Mathematical
Inequalities
& Applications
Volume 14, Number 1 (2011), 79–92

SOME GENERALIZATIONS FOR OPIAL'S INEQUALITY INVOLVING SEVERAL FUNCTIONS AND THEIR DERIVATIVES OF ARBITRARY ORDER ON ARBITRARY TIME SCALES

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(*Communicated by M. Bohner*)

Abstract. In this paper, some various types of Opial's inequality involving several functions and their higher-order derivatives are presented on arbitrary time scales. The well-known Muirhead's inequality is employed to obtain very interesting results. While dealing with higher-order derivatives, the generalized Taylor's formula and the generalized polynomials are used to simplify our proofs too. Our new results generalize and extend the existing results in the literature, and some of the works done by Pachpatte. Moreover, our results are not only new for arbitrary time scales, but also new for the continuous and the discrete cases.

1. Introduction

Opial type inequalities have many applications in the theories of differential and difference equations, for instance such inequalities are used to prove existence of solutions (see [1, § 2.27 and § 2.28]). The readers may find very interesting results about continuous and discrete versions of Opial's inequality in the book [1], which is a very nice collection of the most popular articles on this subject. In this paper, we will study Opial's inequality on arbitrary time scales, which unifies and extends continuous and discrete calculus (see [7]). The readers are referred to the book [3] for fundamentals of the time scale theory. Thus, our results not only cover the continuous and the discrete cases, but also cover and improve the new results on time scales in [4]. For some classical results on Opial's inequality, the readers are referred to the articles [2, 5, 6, 8, 9, 10, 12, 13, 14, 15]. In the literature, one can find so many results on the continuous and/or the discrete cases. However, to the best of knowledge, [4] is the only paper that succeeds to unify some of these results by the means of the time scale theory. In this paper, after a long period of time, we succeed to extend, generalize and improve most results of the papers stated above.

Now, we give some of the easiest versions of the Opial's inequality as follows:

Mathematics subject classification (2010): 34A40, 39A13, 26D15.
Keywords and phrases: Muirhead's inequality; Opial's inequality; Taylor's formula; time scales; weights.

THEOREM A. Continuous Opial's inequality, see [1, 15] *For* $f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{R}},\mathbb{R})$ *with* $f(a)=0$*, we have*

$$\int_a^b \left|f(\xi)f'(\xi)\right|\mathrm{d}\xi \leqslant \frac{b-a}{2}\int_a^b \left[f'(\xi)\right]^2\mathrm{d}\xi$$

with equality $f(t)=c(t-a)$ *for* $t\in[a,b]_{\mathbb{R}}$*, where* c *is a constant.*

The discrete version of the above result due to Lasota reads as follows:

THEOREM B. Discrete Opial's inequality, see [1, 9] *For* f *be a sequence defined on* $[a,b]_{\mathbb{Z}}$ *with* $f(a)=0$*, we have*

$$\sum_{\xi=a}^{b-1}\left|[f(\xi)+f(\xi+1)]\Delta f(\xi)\right| \leqslant (b-a)\sum_{\xi=a}^{b-1}\left[\Delta f(\xi)\right]^2$$

with equality $f(t)=c(t-a)$ *for* $t\in[a,b]_{\mathbb{Z}}$*, where* c *is a constant.*

The time scale unification of the above two results due to Bohner and Kaymakçalan is quoted below.

THEOREM C. Dynamic Opial's inequality, see [4] *For* $f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ *with* $f(a)=0$*, we have*

$$\int_a^b \left|[f(\xi)+f^{\sigma}(\xi)]f^{\Delta}(\xi)\right|\Delta\xi \leqslant (b-a)\int_a^b \left[f^{\Delta}(\xi)\right]^2\Delta\xi$$

with equality $f(t)=c(t-a)$ *for* $t\in[a,b]_{\mathbb{T}}$*, where* c *is a constant.*

Some very interesting generalizations are given by Pachpatte, who works with two functions in Opial type inequalities (see [1, 12, 13, 14]). We quote two important generalizations of Pachpatte's below. First, we give the continuous case.

THEOREM AA. See [1, 12] *For* $f,g \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{R}},\mathbb{R})$ *with* $f(a)=g(a)=0$*, we have*

$$\int_a^b \Big[\left|f(\xi)g'(\xi)\right|+\left|g(\xi)f'(\xi)\right|\Big]\mathrm{d}\xi \leqslant \frac{b-a}{2}\int_a^b \Big[\left[f'(\xi)\right]^2+\left[g'(\xi)\right]^2\Big]\mathrm{d}\xi$$

with equality $f(t)=g(t)=c(t-a)$ *for* $t\in[a,b]_{\mathbb{R}}$*, where* c *is a constant.*

One can easily see that Theorem AA reduces to Theorem A by letting $f=g$.
And next, we give the following result which can be regarded as the discrete corresponding of Theorem AA.

THEOREM BB. See [1, 13, 14] *For* f,g *two sequences defined for* $t\in[a,b]_{\mathbb{Z}}$ *with* $f(a)=g(a)=0$*, we have*

$$\sum_{\xi=a}^{b-1}\Big[\left|f(\xi)\Delta g(\xi)\right|+\left|g(\xi+1)\Delta f(\xi)\right|\Big] \leqslant \frac{b-a}{2}\sum_{\xi=a}^{b-1}\Big[\left[\Delta f(\xi)\right]^2+\left[\Delta g(\xi)\right]^2\Big]$$

with equality $f(t) = g(t) = c(t-a)$ *for* $t \in [a,b]_{\mathbb{Z}}$, *where* c *is a constant.*

Letting $f = g$, one can see that Theorem BB reduces to Theorem B provided that f is of fixed sign. In the case that f alternates in sign, the left-hand side of the inequality in Theorem BB is not less than the left-hand side of the inequality in Theorem B, this fact follows from the following simple inequality:

$$|\alpha+\beta| \leqslant |\alpha|+|\beta| \quad \text{for } \alpha,\beta \in \mathbb{R}.$$

It is not hard to check that the inequality given above holds with equality provided that at least one of α and β is 0 or they have same signs. This reason shows us why Theorem BB reduces to Theorem B when f is of fixed sign.

