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WEIGHTED MIXED NORM INEQUALITIES IN MARTINGALE SPACES

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Abstract. For the maximal geometric mean operator G in martingale spaces, we characterize the weight paries (u,v) for which G is bounded from martingale space $L^p(vd\mu)$ to $wL^q(ud\mu)$ or $L^q(ud\mu), 1 \leqslant p \leqslant q < \infty$.

1. Introduction

Let R^n be the n-dimensional real Euclidean space and f a real valued measurable function, the classical Hardy-Littlewood maximal operator M, the maximal geometric mean operator G and the minimal operator $\mathbb{m}$ are defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

$$G(f)(x) = \sup_{x\in Q} \exp \frac{1}{|Q|}\int_Q \log|f(y)|dy$$

and

$$\mathbb{m}f(x) = \inf_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy.$$

where Q is a non-degenerate cube with its sides paralleled to the coordinate axes and $|Q|$ is the Lebesgue measure of Q.

Let u, v be two weights, i.e. positive measurable functions. As well known, if $u = v$ and $p > 1$, [1] showed that the inequality

$$\int_{R^n} (Mf(x))^p v(x)dx \leqslant C\int_{R^n} |f(x)|^p v(x)dx, \ \forall f \in L^p(v)$$

holds if and only if $\omega \in A_p$, i.e., for any cube Q in R^n with sides parallel to the coordinates

$$\left(\frac{1}{|Q|}\int_Q v(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < C. \tag{1.1}$$

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Let $p \to \infty$ in (1.1), it follows that

$$\left(\frac{1}{|Q|}\int_Q v(x)dx\right)\exp\left(\frac{1}{|Q|}\int_Q \log\left(\frac{1}{v(x)}\right)dx\right) < C, \tag{1.2}$$

which is an alternative definition of A_∞ weight (see [3]). It is known that [4] and [2] used (1.2) to characterize the boundedness of G from $L^1(v)$ to $L^1(v)$. In the case of two weights, [5] gave that

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\exp\left(\frac{1}{|Q|}\int_Q \log\left(\frac{1}{v(x)}\right)dx\right) < C,\ \forall\, Q \Leftrightarrow \sup_{\|f\|_{L^p(v)}=1} \| Gf \|_{\omega L^p(u)} < \infty \tag{1.3}$$

and

$$\int_Q G(v^{-1}\chi_Q)(x)u(x)dx \leqslant C|Q|,\ \forall\, Q \Leftrightarrow \sup_{\|f\|_{L^p(v)}=1} \| Gf \|_{L^p(u)} < \infty, \tag{1.4}$$

which generalize the results of [6]. Recently, [7, 8, 9, 10] also studied the minimal operator $\mathbb{m}$ and maximal geometric mean operator G. Comparing with these results, [11] and [12] examined the weighted inequalities of the minimal operator and the maximal geometric mean operator in martingale spaces.

In this paper, we prove the weighted mixed norm inequalities of the maximal geometric mean operator in martingale spaces which is a generalization of [12]. Our approach is similar to that of [13] and [14] but different from that of [12]. The rest of this section consists of the preliminaries for the second section.

Let $(\Omega,\mathscr{F},\mu)$ be a complete probability space and $(\mathscr{F}_n)_{n\geqslant 0}$ an increasing sequence of sub-σ-fields of $\mathscr{F}$ with $\mathscr{F} = \bigvee_{n\geqslant 0} \mathscr{F}_n$. A weight ω is a random variable with $\omega > 0$ and $E(\omega) < \infty$. In this paper, for $p \geqslant 1$, a martingale $f = (f_n)_{n\geqslant 0} \in L^p(\omega d\mu)$ is meant as $f_n = E(f|\mathscr{F}_n)$, $f \in L^p(\omega d\mu)$. The maximal operator M and the maximal geometric mean operator G for martingale $f = (f_n)$ are defined by

$$Mf = \sup_{n\geqslant 0}|f_n| \text{ and } Gf = \sup_{n\geqslant 0}\exp E(\log|f||\mathscr{F}_n),$$

respectively. For $B \in \mathscr{F}$, we always denote $\int_\Omega \chi_B d\mu$ and $\int_\Omega \chi_B \omega d\mu$ by $|B|$ and $|B|_\omega$, respectively. For $(\Omega,\mathscr{F},\mu)$ and $(\mathscr{F}_n)_{n\geqslant 0}$, the family of all stopping times is denoted by $\mathscr{T}$.

