$. Say that $p(.) \in \mathscr{L}$ if there is a constant $K > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{K}{-\ln|x-y|}$$

for $x, y \in \mathbb{R}$, $0 < |x-y| \leqslant \frac{1}{2}$.

Remind classes $\mathscr{L}^+, \mathscr{L}^-$ from [10]. These classes of local conditions are assumed in proofs of boundedness of the one-sided maximal operator on bounded interval.

DEFINITION 2.2. Let $p(.) \in \mathfrak{B}$. Say that $p(.) \in \mathscr{L}^+$ if there is a constant $K > 0$ such that

$$p(y) \geqslant p(x) - \frac{K}{\ln \frac{1}{y-x}}$$

for $x, y \in \mathbb{R}$, $0 < y - x \leqslant \frac{1}{2}$.

Say that $p(.) \in \mathscr{L}^-$ if $\widetilde{p}(.) \in \mathscr{L}^+$ where $\widetilde{p}(x) = p(-x)$. Note that $\mathscr{L}^+ \cap \mathscr{L}^- = \mathscr{L}$.

Remind now a condition to $p(.)$ at infinity which was investigated in [11].

DEFINITION 2.3. Let $r(.) \in \mathfrak{M}$ be a measurable. Say that $r(.) \in \mathscr{P}$ if there exists a constant $c > 0$ such that

$$\int_{\{x;r(x)\neq 0\}} c^{\frac{1}{r(x)}}\,dx < \infty.$$

THEOREM 1. *Let $q(.) \in \mathfrak{B}$ be a non-increasing function in $\mathbb{R}$. Assume that $p(.) \in \mathscr{L}^+$ and $|p(.) - q(.)| \in \mathscr{P}$. Then M^+ is bounded on $L^{p(.)}(\mathbb{R})$.*

Analogously, M^- is bounded on $L^{p(.)}(\mathbb{R})$ provided $q(.) \in \mathfrak{B}$ is non-decreasing and $p(.) \in \mathscr{L}^-$.

We will use for the second condition functions $\ln x$, $\ln\ln x$, $\ln\ln\ln x$ and so on. Denote these functions by $\ln_k$ where the subscript k means the number of symbols "ln". Define numbers e_k by

$$\mathrm{e}_0 = 1,\ \ \mathrm{e}_{k+1} = (\mathrm{e})^{\mathrm{e}_k} \tag{2.1}$$

and functions $\ln_k x$ on intervals (e_k, ∞) by

$$\ln_0 x = x,\ \ \ln_{k+1} x = \ln(\ln_k x). \tag{2.2}$$

Set for $\alpha > 0$

$$b_{k,\alpha}(x) = -\frac{1}{\alpha}\frac{d}{dx}(\ln_k^{-\alpha} x). \tag{2.3}$$

DEFINITION 2.4. Say that an even function $p(.) \in \mathfrak{B}$ quickly tends to a constant (write $p(.) \in \mathscr{QC}$) if there exist numbers $K > 0$, $k \in \mathbb{N}$ and $\alpha > 0$ such that

(i) $p(.)$ is monotone on (e_k, ∞),

(ii) $\left|\frac{dp}{dx}(x)\right| \leqslant K b_{k,\alpha}(x), \qquad x \geqslant \mathrm{e}_k.$

Remark that each function $p(x)$ which is equal to $\frac{1}{\ln^{\alpha}(x)}$ near infinity belongs to the class $\mathscr{QC}$ for any $\alpha > 0$.

THEOREM 2. *Let $q(.) \in \mathscr{QC}$. Assume that $p(.) \in \mathscr{L}^+$ ($p(.) \in \mathscr{L}^-$) and $|p(.) - q(.)| \in \mathscr{P}$. Then M^+ (M^-) is bounded on $L^{p(.)}(\mathbb{R})$.*

3. Preparatory assertions

Let us start with the well-known theorem on the maximal operator. The proof can be found for instance in [9], Theorem 21.76.

PROPOSITION 3.1. *Let $r \in \mathbb{R}$, $1 < r \leqslant \infty$, then there exists $M_r > 0$ such that*

$$\int_{\mathbb{R}} \big(Mf(x)\big)^r dx \leqslant M_r \int_{\mathbb{R}} |f(x)|^r dx.$$

Proofs of the following two lemmas can be found in [11], see Lemma 1.7 and Lemma 2.12.