The motivation of this paper comes from the works done by Pachpatte, we extend and generalize his key idea to several functions and arbitrary time scales. Since the dynamic unification of Theorem A and Theorem B is given in Theorem C, it is natural to ask the question "Is it possible to give a dynamic generalization to Theorem AA and Theorem BB?". In this paper, we shall give the expected affirmative answer for the question stated above with an important application of the generalized Taylor's formula on time scales. We not only generalize the results by Pachpatte, but also improve some of the very important results about this subject. Unlike to the earlier results due to Pachpatte, we will consider several functions instead of two, and their (Delta) derivatives of higher-order.

Now, we state Muirhead's inequality, which allows us to obtain new interesting results.

THEOREM D. See [11, pp. 338] *Let* S^n *be the symmetry group of the set* $[1,n]_{\mathbb{N}}$, *and* $A := (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $B := (\beta_1, \beta_2, \ldots, \beta_n)$ *be two vectors with nonnegative entries and* $\sum_{j=1}^{k} \alpha_j \geqslant \sum_{j=1}^{k} \beta_j$ *for all* $k \in [1,n-1]_{\mathbb{N}}$ *and* $\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j$, *then it is said that* A *majorizes* B *(we prefer the notation* $A \triangleright B$*), and the following inequality is true:*

$$\sum_{\pi\in S^n} \prod_{j=1}^{n} x_{\pi_j}^{\alpha_j} \geqslant \sum_{\pi\in S^n} \prod_{j=1}^{n} x_{\pi_j}^{\beta_j},$$

where π_j *denotes the* j*-th component of the permutation* π, *and* $x_j \in \mathbb{R}_0^+$ *holds for all* $j \in [1,n]_{\mathbb{N}}$.

One can easily see that for $(2,0) \triangleright (1,1)$ Theorem D gives us the following well-known inequality

$$x_1^2 + x_2^2 \geqslant 2x_1x_2 \quad \text{with} \quad x_1, x_2 \geqslant 0.$$

This inequality gives us the well-known inequality between arithmetic and geometric means by letting $y_1 := 2x_1^2$ and $y_2 := 2x_2^2$, i.e.,

$$\frac{y_1+y_2}{2} \geqslant \sqrt{y_1y_2}.$$

This paper is arranged as follows: in § 2, we give some preliminaries adapted from [3]; in § 3, we start to state our main results on generalizations of Opial's inequality, where we make a very nice application of the generalized Taylor's formula on time scales; in § 4, we consider weighted cases of the results in § 3; and finally in § 5, before we close, we give some remarks about our results. Throughout the paper, for convenience, the empty sum and the empty product are assumed to be 0 and 1, respectively, i.e., for $\alpha,\beta \in \mathbb{Z}$ with $\beta < \alpha$, $\sum_{j=\alpha}^{\beta} f_j = 0$ and $\prod_{j=\alpha}^{\beta} f_j = 1$.

2. Time scales essentials

Here, for completeness in the paper, we quote some definitions and results from [3], which will be applied in our proofs.

DEFINITION 2.1. A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

DEFINITION 2.2. On an arbitrary time scale $\mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t,\infty)_{\mathbb{T}}$ for $t \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty,t)_{\mathbb{T}}$ for $t \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$. For convenience, we set $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$.

DEFINITION 2.3. Let t be a point in $\mathbb{T}$. If $\sigma(t) = t$ holds, then t is called *right-dense*, otherwise ($\sigma(t) > t$) it is called *right-scattered*. Similarly, if $\rho(t) = t$ holds, then t is called *left-dense*, a point which is not left-dense ($\rho(t) < t$) is called *left-scattered*.

DEFINITION 2.4. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided that it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exists (finite) at left-dense points of $\mathbb{T}$. The set of rd-continuous functions is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, and $\mathrm{C}_{\mathrm{rd}}^1(\mathbb{T},\mathbb{R})$ denotes the set of functions of which delta derivative belongs to $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

Now, we give the following formula for differentiation of several functions.

THEOREM 2.1. (See [3, Exercise 1.22]) *Let $n \in \mathbb{N}$ and $f_j : \mathbb{T} \to \mathbb{R}$ be differentiable functions for $j \in [1,n]_{\mathbb{N}}$, then we have*

$$\left[\prod_{j=1}^{n} f_j(t)\right]^{\Delta} = \sum_{j=1}^{n} \left\{ \left[\prod_{i=1}^{j-1} f_i^{\sigma}(t)\right] f_j^{\Delta}(t) \left[\prod_{i=j+1}^{n} f_i(t)\right] \right\} \quad \text{for } t \in \mathbb{T}.$$

THEOREM 2.2. (Existence of antiderivatives [3, Theorem 1.74]) *Let f be a rd-continuous function. Then f has an antiderivative F such that $F^{\Delta} = f$ holds.*