2. Results and Their Proofs

THEOREM 2.1. *Let (u,v) be a couple of weights and $1 < p \leqslant q < \infty$. Suppose that $v^{-\frac{1}{p-1}} \in L^1(\Omega)$, then the following statements are equivalent:*

(1) *There exists a positive constant C_1 such that*

$$\left(\int_{\{\tau<\infty\}} \left(G(v^{-\frac{1}{p}}\chi_{\{\tau<\infty\}})\right)^q u d\mu\right)^{\frac{1}{q}} \leqslant C_1|\{\tau<\infty\}|^{\frac{1}{p}}, \forall \tau \in \mathscr{T}; \tag{2.1}$$

(2) *There exists a positive constant C_2 such that*

$$\Big(\int_\Omega (Gf)^q u d\mu\Big)^{\frac{1}{q}} \leqslant C_2\Big(\int_\Omega |f|^p v d\mu\Big)^{\frac{1}{p}}, \ \forall f=(f_n)\in L^p(vd\mu). \tag{2.2}$$

Proof. To prove the boundedness of G from (2.1), let $f\in L^p(vd\mu)$. For all $k\in Z$, define stopping times

$$\tau_k = \inf\{n : \exp E(\log|f||\mathscr{F}_n) > 2^k\}.$$

Set

$$A_{k,j} = \{\tau_k<\infty\}\cap\{2^j < \exp E\left(\log(v^{-1})|\mathscr{F}_{\tau_k}\right) \leqslant 2^{j+1}\};$$

$$B_{k,j} = \{\tau_k<\infty, \tau_{k+1}=\infty\}\cap\{2^j < \exp E\left(\log(v^{-1})|\mathscr{F}_{\tau_k}\right) \leqslant 2^{j+1}\},\ j\in Z.$$

Then $A_{k,j}\in\mathscr{F}_{\tau_k}, B_{k,j}\subseteq A_{k,j}$. Moreover, $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k < Gf \leqslant 2^{k+1}\} = \{\tau_k<\infty, \tau_{k+1}=\infty\} = \bigcup_{j\in Z} B_{k,j}, k\in Z.$$

Trivially,

$$\exp E\left(\log|f||\mathscr{F}_{\tau_k}\right) = \exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_{\tau_k}\right)\exp E\left(\log(v^{-\frac{1}{p}})|\mathscr{F}_{\tau_k}\right).$$

It follows that

$$\begin{aligned}
2^{kq} &\leqslant \operatorname*{ess\,inf}_{A_{k,j}}\Big(\exp E\left(\log|f||\mathscr{F}_{\tau_k}\right)\Big)^q \\
&\leqslant \operatorname*{ess\,inf}_{A_{k,j}}\Big(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_{\tau_k}\right)\Big)^q \operatorname*{ess\,sup}_{A_{k,j}}\Big(\exp E\left(\log(v^{-\frac{1}{p}})|\mathscr{F}_{\tau_k}\right)\Big)^q \\
&\leqslant 2^{\frac{q}{p}}\operatorname*{ess\,inf}_{A_{k,j}}\Big(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_{\tau_k}\right)\Big)^q |B_{k,j}|_u^{-1}\times \\
&\qquad \times\int_{B_{k,j}}\left(\exp E\left(\log(v^{-\frac{1}{p}})|\mathscr{F}_{\tau_k}\right)\right)^q u d\mu.
\end{aligned}$$