LEMMA 3.2. *Let $p(.) \in \mathfrak{B}$. Then the following statements are equivalent.*

(i) *Then there exists a constant $C > 0$ such that $\|Mf\|_{p(.)} \leqslant C\|f\|_{p(.)}$ for all $f \in L^{p(.)}(\mathbb{R})$.*

(ii) *$\int_{\mathbb{R}^n} |Mf(x)|^{p(x)} dx < \infty$ provided $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leqslant 1$.*

LEMMA 3.3. *Let $|p(.) - q(.)| \in \mathscr{P}$. Assume that $|f(x)| \leqslant 1$ a.e. in $\mathbb{R}$. Then $\int_{\mathbb{R}} |f(x)|^{p(x)} dx < \infty$ if and only if $\int_{\mathbb{R}} |f(x)|^{q(x)} dx < \infty$.*

The next theorem is proved in [13], Theorem 7.2.

THEOREM 3.4. *Assume that $p(.), q(.) \in \mathfrak{B}$, $q(.)$ is an even function. Let*

(i) $p(.) \in \mathscr{L}$,

(ii) $q(.) \in \mathscr{QC}$,

(iii) $|p(.) - q(.)| \in \mathscr{P}$.

Then the operator M is bounded on $L^{p(.)}(\mathbb{R})$.

LEMMA 3.5. *Assume $|p(.) - q(.)| \in \mathscr{P}$ and $|q(.) - r(.)| \in \mathscr{P}$. Then $|p(.) - r(.)| \in \mathscr{P}$.*

Proof. By the assumptions there are $c_1, c_2 > 0$ such that

$$\int_{\{p \neq q\}} c_1^{\frac{1}{|p(x)-q(x)|}} dx < \infty, \quad \int_{\{q \neq r\}} c_2^{\frac{1}{|q(x)-r(x)|}} dx < \infty. \tag{3.1}$$

Without lose of generality we can assume $c_1 < 1, c_2 < 1$. Accepting the convention $a^\infty = 0$ for $a < 1$ we can rewrite (3.1) as

$$\int_{\mathbb{R}} c_1^{\frac{1}{|p(x)-q(x)|}} dx < \infty, \quad \int_{\mathbb{R}} c_2^{\frac{1}{|q(x)-r(x)|}} dx < \infty. \tag{3.2}$$

Choose $c_3 > 0$ such that $\sqrt{c_3} \leqslant \min\{c_1, c_2\}$. Then $c_3 < 1$. Set

$$A = \{x \in \mathbb{R}; |p(x) - q(x)| > |q(x) - r(x)|\}, \quad B = \mathbb{R} \setminus A.$$

Following clear inequalities

$$\frac{1}{|p(x) - r(x)|} \geqslant \frac{1}{|p(x) - q(x)| + |q(x) - r(x)|} \geqslant \frac{1}{2\max\{|p(x) - q(x)|, |q(x) - r(x)|\}}$$

imply

$$\begin{aligned}\int_{\mathbb{R}} c_3^{\frac{1}{|p(x)-r(x)|}}\,dx &\leqslant \int_{\mathbb{R}} (\sqrt{c_3})^{\frac{1}{\max\{|p(x)-q(x)|,|q(x)-r(x)|\}}}\,dx \\ &= \int_A (\sqrt{c_3})^{\frac{1}{\max\{|p(x)-q(x)|,|q(x)-r(x)|\}}}\,dx + \int_B (\sqrt{c_3})^{\frac{1}{\max\{|p(x)-q(x)|,|q(x)-r(x)|\}}}\,dx \\ &= \int_A (\sqrt{c_3})^{\frac{1}{|p(x)-q(x)|}}\,dx + \int_B (\sqrt{c_3})^{\frac{1}{|q(x)-r(x)|}}\,dx \leqslant \int_A c_1^{\frac{1}{|p(x)-q(x)|}}\,dx \\ &\quad + \int_B c_2^{\frac{1}{|q(x)-r(x)|}}\,dx \leqslant \int_{\mathbb{R}} c_1^{\frac{1}{|p(x)-q(x)|}}\,dx + \int_{\mathbb{R}} c_2^{\frac{1}{|q(x)-r(x)|}}\,dx < \infty\end{aligned}$$

which finishes the proof. □

4. Boundedness of M^+

Divide this part in several subsections. First investigate a local condition an then conditions at infinity.