DEFINITION 2.5. If $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and $a \in \mathbb{T}$, then we define the *integral* by

$$F(t) := \int_a^t f(\xi)\Delta\xi \quad \text{for } t \in \mathbb{T}.$$

THEOREM 2.3. ([3, Theorem 1.77]) *Let f,g be rd-continuous functions, $a,b,c \in \mathbb{T}$ and $\alpha,\beta \in \mathbb{R}$. Then, the following statements are true:*

1. $\int_a^b \left[\alpha f(\xi)+\beta g(\xi)\right]\Delta\xi = \alpha\int_a^b f(\xi)\Delta\xi+\beta\int_a^b g(\xi)\Delta\xi$,
2. $\int_a^b f(\xi)\Delta\xi = -\int_b^a f(\xi)\Delta\xi$,
3. $\int_a^c f(\xi)\Delta\xi = \int_a^b f(\xi)\Delta\xi+\int_b^c f(\xi)\Delta\xi$,
4. $\int_a^b f(\xi)g^{\Delta}(\xi)\Delta\xi = f(b)g(b)-f(a)g(a)-\int_a^b f^{\Delta}(\xi)g(\sigma(\xi))\Delta\xi$.

Now, we give the definition of the generalized polynomials as follows:

$$h_n(t,s) := \begin{cases} 1, & n=0 \\ \displaystyle\int_s^t h_{n-1}(\xi,s)\Delta\xi, & n\in\mathbb{N} \end{cases}$$

and

$$g_n(t,s) := \begin{cases} 1, & n=0 \\ \displaystyle\int_s^t g_{n-1}(\sigma(\xi),s)\Delta\xi, & n\in\mathbb{N} \end{cases}$$

for all $s,t\in\mathbb{T}$.

The readers can find very useful and interesting results on the generalized polynomials in [3, § 1.6].

PROPERTY 2.1. *Using induction it is easy to see that* $h_n(t,s)\geqslant 0$ *holds for all* $n\in\mathbb{N}$ *and* $s,t\in\mathbb{T}$ *with* $t\geqslant s$ *and* $(-1)^n h_n(t,s)\geqslant 0$ *holds for all* $n\in\mathbb{N}$ *and* $s,t\in\mathbb{T}$ *with* $t\leqslant s$. *Moreover,* $h_n(t,s)$ *is increasing with respect to its first component for all* $t\geqslant s$.

The following result can be inferred from [3, Lemma 1.119].

LEMMA 2.1. *For* $n\in\mathbb{N}$ *and* $t\in\mathbb{T}$, *we have* $g_n(t,s)=0$ *for all* $s\in[\rho^{n-1}(t),t]_{\mathbb{T}}$.

The following result is found in [3, Theorem 1.112], which will be needed in our proofs.

LEMMA 2.2. *For* $n\in\mathbb{N}$, $t\in\mathbb{T}$ *and* $s\in\mathbb{T}^{\kappa^n}$, *we have* $h_n(t,s)=(-1)^n g_n(s,t)$.

By Lemma 2.1 and Lemma 2.3, we can give the following result.

LEMMA 2.3. *For* $n\in\mathbb{N}$ *and* $t\in\mathbb{T}$, *we have* $h_n(t,s)=0$ *for all* $s\in[\rho^{n-1}(t),t]_{\mathbb{T}}$.

Now, we quote the following well-known result.

THEOREM 2.4. (Taylor's formula [3, Theorem 1.113]) *Let* $n\in\mathbb{N}$ *and* $f\in \mathrm{C}^n_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *be an* n *times differentiable function. For* $s\in\mathbb{T}^{\kappa^{n-1}}$, *we have*

$$f(t)=\sum_{j=0}^{n-1} h_j(t,s)f^{\Delta^j}(s)+\int_s^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\xi))f^{\Delta^n}(\xi)\Delta\xi \quad \textit{for all } t\in\mathbb{T}.$$

Also, we need the following integral inequality.

LEMMA 2.4. (Hölder's inequality [3, Theorem 6.13]) *Let* $a,b \in \mathbb{T}$. *For* $f,g \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, *we have*

$$\int_a^b |f(\xi)g(\xi)|\Delta\xi \leqslant \left(\int_a^b |f(\xi)|^p \Delta\xi\right)^{\frac{1}{p}} \left(\int_a^b |g(\xi)|^q \Delta\xi\right)^{\frac{1}{q}},$$

where $p>1$ *and* $1/p+1/q=1$.

3. Generalizations of Opial's inequality on time scales

In this section, we give our generalized Opial's inequalities, which also generalize some of the important works done by Pachpatte.

THEOREM 3.1. *Let* $n \in \mathbb{N}$, $a,b \in \mathbb{T}$ *and* $f_j \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *for all* $j \in [1,n+1]_{\mathbb{N}}$ *with* $f_j(a)=0$ *for all* $j \in [1,n+1]_{\mathbb{N}}$. *Then, the following inequality holds*

$$\int_a^b \sum_{j=1}^{n+1} \left| \left[\prod_{i=1}^{j-1} f_i^{\sigma}(\xi)\right] f_j^{\Delta}(\xi) \left[\prod_{i=j+1}^{n+1} f_i(\xi)\right] \right| \Delta\xi \leqslant \frac{(b-a)^n}{n+1} \int_a^b \sum_{j=1}^{n+1} |f_j^{\Delta}(\xi)|^{n+1} \Delta\xi \quad (1)$$

with equality $f_j(t)=c(t-a)$ *for all* $j \in [1,n+1]_{\mathbb{N}}$, *where* c *is a constant.*

Proof. The proof of this theorem can be done by following very similar steps to that of the following one, and we skip it here. □

REMARK 3.1. Theorem 3.1 reduces to Theorem AA and Theorem BB by letting $n=1$, $\mathbb{T}=\mathbb{R}$, and $n=1$, $\mathbb{T}=\mathbb{Z}$, respectively. Also, note that Theorem AA and Theorem BB generalize Theorem A and Theorem B, respectively. Theorem 3.1 includes Theorem C with $n=1$ and $f_1=f_2$ for arbitrary time scales.