To estimate $\int_\Omega (Gf)^q u d\mu$, firstly we have

$$\begin{aligned}
\int_\Omega (Gf)^q u d\mu &= \sum_{k\in Z}\int_{\{2^k<Gf\leqslant 2^{k+1}\}}(Gf)^q u d\mu \\
&\leqslant 2^q\sum_{k\in Z}\int_{\{2^k<Gf\leqslant 2^{k+1}\}} 2^{kq} u d\mu \\
&= 2^q\sum_{k\in Z, j\in Z}\int_{B_{k,j}} 2^{kq} u d\mu \\
&\leqslant 2^q 2^{\frac{q}{p}}\sum_{k\in Z, j\in Z}\operatorname*{ess\,inf}_{A_{k,j}}\Big(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_{\tau_k}\right)\Big)^q \\
&\qquad\times\int_{B_{k,j}}\left(\exp E\left(\log(v^{-\frac{1}{p}})|\mathscr{F}_{\tau_k}\right)\right)^q u d\mu.
\end{aligned}$$

It is clear that ϑ is a measure on $X = Z^2$ with

$$\vartheta(k,j) = \int_{B_{k,j}} \left(\exp E\left(\log(v^{-\frac{1}{p}})|\mathscr{F}_{\tau_k}\right)\right)^q u d\mu.$$

For the above $f \in L^p(vd\mu)$, define

$$Tf(k,j) = \operatorname*{ess\,inf}_{A_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_{\tau_k}\right)\right)^q$$

and denote

$$E_\lambda = \left\{(k,j) : \operatorname*{ess\,inf}_{A_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_{\tau_k}\right)\right)^q > \lambda\right\} \text{ and } G_\lambda = \bigcup_{(k,j)\in E_\lambda} A_{k,j}$$

for each $\lambda > 0$. Then we have

$$\begin{aligned}
|\{Tf > \lambda\}|_\vartheta &= \sum_{(k,j)\in E_\lambda} \int_{B_{k,j}} \left(\exp E\left(\log(v^{-\frac{1}{p}})|\mathscr{F}_{\tau_k}\right)\right)^q u d\mu. \\
&\leqslant \sum_{(k,j)\in E_\lambda} \int_{B_{k,j}} \left(\exp E\left(\log(v^{-\frac{1}{p}}\chi_{G_\lambda})|\mathscr{F}_{\tau_k}\right)\right)^q u d\mu. \\
&\leqslant \int_{G_\lambda} \left(G(v^{-\frac{1}{p}}\chi_{G_\lambda})\right)^q u d\mu.
\end{aligned}$$

Let $\tau = \inf\left\{n : \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right)\right)^q > \lambda\right\}$, we have

$$G_\lambda \subseteq \left\{\left(G(fv^{\frac{1}{p}})\right)^q > \lambda\right\} = \{\tau < \infty\}.$$