4.1. Local condition

Proofs in this part modify proofs from [3], [11] and [10]. Given $M \subset \mathbb{R}$ and $p(.) : \mathbb{R} \to \mathbb{R}$ we adopt the notation $p_M^- = \text{ess inf}\{p(x); x \in M\}$ and $p_M^+ = \text{ess sup}\{p(x); x \in M\}$ and denote in the next $p^* := p_{\mathbb{R}}^+$.

LEMMA 4.1. *Let $p(.)$ be given. Then the following statements are equivalent:*

(i) *There exists a constant $C > 0$ such that the inequality*

$$h^{p^-_{(x,x+h)} - p(x)} \leqslant C$$

holds for a. e. $x \in \mathbb{R}$ and $0 < h \leqslant \frac{1}{2}$.

(ii) *There exists a constant $L > 0$ such that the inequality*

$$h^{p(x+h)-p(x)} \leqslant L$$

holds for all $x \in \mathbb{R}$ and $0 < h \leqslant \frac{1}{2}$.

(iii) $p(.) \in \mathscr{L}^+$.

Proof. The proof is done in [10], Proposition B'. □

DEFINITION 4.2. Let $p(.) \in \mathscr{L}^+$. Say that a function f belongs to a class $\mathscr{G}_{p(.)}$ (write $f \in \mathscr{G}_{p(.)}$) if $f(x) = 0$ or $|f(x)| \geqslant 1$ for each $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} |f(x)|^{p(x)} dx \leqslant 1.$$

LEMMA 4.3. *Let $p(.) \in \mathscr{L}^+$. Then there exists a constant $C_p > 0$ such that the inequality*

$$|M^+ f(x)|^{p(x)} \leqslant C_p M^+(|f(.)|^{p(.)})(x)$$

holds for all $f \in \mathscr{G}_{p(.)}$ and $x \in \mathbb{R}^n$.

Proof. Set for $h > 0$

$$M_h^+ f(x) = \frac{1}{h} \int_x^{x+h} |f(y)| dy.$$

Suppose $f \in \mathscr{G}_{p(.)}$ and $x \in \mathbb{R}^n$. Fix for a moment $x \in \mathbb{R}$, $h > 0$ and denote $p^- := p^-_{(x,x+h)}$.

Assume first $h \leqslant \frac{1}{2}$. By Jensen's inequality we obtain

$$\big(M_h^+ f(x)\big)^{p(x)} = \Big(\frac{1}{h} \int_x^{x+h} |f(y)| dy\Big)^{p(x)} \leqslant \Big(\frac{1}{h} \int_x^{x+h} |f(y)|^{p^-} dy\Big)^{\frac{p(x)}{p^-}} := I. \tag{4.1}$$

Since $f \in \mathscr{G}_{p(.)}$ and $p^- \leqslant p(y)$ for $y \in (x, x+h)$ we have $|f(y)|^{p^-} \leqslant |f(y)|^{p(y)}$ which gives

$$I \leqslant h^{-\frac{p(x)}{p^-}} \Big(\int_x^{x+h} |f(y)|^{p(y)} dy\Big)^{\frac{p(x)}{p^-}}.$$

Clearly

$$\int_x^{x+h} |f(y)|^{p(y)} dy \leqslant \int_{\mathbb{R}} |f(y)|^{p(y)} dy \leqslant 1$$

and using $f \in \mathscr{G}_{p(.)}$ with $\frac{p(x)}{p^-} \geqslant 1$ we obtain

$$I \leqslant h^{-\frac{p(x)}{p^-}} \int_x^{x+h} |f(y)|^{p(y)} dy = h^{1-\frac{p(x)}{p^-}} \Big(\frac{1}{h} \int_x^{x+h} |f(y)|^{p(y)} dy\Big).$$

By Lemma 4.1 we have

$$h^{\frac{p^- - p(x)}{p^-}} \leqslant C^{\frac{1}{p^-}} \leqslant \max\{1, C\}^{\frac{1}{p^-}} \leqslant \max\{1, C\}.$$

Thus, $I \leqslant \max\{1, C\} M_h\big(|f(.)|^{p(.)}\big)(x)$ which gives with (4.1)

$$\big|M_h^+ f(x)\big|^{p(x)} \leqslant \max\{1, C\} M_h^+\big(|f(.)|^{p(.)}\big)(x). \tag{4.2}$$