THEOREM 3.2. *Let* $m,n \in \mathbb{N}$, $a,b \in \mathbb{T}$ *and* $f_j \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *with* $f_j(a)=0$ *for all* $j \in [1,n+1]_{\mathbb{N}}$. *Then, the following inequality holds*

$$\int_a^b \sum_{j=0}^{m} \left[\prod_{i=1}^{n+1} f_i^{\sigma}(\xi)\right]^j \left[\prod_{i=1}^{n+1} f_i(\xi)\right]^{m-j} \left| \sum_{j=1}^{n+1} \left| \left[\prod_{i=1}^{j-1} f_i^{\sigma}(\xi)\right] f_j^{\Delta}(\xi) \left[\prod_{i=j+1}^{n+1} f_i(\xi)\right] \right| \right| \Delta\xi$$
$$\leqslant \frac{(b-a)^{(n+1)(m+1)-1}}{n+1} \int_a^b \left[\sum_{j=1}^{n+1} |f_j^{\Delta}(\xi)|\right]^{(n+1)(m+1)} \Delta\xi \quad (2)$$

with equality $f_j(t)=c(t-a)$ *for all* $j \in [1,n+1]_{\mathbb{N}}$, *where* c *is a constant.*

Proof. Set $F_j(t) := \int_a^t |f_j^\Delta(\xi)|\Delta\xi$ for $t \in [a,b]_\mathbb{T}$ and all $j \in [1,n+1]_\mathbb{N}$. Then, on $[a,b]_\mathbb{T}$, we have $F_j^\Delta = |f_j^\Delta|$ and $F_j \geqslant |f_j|$ for all $j \in [1,n+1]_\mathbb{N}$. Now, set $F(t) := \prod_{j=1}^{n+1} F_j(t)$ for $t \in [a,b]_\mathbb{T}$. It follows that

$$\begin{aligned}
&\int_a^b \sum_{j=0}^{m} \left|\left[\prod_{i=1}^{n+1} f_i^\sigma(\xi)\right]^j \left[\prod_{i=1}^{n+1} f_i(\xi)\right]^{m-j}\right| \left|\sum_{j=1}^{n+1}\left[\prod_{i=1}^{j-1} f_i^\sigma(\xi)\right] f_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1} f_i(\xi)\right]\right|\Delta\xi \\
\leqslant &\int_a^b \sum_{j=0}^{m} \left[\prod_{i=1}^{n+1} F_i^\sigma(\xi)\right]^j \left[\prod_{i=1}^{n+1} F_i(\xi)\right]^{m-j} \sum_{j=1}^{n+1}\left\{\left[\prod_{i=1}^{j-1} F_i^\sigma(\xi)\right] F_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1} F_i(\xi)\right]\right\}\Delta\xi \\
= &\int_a^b \sum_{j=0}^{m} \left[F^\sigma(\xi)\right]^j \left[F(\xi)\right]^{m-j} F^\Delta(\xi)\Delta\xi = \int_a^b \left[\left[F(\xi)\right]^{(m+1)}\right]^\Delta \Delta\xi \\
= &\left[F(b)\right]^{m+1} = \prod_{j=1}^{n+1}\left[F_j(b)\right]^{m+1} \leqslant \frac{1}{n+1}\sum_{j=1}^{n+1}\left[F_j(b)\right]^{(n+1)(m+1)}
\end{aligned} \tag{3}$$

is true, where we have applied the arithmetic mean and geometric mean inequalities at the last step. On the other hand, for $j \in [1,n+1]_\mathbb{N}$, we have

$$\begin{aligned}
\left[F_j(b)\right]^{(n+1)(m+1)} &= \left(\int_a^b |f_j^\Delta(\xi)|\Delta\xi\right)^{(n+1)(m+1)} \\
&\leqslant (b-a)^{(n+1)(m+1)-1}\int_a^b |f_j^\Delta(\xi)|^{(n+1)(m+1)}\Delta\xi
\end{aligned} \tag{4}$$

by applying Hölder's inequality. Considering (4) in (3), we get (2).

On the other hand, by letting $f_j(t) = c(t-a)$ for $t \in [a,b]_\mathbb{T}$ and all $j \in [1,n+1]_\mathbb{N}$ for some constant c, one can easily see that (2) holds with equality. The proof is hence completed. □

REMARK 3.2. By letting $n = 1$, $\mathbb{T} = \mathbb{R}$, Theorem 3.2 reduces to [12, Theorem 4], which generalizes some results in [8]. To the best our knowledge, this result for $\mathbb{T} \neq \mathbb{R}$ is new.