It follows from (2.1) that

$$\begin{aligned}
|\{Tf > \lambda\}|_\vartheta &\leqslant \int_{G_\lambda} \left(G(v^{-\frac{1}{p}}\chi_{G_\lambda})\right)^q u d\mu \\
&\leqslant \int_{\{\tau<\infty\}} \left(G(v^{-\frac{1}{p}}\chi_{\{\tau<\infty\}})\right)^q u d\mu \\
&\leqslant C_1^q |\{\tau < \infty\}|^{\frac{q}{p}} \\
&= C_1^q \left|\left\{\left(G(fv^{\frac{1}{p}})\right)^q > \lambda\right\}\right|^{\frac{q}{p}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_\Omega (Gf)^q u d\mu &\leqslant 2^q 2^{\frac{q}{p}} \int_X Tf d\vartheta = 2^q 2^{\frac{q}{p}} \int_0^\infty |\{Tf > \lambda\}|_\vartheta d\lambda \qquad (2.3) \\
&= 2^q 2^{\frac{q}{p}} \sum_{l\in Z} \int_{2^l}^{2^{l+1}} |\{Tf > \lambda\}|_\vartheta d\lambda \\
&\leqslant 2^q 2^{\frac{q}{p}} \sum_{l\in Z} 2^l |\{Tf > 2^l\}|_\vartheta
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2^q 2^{\frac{q}{p}} C_1^q \sum_{l\in Z} 2^l |\{(G(fv^{\frac{1}{p}}))^q > 2^l\}|^{\frac{q}{p}} \\
&= 2^q 2^{\frac{q}{p}} C_1^q \sum_{l\in Z} \Big(2^{\frac{p}{q}\cdot l} |\{(G(fv^{\frac{1}{p}}))^p > 2^{\frac{p}{q}\cdot l}\}|\Big)^{\frac{q}{p}} \\
&\leqslant 2^q 2^{\frac{q}{p}} C_1^q \Big(\sum_{l\in Z} 2^{\frac{p}{q}\cdot l} |\{(G(fv^{\frac{1}{p}}))^p > 2^{\frac{p}{q}\cdot l}\}|\Big)^{\frac{q}{p}} \\
&= 2^q 2^{\frac{q}{p}} \left(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\right)^{\frac{q}{p}} C_1^q \Big(\sum_{l\in Z} \Big(2^{\frac{p}{q}\cdot l} - 2^{\frac{p}{q}(l-1)}\Big) \times \\
&\qquad\qquad \times \Big|\Big\{(G(fv^{\frac{1}{p}}))^p > 2^{\frac{p}{q}\cdot l}\Big\}\Big|\Big)^{\frac{q}{p}} \\
&\leqslant 2^q 2^{\frac{q}{p}} \left(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\right)^{\frac{q}{p}} C_1^q \Big(\sum_{l\in Z} \int_{2^{\frac{p}{q}(l-1)}}^{2^{\frac{p}{q}\cdot l}} |\{(G(fv^{\frac{1}{p}}))^p > \lambda\}| d\lambda\Big)^{\frac{q}{p}} \\
&\leqslant 2^q 2^{\frac{q}{p}} \left(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\right)^{\frac{q}{p}} C_1^q \Big(\int_0^\infty |\{(G(fv^{\frac{1}{p}}))^p > \lambda\}| d\lambda\Big)^{\frac{q}{p}} \\
&= 2^q 2^{\frac{q}{p}} \left(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\right)^{\frac{q}{p}} C_1^q \Big(\int_\Omega (G(fv^{\frac{1}{p}}))^p d\mu\Big)^{\frac{q}{p}},
\end{aligned}
$$

where we have used $p \leqslant q$. Note that $p > 1$, in virtue of Jensen's inequality and Doob's inequality, we have

$$
\begin{aligned}
\int_\Omega (Gf)^q d\mu &\leqslant 2^q 2^{\frac{q}{p}} \left(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\right)^{\frac{q}{p}} C_1^q \Big(\int_\Omega \Big(M(fv^{\frac{1}{p}})\Big)^p d\mu\Big)^{\frac{q}{p}} \qquad (2.4)\\
&\leqslant 2^q 2^{\frac{q}{p}} (p')^q \left(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\right)^{\frac{q}{p}} C_1^q \Big(\int_\Omega |f|^p v d\mu\Big)^{\frac{q}{p}}.
\end{aligned}
$$

Whence (2.2) is valid with $C_2 = 2\cdot 2^{\frac{1}{p}} p' \left(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\right)^{\frac{1}{p}} C_1$.

For the converse, fix $\tau \in \mathscr{T}$ and substitute $f = v^{-\frac{1}{p}}\chi_{\{\tau<\infty\}}$ into inequality (2.2), we have

$$
\Big(\int_{\{\tau<\infty\}} \Big(G(v^{-\frac{1}{p}}\chi_{\{\tau<\infty\}})\Big)^q u d\mu\Big)^{\frac{1}{q}} \leqslant \Big(\int_\Omega \Big(G(v^{-\frac{1}{p}}\chi_{\{\tau<\infty\}})\Big)^q u d\mu\Big)^{\frac{1}{q}} \leqslant C_2 |\{\tau<\infty\}|^{\frac{1}{p}},
$$

thus (2.1) is valid with $C_1 = C_2$.

COROLLARY 2.2. *Let (u,v) be a couple of weights and $1 < p < \infty$. Suppose that $v^{-\frac{1}{p-1}} \in L^1(\Omega)$, then the following statements are equivalent:*

(1) *There exists a positive constant* C_{11} *such that*

$$\int_{\{\tau<\infty\}} G(v^{-1}\chi_{\{\tau<\infty\}})ud\mu \leqslant C_{11}|\{\tau<\infty\}|, \forall \tau \in \mathscr{T}; \tag{2.5}$$

(2) *There exists a positive constant* C_{21} *such that*

$$\int_{\Omega}(Gf)^p ud\mu \leqslant C_{21}\int_{\Omega}|f|^p vd\mu,\ \forall f=(f_n)\in L^p(vd\mu). \tag{2.6}$$

The Corollary 2.2 is a special case of the Theorem 2.1. Using it, we have the following corollary.