Assume now $h > \frac{1}{2}$. Clearly, using $f \in \mathscr{G}_{p(.)}$ we have

$$\begin{aligned}
\big(M_h^+ f(x)\big)^{p(x)} &= \Big(\frac{1}{h} \int_x^{x+h} |f(y)| dy\Big)^{p(x)} = h^{-p(x)} \Big(\int_x^{x+h} |f(y)| dy\Big)^{p(x)} \\
&\leqslant 2^{p(x)} (2h)^{-p(x)} \Big(\int_x^{x+h} |f(y)|^{p(y)} dy\Big)^{p(x)} \leqslant 2^{p^*} (2h)^{-1} \int_x^{x+h} |f(y)|^{p(y)} dy \\
&= 2^{p^*-1} \, \frac{1}{h} \int_x^{x+h} |f(y)|^{p(y)} dy = 2^{p^*-1} M_h^+\big(|f(.)|^{p(.)}\big)(x)
\end{aligned}$$

which proves with (4.2)

$$\left|M_h^+ f(x)\right|^{p(x)} \leqslant C_p M_h^+\left(|f(.)|^{p(.)}\right)(x)$$

with some positive constant C_p.

Taking supremum on both sides we obtain the required inequality. □

LEMMA 4.4. *Let $p(.) \in \mathscr{L}^+$ and $f \in \mathscr{G}_{p(.)}$. Then*

$$\int_{\mathbb{R}} |M^+ f(x)|^{p(x)} dx < \infty.$$

Proof. Set $q(x) = \frac{p(x)}{p^*}$ and $h(x) = |f(x)|^{q(x)}$. Then $h \in L^{p^*}$ and according to Theorem 3.1 we have

$$\int_{\mathbb{R}^n} |Mh(x)|^{p^*} dx \leqslant M_{p^*} \int_{\mathbb{R}^n} |h(x)|^{p^*} dx < \infty.$$

It yields with an easy fact $q(.) \in \mathscr{L}^+$, Proposition 3.1 and Lemma 4.3

$$\begin{aligned}\int_{\mathbb{R}^n} |Mf(x)|^{p(x)} dx &= \int_{\mathbb{R}^n} (|Mf(x)|^{q(x)})^{p^*} dx \leqslant C_q^{p^*} \int_{\mathbb{R}^n} \left(M\big(f(.)^{q(.)}\big)(x)\right)^{p^*} dx \\ &\leqslant C_q^{p^*} \int_{\mathbb{R}^n} \left(M(f(.)^{q(.)})(x)\right)^{p^*} dx \leqslant C_q^{p^*} M_{p^*} \int_{\mathbb{R}^n} \left((f(.)^{q(.)})(x)\right)^{p^*} dx \\ &= C_q^{p^*} M_{p^*} \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx\end{aligned}$$

which finishes the proof. □

The following two sections contain the main results of this paper. Proofs of Theorem 1 and Theorem 2 are given here.

4.2. First condition at infinity

LEMMA 4.5. *Let $p(.) \in \mathfrak{B}$ be non-increasing and $|f(x)| \leqslant 1$. Then a point-wise inequality*

$$\left(M^+ f(x)\right)^{p(x)} \leqslant \left(M^+ f(.)^{p(.)}\right)(x)$$

holds for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$ be an arbitrary and fix $h > 0$. Then

$$\left(\frac{1}{h}\int_x^{x+h} |f(t)| dt\right)^{p(x)} \leqslant \left(\frac{1}{h}\int_x^{x+h} |f(t)|^{p(x)} dt\right) \leqslant \left(\frac{1}{h}\int_x^{x+h} |f(t)|^{p(t)} dt\right).$$

Taking supremum on both sides we obtain

$$\left(M^+ f(x)\right)^{p(x)} \leqslant \left(M^+ f(.)^{p(.)}\right)(x).$$

which finishes the proof. □

LEMMA 4.6. *Let $p(.) \in \mathfrak{B}$ be non-increasing and $|f(x)| \leqslant 1$. Then*

$$\int_{\mathbb{R}} |f(x)|^{p(x)} dx < \infty \implies \int_{\mathbb{R}} |M^+ f(x)|^{p(x)} dx < \infty$$