THEOREM 3.3. *Let $\ell, n \in \mathbb{N}$, $a, b \in \mathbb{T}$ and $f_j \in \mathrm{C}_{\mathrm{rd}}^\ell(\mathbb{T},\mathbb{R})$ with $f_j^{\Delta^i}(a) = 0$ for all $j \in [1,n+1]_\mathbb{N}$ and all $i \in [0,\ell)_{\mathbb{N}_0}$. Then, the following inequality holds*

$$\begin{aligned}
&\int_a^b \sum_{j=1}^{n+1}\left|\left[\prod_{i=1}^{j-1} f_i^\sigma(\xi)\right] f_j^{\Delta^\ell}(\xi)\left[\prod_{i=j+1}^{n+1} f_i(\xi)\right]\right|\Delta\xi \\
\leqslant &\left(\frac{1}{n+1}\int_a^b\left(\int_a^{\sigma(\xi)}\left[h_{\ell-1}(\sigma(\xi),\sigma(\zeta))\right]^2\Delta\zeta\right)^n\Delta\xi\right)^{\frac{1}{2}} \sum_{j=1}^{n+1}\left(\int_a^b\left[f_j^{\Delta^\ell}(\xi)\right]^2\Delta\xi\right)^{\frac{n+1}{2}}.
\end{aligned}$$

Proof. By the initial values, Lemma 2.3 and Taylor's formula, we have

$$f_j(t)=\int_a^t h_{\ell-1}(t,\sigma(\xi))f_j^{\Delta^\ell}(\xi)\Delta\xi \quad \text{for all } t\in[a,b]_{\mathbb{T}} \text{ and all } j\in[0,n+1]_{\mathbb{N}}. \tag{5}$$

Now, set

$$F_j(t):=\int_a^t \left[f_j^{\Delta^\ell}(\xi)\right]^2\Delta\xi \quad \text{for all } t\in[a,b]_{\mathbb{T}} \text{ and all } j\in[0,n+1]_{\mathbb{N}}. \tag{6}$$

Then, from Property 2.1, Hölder's inequality and (5), we have

$$\begin{aligned}
&\left|\left[\prod_{i=1}^{j-1}f_i^\sigma(t)\right]f_j^{\Delta^\ell}(t)\left[\prod_{i=j+1}^{n+1}f_i(t)\right]\right|\\
=&\left|\left[\prod_{i=1}^{j-1}\int_a^{\sigma(t)}h_{\ell-1}(\sigma(t),\sigma(\xi))f_i^{\Delta^\ell}(\xi)\Delta\xi\right]\left[F_j^\Delta(t)\right]^{\frac{1}{2}}\left[\prod_{i=j+1}^{n+1}\int_a^t h_{\ell-1}(t,\sigma(\xi))f_i^{\Delta^\ell}(\xi)\Delta\xi\right]\right|\\
\leqslant& H(t)\left(\left[\prod_{i=1}^{j-1}F_i^\sigma(t)\right]F_j^\Delta(t)\left[\prod_{i=j+1}^{n+1}F_i(t)\right]\right)^{\frac{1}{2}}
\end{aligned} \tag{7}$$

for all $j\in[1,n+1]_{\mathbb{N}}$, where

$$H(t):=\left(\int_a^{\sigma(t)}\left[h_{\ell-1}(\sigma(t),\sigma(\xi))\right]^2\Delta\xi\right)^{\frac{n}{2}} \quad \text{for all } t\in[a,b]_{\mathbb{T}}. \tag{8}$$

Integrating (7) from a to b and applying Hölder's inequality, we get

$$\begin{aligned}
&\int_a^b\left|\left[\prod_{i=1}^{j-1}f_i^\sigma(\xi)\right]f_j^{\Delta^\ell}(\xi)\left[\prod_{i=j+1}^{n+1}f_i(\xi)\right]\right|\Delta\xi\\
\leqslant&\left(\int_a^b\left[H(\xi)\right]^2\Delta\xi\right)^{\frac{1}{2}}\left(\int_a^b\left[\prod_{i=1}^{j-1}F_i^\sigma(\xi)\right]F_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1}F_i(\xi)\right]\Delta\xi\right)^{\frac{1}{2}}.
\end{aligned}$$

Then, summing the resulting inequality over $j\in[1,n+1]_{\mathbb{N}}$, we get

$$\begin{aligned}
&\int_a^b\sum_{j=1}^{n+1}\left|\left[\prod_{i=1}^{j-1}f_i^\sigma(\xi)\right]f_j^{\Delta^\ell}(\xi)\left[\prod_{i=j+1}^{n+1}f_i(\xi)\right]\right|\Delta\xi\\
\leqslant&\left(\int_a^b\left[H(\xi)\right]^2\Delta\xi\right)^{\frac{1}{2}}\sum_{j=1}^{n+1}\left(\int_a^b\left[\prod_{i=1}^{j-1}F_i^\sigma(\xi)\right]F_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1}F_i(\xi)\right]\Delta\xi\right)^{\frac{1}{2}}\\
\leqslant&\left(\int_a^b\left[H(\xi)\right]^2\Delta\xi\right)^{\frac{1}{2}}\left((n+1)\sum_{j=1}^{n+1}\int_a^b\left[\prod_{i=1}^{j-1}F_i^\sigma(\xi)\right]F_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1}F_i(\xi)\right]\Delta\xi\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&=\left((n+1)\int_a^b\Big[H(\xi)\Big]^2\Delta\xi\right)^{\frac{1}{2}}\left(\int_a^b\left[\prod_{j=1}^{n+1}F_j(\xi)\right]^{\Delta}\Delta\xi\right)^{\frac{1}{2}}\\
&=\left((n+1)\int_a^b\Big[H(\xi)\Big]^2\Delta\xi\right)^{\frac{1}{2}}\left(\prod_{j=1}^{n+1}F_j(b)\right)^{\frac{1}{2}}\\
&\leqslant\left(\frac{1}{n+1}\int_a^b\Big[H(\xi)\Big]^2\Delta\xi\right)^{\frac{1}{2}}\sum_{j=1}^{n+1}\Big[F_j(b)\Big]^{\frac{n+1}{2}},
\end{aligned}
\tag{9}
$$