COROLLARY 2.3. *Let* (u,v) *be a couple of weights. Suppose that* $v^{-1}\in L^\infty(\Omega)$, *then the following statements are equivalent:*

(1) *There exists a positive constant* C_{12} *such that*

$$\int_{\{\tau<\infty\}} G(v^{-1}\chi_{\{\tau<\infty\}})ud\mu \leqslant C_{12}|\{\tau<\infty\}|, \forall \tau \in \mathscr{T}; \tag{2.7}$$

(2) *There exists a positive constant* C_{22} *such that*

$$\int_{\Omega}(Gf)ud\mu \leqslant C_{22}\int_{\Omega}|f|vd\mu,\ \forall f=(f_n)\in L^1(vd\mu). \tag{2.8}$$

Proof. Fix $p=2$, then $v^{-\frac{1}{p-1}}\in L^1(\Omega)$.

$(1)\Rightarrow(2)$. Suppose that $f=(f_n)\in L^1(vd\mu)$. Let $g=f^{\frac{1}{2}}$, then $g=(g_n)\in L^2(\Omega)$ and $Gf=(Gg)^2$. It follows from Corollary 2.2 that

$$\int_{\Omega}(Gg)^2 ud\mu \leqslant C_{21}\int_{\Omega}|g|^2 vd\mu,$$

that is

$$\int_{\Omega}(Gf)ud\mu \leqslant C_{21}\int_{\Omega}|f|vd\mu.$$

$(2)\Rightarrow(1)$. Suppose that $f=(f_n)\in L^2(vd\mu)$. Let $g=f^2$, then $g=(g_n)\in L^1(\Omega)$ and $Gg=(Gf)^2$. Thus

$$\int_{\Omega}(Gg)ud\mu \leqslant C_{22}\int_{\Omega}|g|vd\mu,$$

that is

$$\int_{\Omega}(Gf)^2 ud\mu \leqslant C_{22}\int_{\Omega}|f|^2 vd\mu.$$

Combing with Corollary 2.2, we have (2.7). □

THEOREM 2.4. *Let* (u,v) *be a couple of weights and* $1<p\leqslant q<\infty$. *Suppose that* $v^{-\frac{1}{p-1}}\in L^1(\Omega)$, *then the following statements are equivalent:*

(1) *There exists a positive constant C_3 such that*

$$\Big(\int_{\{\tau<\infty\}} \exp E\,(\log(|f|^q)|\mathscr{F}_\tau)\,ud\mu\Big)^{\frac{1}{q}} \leqslant C_3\Big(\int_\Omega |f|^p v d\mu\Big)^{\frac{1}{p}},\ \forall \tau\in\mathscr{T},$$
$$f=(f_n)\in L^p(vd\mu); \tag{2.9}$$

(2) *There exists a positive constant C_4 such that*

$$\lambda|\{Gf>\lambda\}|_u^{\frac{1}{q}} \leqslant C_4\Big(\int_\Omega |f|^p v d\mu\Big)^{\frac{1}{p}},\ \forall\lambda>0,\ f=(f_n)\in L^p(vd\mu); \tag{2.10}$$

(3) *There exists a positive constant C_5 such that*

$$\Big(\int_B \exp E\left(\log(v^{-\frac{q}{p}})|\mathscr{F}_n\right) u_n d\mu\Big)^{\frac{1}{q}} \leqslant C_5|B|^{\frac{1}{p}},\ \forall B\in\mathscr{F}_n,\ n\in N. \tag{2.11}$$

Proof. We shall follow the scheme: $(2)\Leftrightarrow(1)\Leftrightarrow(3)$.