Proof. Since $p(.)$ is non-increasing there exists $p_\infty := \lim_{x\to\infty} p(x)$. Set $q(x) = \frac{p(x)}{p_\infty}$. Then $q(.)$ is non-increasing and Lemma 4.5 yields

$$\begin{aligned}\int_{\mathbb{R}} |M^+ f(x)|^{p(x)} dx &= \int_{\mathbb{R}} \left(|M^+ f(x)|^{q(x)}\right)^{p_\infty} dx \leqslant \int_{\mathbb{R}} \left[M^+\left(f(.)^{q(.)}\right)(x)\right]^{p_\infty} dx \\ &\leqslant C \int_{\mathbb{R}} \left(f(x)^{q(x)}\right)^{p_\infty} dx = C \int_{\mathbb{R}} f(x)^{p(x)} dx < \infty\end{aligned}$$

and our lemma follows. □

THEOREM 4.7. *Let $q(.)$ be non-increasing, $|p(.) - q(.)| \in \mathscr{P}$ and $|f(x)| \leqslant 1$. Then*

$$\int_{\mathbb{R}} |f(x)|^{p(x)} dx < \infty \implies \int_{\mathbb{R}} |M^+ f(x)|^{p(x)} dx < \infty$$

Proof. Assume $\int_{\mathbb{R}} |f(x)|^{p(x)} dx < \infty$. By Lemma 3.3 we have $\int_{\mathbb{R}} |f(x)|^{q(x)} dx < \infty$. Using Lemma 4.6 we obtain $\int_{\mathbb{R}} |M^+ f(x)|^{q(x)} dx < \infty$ and again by Lemma 3.3 we conclude $\int_{\mathbb{R}} |M^+ f(x)|^{p(x)} dx < \infty$ which finishes the proof. □

We prove now First theorem. Recall its formulation here.

THEOREM 1. *Let $q(.) \in \mathfrak{B}$ be a non-increasing function in $\mathbb{R}$. Assume that $p(.) \in \mathscr{L}^+$ and $|p(.) - q(.)| \in \mathscr{P}$. Then M^+ is bounded on $L^{p(.)}(\mathbb{R})$.*

Analogously, M^- is bounded on $L^{p(.)}(\mathbb{R})$ provided $q(.) \in \mathfrak{B}$ is non-decreasing and $p(.) \in \mathscr{L}^-$.

Proof. We will investigate only the boundedness of M^+, the boundedness of M^- can be proved by the same way. Assume

$$\int_{\mathbb{R}} |f(x)|^{p(x)} dx \leqslant 1.$$

Split the function f in two parts as follows:

$$f(x) = f(x)\chi_{\{|f|>1\}}(x) + f(x)\chi_{\{|f|\leqslant 1\}}(x) := f_1(x) + f_2(x).$$

Clearly, an easy fact $f_1(x) f_2(x) = 0$ for all $x \in \mathbb{R}$ gives $|f(x)|^{p(x)} = |f_1(x)|^{p(x)} + |f_2(x)|^{p(x)}$ which gives

$$\int_{\mathbb{R}} |f_1(x)|^{p(x)} dx \leqslant 1, \qquad \int_{\mathbb{R}} |f_2(x)|^{p(x)} dx \leqslant 1.$$

Consequently, $f_1 \in \mathscr{G}_{p(.)}$ and using the assumption $p(.) \in \mathscr{L}^+$ we conclude by Lemma 4.4

$$\int_{\mathbb{R}} |M^+ f_1(x)|^{p(x)} dx < \infty. \tag{4.3}$$

Since $q(.)$ is non-increasing, $|p(.) - q(.)| \in \mathscr{P}$ by the assumptions and $|f_2(x)| \leqslant 1$ we have by Lemma 4.7

$$\int_{\mathbb{R}} |M^+ f_2(x)|^{p(x)} dx < \infty.$$

Finally, using this last inequality, (1.1) and (4.3) we obtain

$$\begin{aligned}\int_{\mathbb{R}} |M^+ f(x)|^{p(x)} dx &\leqslant \int_{\mathbb{R}} |M^+ f_1(x) + M^+ f_2(x)|^{p(x)} dx \\ &\leqslant 2^{p^*-1} \Big(\int_{\mathbb{R}} |M^+ f_1(x)|^{p(x)} dx + \int_{\mathbb{R}} |M^+ f_2(x)|^{p(x)} dx \Big) < \infty\end{aligned}$$

and Lemma 3.2 finishes the proof. □

4.3. Second condition at infinity

We prove here Theorem 2. Recall its formulation.