where the elementary inequalities

$$
\sum_{j=1}^{n+1}\lambda_j^{\frac{1}{2}}\leqslant\left((n+1)\sum_{j=1}^{n+1}\lambda_j\right)^{\frac{1}{2}}\quad\text{and}\quad\left(\prod_{j=1}^{n+1}\lambda_j\right)^{\frac{1}{2}}\leqslant\frac{1}{n+1}\sum_{j=1}^{n+1}\lambda_j^{\frac{n+1}{2}}
$$
$$
\text{for all } \lambda_j\in\mathbb{R}_0^+ \text{ and } j\in[1,n+1]_{\mathbb{N}}
$$

are employed while passing from the second step to the third one, and passing to the last step, respectively. Obviously, from (6) and (8), we see that (9) is the desired inequality, and the proof is hence completed. □

THEOREM 3.4. *Let $\ell,m,n\in\mathbb{N}$, $a,b\in\mathbb{T}$ and $f_j\in \mathrm{C}_{\mathrm{rd}}^{\ell}(\mathbb{T},\mathbb{R})$ with $f_j^{\Delta^i}(a)=0$ for all $j\in[1,n+1]_{\mathbb{N}}$ and all $i\in[0,\ell)_{\mathbb{N}}$. Then, the following inequality holds*

$$
\begin{aligned}
&\int_a^b\sum_{j=0}^{m}\left|\left[\prod_{i=1}^{n+1}f_i^{\sigma}(\xi)\right]^j\left[\prod_{i=1}^{n+1}f_i(\xi)\right]^{m-j}\right|\sum_{j=1}^{n+1}\left|\left[\prod_{i=1}^{j-1}f_i^{\sigma}(\xi)\right]f_j^{\Delta^\ell}(\xi)\left[\prod_{i=j+1}^{n+1}f_i(\xi)\right]\right|\Delta\xi\\
&\leqslant\left(\frac{m+1}{n+1}\int_a^b\left(\int_a^{\sigma(\xi)}\Big[h_{\ell-1}(\sigma(\xi),\sigma(\zeta))\Big]^2\Delta\zeta\right)^{m+n}\Delta\xi\right)^{\frac{1}{2}}\\
&\quad\times\sum_{j=1}^{n+1}\left(\int_a^b\Big[f_j^{\Delta^\ell}(\xi)\Big]^2\Delta\xi\right)^{\frac{(m+1)(n+1)}{2}}.
\end{aligned}
$$

Proof. It is very easy to prove this theorem by following very similar steps to the proofs of Theorem 3.2 and Theorem 3.3. □

4. Weighted cases

In this section, we give weighted cases of the results given in § 3.

THEOREM 4.1. *Let $n\in\mathbb{N}$, $a,b\in\mathbb{T}$ and $f_j\in\mathrm{C}_{\mathrm{rd}}^1(\mathbb{T},\mathbb{R})$ with $f_j(a)=0$ for all $j\in[1,n+1]_{\mathbb{N}}$, and that $p\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+)$ with $\int_a^b\big[p(\xi)\big]^{-1/n}\Delta\xi<\infty$ and $q\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}_0^+)$*

be a nonincreasing function. Then, the following inequality holds

$$\begin{aligned}&\int_a^b q^\sigma(\xi)\sum_{j=1}^{n+1}\left|\left[\prod_{i=1}^{j-1} f_i^\sigma(\xi)\right] f_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1} f_i(\xi)\right]\right|\Delta\xi \\ &\leqslant \frac{1}{n+1}\left(\int_a^b \frac{1}{\left[p(\xi)\right]^{\frac{1}{n}}}\Delta\xi\right)^n \left(\int_a^b p(\xi)q^\sigma(\xi)\sum_{j=1}^{n+1}\left|f_j^\Delta(\xi)\right|^{n+1}\Delta\xi\right).\end{aligned} \tag{10}$$

Proof. First, we set

$$F_j(t) := \int_a^t \left[q^\sigma(\xi)\right]^{\frac{1}{n+1}}\left|f_j^\Delta(\xi)\right|\Delta\xi, \quad \text{then } F_j^\Delta(t) = \left[q^\sigma(t)\right]^{\frac{1}{n+1}}\left|f_j^\Delta(t)\right|$$

for $t\in[a,b]_{\mathbb{T}}$ and all $j\in[1,n+1]_{\mathbb{N}}$. Then we have

$$F_j(t) \geqslant q^{\frac{1}{n+1}}(t)\int_a^t |f_j^\Delta(\xi)|\Delta\xi \geqslant q^{\frac{1}{n+1}}(t)\left|\int_a^t f_j^\Delta(\xi)\Delta\xi\right| \geqslant q^{\frac{1}{n+1}}(t)\left|f_j(t)\right|$$

for $t\in[a,b]_{\mathbb{T}}$ and all $j\in[1,n+1]_{\mathbb{N}}$ (note here that for any $t\in[a,b]_{\mathbb{T}}$, $\xi\in[a,t)_{\mathbb{T}}$ implies $\sigma(\xi)\leqslant t$ and thus $q^\sigma(\xi)\leqslant q(t)$). Now, set $F(t) := \prod_{j=1}^{n+1} F_j(t)$ for $t\in[a,b]_{\mathbb{T}}$. It follows very similar to the proof of Theorem 3.2 that