$(1)\Rightarrow(2)$. Let $f=(f_n)_{n\geqslant 0}\in L^p(v)$. For $\lambda>0$, define

$$\tau=\inf\{n:\ \exp E(\log|f||\mathscr{F}_n)>\lambda\}.$$

It follows from (2.9) that

$$\begin{aligned}\lambda|\{Gf>\lambda\}|_u^{\frac{1}{q}} &= (\int_{\{Gf>\lambda\}}\lambda^q u d\mu)^{\frac{1}{q}}\\ &\leqslant (\int_{\{\tau<\infty\}}\exp E(\log(|f|^q)|\mathscr{F}_\tau)ud\mu)^{\frac{1}{q}}\\ &\leqslant C_3^q\Big(\int_\Omega |f|^p v d\mu\Big)^{\frac{q}{p}}.\end{aligned}$$

Thus (2.10) is valid with $C_4=C_3$.

$(2)\Rightarrow(1)$. Fix $n\in N$ and $B\in\mathscr{F}_n$. For $f=(f_n)\in L^p(vd\mu)$, let $g=f\chi_B$, then $E\,(\log|g||\mathscr{F}_n)=E\,(\log(|f|\chi_B)|\mathscr{F}_n)$. Moreover

$$Gg\geqslant \exp E\,(\log(|f|\chi_B)|\mathscr{F}_n).$$

Combing with (2.10), we have

$$\begin{aligned}\lambda^q\int_{B\cap\{\exp E(\log|f||\mathscr{F}_n)>\lambda\}} ud\mu &\leqslant \lambda^q\int_{\{Gg>\lambda\}} ud\mu\\ &\leqslant C_4^q\Big(\int_\Omega |g|^p v d\mu\Big)^{\frac{q}{p}} = C_4^q\Big(\int_B |f|^p v d\mu\Big)^{\frac{q}{p}}.\end{aligned}$$

For $k\in Z$, let

$$B_k=\{2^k<\exp E\,(\log|f||\mathscr{F}_n)\leqslant 2^{k+1}\}.$$

Note that

$$\{2^k < \exp E\left(\log|f||\mathscr{F}_n\right) \leqslant 2^{k+1}\} \subseteq \{\exp E\left(\log|f||\mathscr{F}_n\right) > 2^k\},$$

then

$$\begin{aligned}
\int_\Omega \Big(\exp E\left(\log|f||\mathscr{F}_n\right)\Big)^q u\,d\mu &= \sum_{k\in Z}\int_{B_k}\Big(\exp E\left(\log|f||\mathscr{F}_n\right)\Big)^q u\,d\mu \\
&\leqslant 2^q \sum_{k\in Z}\int_{B_k\cap\{\exp E(\log|f||\mathscr{F}_n)>2^k\}} 2^{kq} u\,d\mu \\
&\leqslant 2^q C_4^q \sum_{k\in Z}\left(\int_{B_k}|f|^p v\,d\mu\right)^{\frac{q}{p}} \\
&\leqslant 2^q C_4^q \left(\sum_{k\in Z}\int_{B_k}|f|^p v\,d\mu\right)^{\frac{q}{p}} \\
&\leqslant 2^q C_4^q \left(\int_\Omega |f|^p v\,d\mu\right)^{\frac{q}{p}},
\end{aligned}$$

where we have used $1 \leqslant \frac{q}{p}$. As for $\tau \in \mathscr{T}$, it is easy to see that

$$\begin{aligned}
\int_{\{\tau<\infty\}} \exp E\left(\log(|f|^q)|\mathscr{F}_\tau\right) u\,d\mu &= \sum_{k\in Z}\int_{\{\tau=n\}} \exp E\left(\log(|f\chi_{\{\tau=n\}}|^q)|\mathscr{F}_n\right) u\,d\mu \\
&\leqslant 2^q C_4^q \sum_{k\in Z}\left(\int_\Omega |f\chi_{\{\tau=n\}}|^p v\,d\mu\right)^{\frac{q}{p}} \\
&\leqslant 2^q C_4^q \left(\sum_{k\in Z}\int_\Omega |f\chi_{\{\tau=n\}}|^p v\,d\mu\right)^{\frac{q}{p}} \\
&\leqslant 2^q C_4^q \Big(\int_\Omega |f|^p v\,d\mu\Big)^{\frac{q}{p}}.
\end{aligned}$$

Therefore,

$$\Big(\int_{\{\tau<\infty\}} \exp E\left(\log(|f|^q)|\mathscr{F}_\tau\right) u\,d\mu\Big)^{\frac{1}{q}} \leqslant C_3\Big(\int_\Omega |f|^p v\,d\mu\Big)^{\frac{1}{p}}$$

with $C_3 = 2C_4$.