THEOREM 2. *Let $q(.) \in \mathscr{QC}$. Assume that $p(.) \in \mathscr{L}^+$ ($p(.) \in \mathscr{L}^-$) and $|p(.) - q(.)| \in \mathscr{P}$. Then M^+ (M^-) is bounded on $L^{p(.)}(\mathbb{R})$.*

Proof. Since $q(.) \in \mathscr{QC}$ it has a derivative and so, $q(t)$ is continuous for large t. Due to the monotony of $q(.)$ there exists $a := \lim_{x\to\infty} q(x)$. Again by $q(.) \in \mathscr{QC}$ we have $a > 1$ and $\lim_{x\to\infty} \left|\frac{dq}{dx}(x)\right| = 0$. Then there is $x_0 > 0$ large enough with $q(x_0) > 1$ and $\left|\frac{dq}{dx}(x)\right| \leqslant 1$ for $|x| \geqslant x_0$. Set

$$r(x) = \begin{cases} q(x) & \text{for } |x| \geqslant x_0, \\ q(x_0) & \text{for } |x| < x_0. \end{cases}$$

Thus, $r(.)$ is Lipschitz function (even with a constant 1) and so, $r(.) \in \mathscr{L}$. Moreover, since $r(x) = q(x)$ for large x we have $r(.) \in \mathscr{QC}$, even and monotone. By Theorem 3.4 we have that M is bounded on $L^{r(.)}(\mathbb{R})$, i. e. the implication

$$\int_{\mathbb{R}} |g(x)|^{r(x)} dx \leqslant 1 \implies \int_{\mathbb{R}} |Mg(x)|^{r(x)} dx < \infty \tag{4.4}$$

holds.

Assume

$$\int_{\mathbb{R}} |f(x)|^{p(x)} dx \leqslant 1.$$

Split f into two parts as in the proof of Theorem 1. Then

$$f(x) = f(x)\chi_{\{|f|>1\}}(x) + f(x)\chi_{\{|f|\leqslant 1\}}(x) := f_1(x) + f_2(x)$$

and

$$\int_{\mathbb{R}} |f_1(x)|^{p(x)} dx \leqslant 1, \qquad \int_{\mathbb{R}} |f_2(x)|^{p(x)} dx \leqslant 1.$$

Since $f_1 \in \mathscr{G}$ and $p(.) \in \mathscr{L}^+$ by the assumptions we have by Lemma 4.4

$$\int_{\mathbb{R}} |M^+ f_1(x)|^{p(x)} dx. \tag{4.5}$$

Estimate now $\int_{\mathbb{R}} |M^+ f_2(x)|^{p(x)} dx$. Since $r(x) = q(x)$ for $|x| \geqslant x_0$ we obtain clearly $|q(.) - r(.)| \in \mathscr{P}$. The assumption $|p(.) - q(.)| \in \mathscr{P}$ and Lemma 3.5 yield $|p(.) - r(.)| \in \mathscr{P}$. The relation $\int_{\mathbb{R}} |f_2(x)|^{p(x)} dx < \infty$ with the fact $|f_2(x)| \leqslant 1$ gives $\int_{\mathbb{R}} |f_2(x)|^{r(x)} dx < \infty$ by Lemma 3.3. Using Lemma 4.4 we obtain

$$\int_{\mathbb{R}} |M^+ f_2(x)|^{r(x)} dx \leqslant \int_{\mathbb{R}} |M f_2(x)|^{r(x)} dx < \infty$$

and again by Lemma 3.3 we conclude

$$\int_{\mathbb{R}} |M^+ f_2(x)|^{p(x)} dx < \infty.$$

Now, (1.1), (4.5) and the last inequality give

$$\int_{\mathbb{R}} |M^+ f(x)|^{p(x)} dx \leqslant \int_{\mathbb{R}} |M^+ f_1(x)|^{p(x)} dx + \int_{\mathbb{R}} |M^+ f_2(x)|^{p(x)} dx < \infty$$

which finishes with Lemma 3.2 the proof. □

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