$$\begin{aligned}&\int_a^b q^\sigma(\xi)\sum_{j=1}^{n+1}\left|\left[\prod_{i=1}^{j-1} f_i^\sigma(\xi)\right] f_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1} f_i(\xi)\right]\right|\Delta\xi \\ &\leqslant \int_a^b \sum_{j=1}^{n+1}\left\{\left[\prod_{i=1}^{j-1} F_i^\sigma(\xi)\right] F_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1} F_i(\xi)\right]\right\}\Delta\xi \\ &= \int_a^b F^\Delta(\xi)\Delta\xi = F(b) = \prod_{j=1}^{n+1} F_j(b) \leqslant \frac{1}{n+1}\sum_{j=1}^{n+1}\left[F_j(b)\right]^{n+1} \\ &= \frac{1}{n+1}\sum_{j=1}^{n+1}\left\{\int_a^b \frac{1}{\left[p(\xi)\right]^{\frac{1}{n+1}}}\left[p(\xi)q^\sigma(\xi)\right]^{\frac{1}{n+1}}\left|f_j^\Delta(\xi)\right|\Delta\xi\right\}^{n+1} \\ &\leqslant \frac{1}{n+1}\left(\int_a^b \frac{1}{\left[p(\xi)\right]^{\frac{1}{n}}}\Delta\xi\right)^n \sum_{j=1}^{n}\left\{\int_a^b p(\xi)q^\sigma(\xi)\left|f_j^\Delta(\xi)\right|^{n+1}\Delta\xi\right\},\end{aligned} \tag{11}$$

where we have applied Hölder's inequality in the last step. It is clear that (10) equal to (11), and this completes the proof. □

REMARK 4.1. Theorem 4.1 reduces to [12, Theorem 2] and [13, Theorem 2] by letting $n=1$, $\mathbb{T}=\mathbb{R}$, and $n=1$, $\mathbb{T}=\mathbb{Z}$, respectively. Also Theorem 4.1 includes [4, Theorem 4.1] with $n=1$ and $f_1=f_2$ on arbitrary time scales provided that $f_1 f_1^\sigma \geqslant 0$ holds on $[a,b]_{\mathbb{T}}^{\kappa}$.

THEOREM 4.2. *Let* $n \in \mathbb{N}$, $a,b \in \mathbb{T}$ *and* $f_j \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *with* $f_j(a) = 0$ *for all* $j \in [1,n+1]_{\mathbb{N}}$, *and that* $p \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+)$ *with* $\int_a^b [p(\xi)]^{-1/(m+n)}\Delta\xi < \infty$ *and* $q \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+_0)$ *be a nonincreasing function. Then, the following inequality holds*

$$\begin{aligned}&\int_a^b q^\sigma(\xi)\sum_{j=0}^m\left|\left[\prod_{i=1}^{n+1} f_i^\sigma(\xi)\right]^j\left[\prod_{i=1}^{n+1} f_i(\xi)\right]^{m-j}\right|\\&\qquad\times\sum_{j=1}^{n+1}\left|\left[\prod_{i=1}^{j-1} f_i^\sigma(\xi)\right]f_j^\Delta(\xi)\left[\prod_{i=j+1}^{n+1} f_i(\xi)\right]\right|\Delta\xi\\&\leqslant\frac{1}{n+1}\left(\int_a^b\frac{1}{[p(\xi)]^{\frac{1}{m+n}}}\Delta\xi\right)^{\frac{(m+n)(m+1)(n+1)}{m+n+1}}\\&\qquad\times\sum_{j=1}^{n+1}\left(\int_a^b p(\xi)q^\sigma(\xi)\left|f_j^\Delta(\xi)\right|^{m+n+1}\Delta\xi\right)^{\frac{(m+1)(n+1)}{m+n+1}}.\end{aligned}$$

Proof. Proof for this result can be obtained easily, so we prefer not to give it here. □

The following two weighted versions of Theorem 3.3 and Theorem 3.4 can be obtained easily, and we omit the proofs.

THEOREM 4.3. *Let* $\ell,n \in \mathbb{N}$, $a,b \in \mathbb{T}$ *and* $f_j \in \mathrm{C}^\ell_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *with* $f_j^{\Delta^i}(a) = 0$ *for all* $j \in [1,n+1]_{\mathbb{N}}$ *and all* $i \in [0,\ell)_{\mathbb{N}_0}$, *and that* $q \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+_0)$. *Then, the following inequality holds*

$$\begin{aligned}&\int_a^b q(\xi)\sum_{j=1}^{n+1}\left\{\left[\prod_{i=1}^{j-1} f_i^\sigma(\xi)\right]f_j^{\Delta^\ell}(\xi)\left[\prod_{i=j+1}^{n+1} f_i(\xi)\right]\right\}\Delta\xi\\&\leqslant\left(\frac{1}{n+1}\int_a^b\left([q(\xi)]^2\int_a^{\sigma(\xi)}\left[h_{\ell-1}(\sigma(\xi),\sigma(\zeta))\right]^2\Delta\zeta\right)^n\Delta\xi\right)^{\frac{1}{2}}\\&\qquad\times\sum_{j=1}^{n+1}\left(\int_a^b\left[f_j^{\Delta^\ell}(\xi)\right]^2\Delta\xi\right)^{\frac{n+1}{2}}.\end{aligned}$$

REMARK 4.2. Theorem 4.3 reduces to [12, Theorem 5] by letting $n = 1$, $\mathbb{T} = \mathbb{R}$, which generalizes [6, Theorem 1]. To the best our knowledge, this result for $\mathbb{T} \neq \mathbb{R}$ is not stated in the literature yet.