$(1) \Rightarrow (3)$. Fix $n \in N$ and $B \in \mathscr{F}_n$. Substituting $f = v^{-\frac{1}{p}}\chi_B$ and $\tau \equiv n \in \mathscr{T}$ into inequality (2.9), we have

$$\Big(\int_B \exp E\left(\log(v^{-\frac{q}{p}})|\mathscr{F}_n\right) u\,d\mu\Big)^{\frac{1}{q}} \leqslant C_5|B|^{\frac{1}{p}},$$

where $C_5 = C_3$.

$(3) \Rightarrow (1)$. If $\tau \equiv n$ for some $n \in N$, we shall show that (2.9) is valid. Fix $f \in L^p(v)$. For each $k \in Z$ and $j \in Z$, let

$$B_{k,j} = \{2^k < \exp E(\log|f||\mathscr{F}_n) \leqslant 2^{k+1}\} \cap \{2^j < \exp E(\log(v^{-1})|\mathscr{F}_n) \leqslant 2^{j+1}\}.$$

Then $B_{k,j} \in \mathscr{F}_n$. Moreover, $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k < \exp E(\log|f||\mathscr{F}_n) \leqslant 2^{k+1}\} = \bigcup_{j \in Z} B_{k,j}.$$

Trivially,

$$\exp E(\log|f||\mathscr{F}_n) = \exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right) \exp E\left(\log(v^{-\frac{1}{p}})|\mathscr{F}_n\right).$$

It follows that

$$\begin{aligned} 2^{kq} &\leqslant \operatorname*{ess\,inf}_{B_{k,j}} |f_n|^q \\ &\leqslant \operatorname*{ess\,inf}_{B_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right)\right)^q \operatorname*{ess\,sup}_{B_{k,j}} \left(\exp E\left(\log(v^{-1})|\mathscr{F}_n\right)\right)^{\frac{q}{p}} \\ &\leqslant 2^{\frac{q}{p}} \operatorname*{ess\,inf}_{B_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right)\right)^q |B_{k,j}|_u^{-1} \int_{B_{k,j}} \left(\exp E\left(\log(v^{-1})|\mathscr{F}_n\right)\right)^{\frac{q}{p}} u\,d\mu. \end{aligned}$$

For the above τ, we estimate $\int_{\{\tau<\infty\}} \exp E(\log(|f|^q)|\mathscr{F}_\tau)\, u\,d\mu$. Note that

$$\begin{aligned} &\int_{\{\tau<\infty\}} \exp E(\log(|f|^q)|\mathscr{F}_\tau)\, u\,d\mu \\ &= \int_\Omega \exp E(\log(|f|^q)|\mathscr{F}_n)\, u\,d\mu \leqslant 2^q \sum_{k\in Z, j\in Z} \int_{B_{k,j}} 2^{kq} u\,d\mu \\ &\leqslant 2^q 2^{\frac{q}{p}} \sum_{k\in Z, j\in Z} \operatorname*{ess\,inf}_{B_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right)\right)^q \int_{B_{k,j}} \left(\exp E\left(\log(v^{-1})|\mathscr{F}_n\right)\right)^{\frac{q}{p}} u\,d\mu. \end{aligned}$$

Following (2.11), we have

$$\begin{aligned} &\int_{\{\tau<\infty\}} \exp E(\log(|f|^q)|\mathscr{F}_\tau)\, u\,d\mu \\ &\qquad\qquad \leqslant 2^q 2^{\frac{q}{p}} C_5^q \sum_{k\in Z, j\in Z} \operatorname*{ess\,inf}_{B_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right)\right)^q |B_{k,j}|^{\frac{q}{p}}. \end{aligned}$$