THEOREM 4.4. *Let* $\ell,m,n \in \mathbb{N}$, $a,b \in \mathbb{T}$ *and* $f_j \in \mathrm{C}^\ell_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *for all* $j \in [1,n+1]_{\mathbb{N}}$ *with* $f_j^{\Delta^i}(a) = 0$ *for all* $j \in [1,n+1]_{\mathbb{N}}$ *and all* $i \in [0,\ell)_{\mathbb{N}_0}$, *and that* $q \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+)$.

Then, the following inequality holds

$$\int_a^b q(\xi)\sum_{j=0}^{m}\left\{\left[\prod_{i=1}^{n+1} f_i^{\sigma}(\xi)\right]^j\left[\prod_{i=1}^{n+1} f_i(\xi)\right]^{m-j}\right\}$$
$$\times\sum_{j=1}^{n+1}\left\{\left[\prod_{i=1}^{j-1} f_i^{\sigma}(\xi)\right] f_j^{\Delta^\ell}(\xi)\left[\prod_{i=j+1}^{n+1} f_i(\xi)\right]\right\}\Delta\xi$$
$$\leqslant\left(\frac{m+1}{n+1}\int_a^b\left(\left[q(\xi)\right]^2\int_a^{\sigma(\xi)}\left[h_{\ell-1}(\sigma(\xi),\sigma(\zeta))\right]^2\Delta\zeta\right)^{m+n}\Delta\xi\right)^{1/2}$$
$$\times\sum_{j=1}^{n+1}\left(\int_a^b\left[f_j^{\Delta^\ell}(\xi)\right]^2\Delta\xi\right)^{\frac{(m+1)(n+1)}{2}}.$$

REMARK 4.3. Theorem 4.4 reduces to [12, Theorem 4] by letting $n=1$, $\mathbb{T}=\mathbb{R}$. To the best our knowledge, this result for $\mathbb{T}\neq\mathbb{R}$ is also new for the literature.

5. Final comments and remarks

Our results can be extended by the applying of Muirhead's inequality stated in Theorem D, for instance the result of Theorem 3.2 can be arranged as follows:

COROLLARY 5.1. *In additions to assumptions of Theorem* 3.2*, suppose that there exists* $(\alpha_1,\alpha_2,\dots,\alpha_{n+1})$ *such that* $\sum_{j=1}^{n+1}\alpha_j=n+1$ *and* $\sum_{j=1}^{k}\alpha_j\geqslant k$ *for all* $k\in[1,n]_{\mathbb{N}}$*, then the right-hand side of* (2) *can be replaced by the following one*

$$\frac{1}{(n+1)!}\sum_{\pi\in S^{n+1}}\left\{\prod_{j=1}^{n+1}(b-a)^{\alpha_j(m+1)}\int_a^b\left|\left[\prod_{j=1}^{n+1} f_{\pi_j}^{\Delta}(\xi)\right]^{\alpha_j(m+1)}\right|\Delta\xi\right\},$$

where S^{n+1} *is the set of all permutations of the set* $[1,n+1]_{\mathbb{N}}$*, and* π_j *stands for the* j*-th component of the permutation* π*.*

Proof. By the second term in (3) and Muirhead's inequality, we have

$$\prod_{j=1}^{n+1}\left[F_j(b)\right]^{m+1}=\frac{1}{(n+1)!}\sum_{\pi\in S^{n+1}}\prod_{j=1}^{n+1}\left[F_{\pi_j}(b)\right]^{m+1}$$
$$\leqslant\frac{1}{(n+1)!}\sum_{\pi\in S^{n+1}}\prod_{j=1}^{n+1}\left[F_{\pi_j}(b)\right]^{\alpha_j(m+1)},$$

where $(\alpha_1,\alpha_2,\dots,\alpha_{n+1})\triangleright(1,1,\dots,1)$. The rest of the proof is similar to that of Theorem 3.2. □

REMARK 5.1. It is not hard to see that Corollary 5.1 reduces to Theorem 3.2 by letting $\beta_j = 1$ for all $j \in [1, n+1]_{\mathbb{N}}$.

COROLLARY 5.2. (See [12, Theorem 6] and [1, Theorem 2.14.4]) *In additions to assumptions of Theorem* 3.2*, suppose that there exists an integer* $k \geqslant 2$ *such that* $a + j(b-a)/k \in [a,b]_{\mathbb{T}}$ *holds for* $j \in [1, k-1]_{\mathbb{N}}$*, then the right-hand side of* (2) *can be replaced by the following smaller one*

$$\frac{1}{n+1}\left(\frac{b-a}{k}\right)^{(m+1)(n+1)-1} \int_a^b \left[\sum_{j=1}^{n+1} \left|f_j^{\Delta}(\xi)\right|\right]^{(m+1)(n+1)} \Delta\xi.$$

Proof. In this case, we apply Theorem 3.2 on each one of the intervals $[a + j(b-a)/k, a + (j+1)(b-a)/k]_{\mathbb{T}}$ for $j \in [0, k-1]_{\mathbb{N}_0}$, and then sum the resulting inequalities. □

REMARK 5.2. Our results in § 3 and § 4 can also be easily improved as done for Theorem 3.1 by considering Corollary 5.1 and Corollary 5.2.

REMARK 5.3. Our results in § 3 and § 4 can be extended by applying Hölder's inequality with different p values.

REMARK 5.4. Arbitrary functions can be inserted into the inequalities of Theorem 4.3 and Theorem 4.4 to improve their results as in Theorem 4.1 and Theorem 4.2.

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(Received February 20, 2009)

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