It is obvious that $\vartheta(k,j) = |B_{k,j}|^{\frac{q}{p}}$ is a measure on $X = Z^2$. For the above $f \in L^p(v\,d\mu)$, denote

$$Tf(k,j) = \operatorname*{ess\,inf}_{B_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right)\right)^q,$$

$$E_\lambda = \{(k,j):\ \operatorname*{ess\,inf}_{B_{k,j}} \left(\exp E\left(\log|fv^{\frac{1}{p}}||\mathscr{F}_n\right)\right)^q > \lambda\}$$

and

$$G_\lambda = \bigcup_{(k,j)\in E_\lambda} B_{k,j} \text{ for each } \lambda > 0.$$

Thus

$$|\{Tf>\lambda\}|_\vartheta = \sum_{(k,j)\in E_\lambda} |B_{k,j}|^{\frac{q}{p}} \leqslant (\sum_{(k,j)\in E_\lambda} |B_{k,j}|)^{\frac{q}{p}} = |G_\lambda|^{\frac{q}{p}} \leqslant \left|\left\{\left(G(fv^{\frac{1}{p}})\right)^q > \lambda\right\}\right|^{\frac{q}{p}}.$$

In the same way of (2.3) and (2.4), we get

$$\left(\int_{\{\tau<\infty\}} \exp E\left(\log(|f|^q)|\mathscr{F}_\tau\right) u d\mu\right)^{\frac{1}{q}} \leqslant C_3 \left(\int_\Omega |f|^p v d\mu\right)^{\frac{1}{p}}$$

with $C_3 = 2\cdot 2^{\frac{1}{p}} p' 2^{\frac{1}{q}} (2^{\frac{p}{q}}-1)^{-\frac{1}{p}} C_5$.

If $\tau \in \mathscr{T}$ is arbitrary, as we have done in the process of $(2) \Rightarrow (1)$, (2.9) is still valid. □

COROLLARY 2.5. *Let (u,v) be a couple of weights and $1<p<\infty$. Suppose that $v^{-\frac{1}{p-1}} \in L^1(\Omega)$, then the following statements are equivalent:*

(1) *There exists a positive constant C_{31} such that*

$$\int_{\{\tau<\infty\}} \exp E\left(\log(|f|^p)|\mathscr{F}_\tau\right) u d\mu \leqslant C_{31} \int_\Omega |f|^p v d\mu, \ \forall \tau \in \mathscr{T},\ f=(f_n) \in L^p(v d\mu); \tag{2.12}$$

(2) *There exists a positive constant C_{41} such that*

$$\lambda |\{(Gf)^p > \lambda\}|_u \leqslant C_{41} \int_\Omega |f|^p v d\mu, \ \forall \lambda > 0,\ f=(f_n)\in L^p(v d\mu); \tag{2.13}$$

(3) *There exists a positive constant C_{51} such that*

$$\exp E\left(\log(v^{-1})|\mathscr{F}_n\right) u_n \leqslant C_{51}, \ \forall n \in N. \tag{2.14}$$

COROLLARY 2.6. *Let (u,v) be a couple of weights. Suppose that $v^{-1} \in L^\infty(\Omega)$, then the following statements are equivalent:*

(1) *There exists a positive constant C_{32} such that*

$$\int_{\{\tau<\infty\}} \exp E\left(\log|f||\mathscr{F}_\tau\right) u d\mu \leqslant C_{32}\left(\int_\Omega |f| v d\mu\right), \ \forall \tau \in \mathscr{T},\ f=(f_n) \in L^1(v d\mu); \tag{2.15}$$

(2) *There exists a positive constant C_{42} such that*

$$\lambda |\{Gf > \lambda\}|_u \leqslant C_{42} \int_\Omega |f| v d\mu, \ \forall \lambda > 0,\ f=(f_n)\in L^1(v d\mu); \tag{2.16}$$

(3) *There exists a positive constant C_{52} such that*

$$\exp E\left(\log(v^{-1})|\mathscr{F}_n\right) u_n \leqslant C_{52}, \ \forall n \in N. \tag{2.17}$$

Proofs of Corollary 2.5 and Corollary 2.6 are evident and we omit them.

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