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L–OPERATOR INTEGRO–DIFFERENTIAL INEQUALITY FOR DISSIPATIVITY OF STOCHASTIC INTEGRO–DIFFERENTIAL EQUATIONS

LIGUANG XU AND FAJIN QIN

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Abstract. In this paper, Itô stochastic integro-differential equations are considered. By establishing an L-operator integro-differential inequality and using the properties of M-cone and stochastic analysis technique, we obtain some new sufficient conditions ensuring the exponential p-dissipativity of the stochastic integro-differential equations. An example is also discussed to illustrate the efficiency of the obtained results.

1. Introduction

Integro-differential equations arise widely in scientific fields, where it is necessary to take into account aftereffect or delay such as control theory, biology, ecology, medicine, etc. (cf. [1–4]). Especially, one always describes a model which possesses hereditary properties by integro-differential equations in practice.

Recently, the asymptotic behavior of dynamical systems has been widely studied (for instance, see [5–9]). As is well known, dissipativity is one of the most important components in the theory of asymptotic behavior. It has good application in many areas, such as stability theory, chaos and synchronization theory, system norm estimation and robust control [10]. In particular, delay effect on the dissipativity and other behaviors of integro-differential equations have been attracted the interest of many authors and various results are reported (for instance, see [4, 11–14]). However, besides delay effect, stochastic effect likewise exists in real system. A lot of dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. Many interesting results on stochastic effect have been reported, e.g., see [15–21].

Therefore, it is necessary to consider both stochastic and delay effect on the dissipativity of integro-differential equations. However, to the best of our knowledge, there are no results on the problems of the exponential p-dissipativity of stochastic

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integro-differential equations. This paper presents one such method by establishing an L-operator integro-differential inequality and employing M-cone. Based on the obtained method, we shall give sufficient conditions for the exponential p-dissipativity of a class of stochastic integro-differential equations. An example is given to illustrate the efficiency of the results.

2. Model and preliminaries

To begin with, we introduce some notation and recall some basic definitions. Let I denote the n-dimensional unit matrix, $\mathscr{N} \triangleq \{1,2,\cdots,n\}$, $R_+ = [0,\infty)$. For $A,B \in R^{m\times n}$ or $A,B \in R^n$, $A \geqslant B (A \leqslant B, A > B, A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geqslant$ $(\leqslant, >, <)$". Especially, A is called a nonnegative matrix if $A \geqslant 0$, and z is called a positive vector if $z > 0$.

$C[X,Y]$ denote the space of continuous mappings from the topological space X to the topological space Y. In particular, let $C \triangleq C[(-\infty,0],R^n]$ denote the family of all bounded continuous R^n-valued functions φ defined on $(-\infty,0]$ with the norm $\|\varphi\| = \sup_{-\infty<\theta\leqslant 0}|\varphi(\theta)|$, where $|\cdot|$ is Euclidean norm on R^n.

For $x \in R^n$, $\varphi \in C$, $p > 0$, we define

$$[x]_+^p = (|x_1|^p,\cdots,|x_n|^p)^T, \quad \text{especially, } [x]_+^1 = (|x_1|,\cdots,|x_n|)^T,$$
$$col\{x_i\}_n = col(x_1,x_2,\cdots,x_n), \quad [\varphi(t)]_\infty = ([\varphi_1(t)]_\infty,\cdots,[\varphi_n(t)]_\infty)^T,$$
$$[\varphi(t)]_+^\infty = [[\varphi(t)]_+^1]_\infty, \quad [\varphi_i(t)]_\infty = \sup_{-\infty<s\leqslant 0}\{\varphi_i(t+s)\},\ i \in \mathscr{N},$$

and $D^+\varphi(t)$ denote the upper right derivative of $\varphi(t)$ at time t.

Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant t_0},P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant t_0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_{t_0}$ contains all P-null sets). $w(t) = (w_1(t),\cdots,w_m(t))^T$ is an m-dimensional Brownian motion defined on $(\Omega,\ \mathscr{F},\ \{\mathscr{F}_t\}_{t\geqslant t_0},\ P)$. Let $C^b_{\mathscr{F}_{t_0}}[(-\infty,0],R^n]$ denote the family of all bounded $\mathscr{F}_{t_0}$-measurable, $C[(-\infty,0],R^n]$-valued random variables φ, satisfying $\|\varphi\|^p_{L^p} = \sup_{-\infty<\theta\leqslant 0}E|\varphi(\theta)|^p < \infty$, where E denotes the expectation of stochastic process.

In this paper, we consider the following $It\hat{o}$ stochastic integro-differential equations:

$$dx_i(t) = \left[-a_ix_i(t) + \sum_{j=1}^n a_{ij}f_j(x_j(t)) + \sum_{j=1}^n \int_{-\infty}^t p_{ij}(t-s)g_j(x_j(s))ds + J_i\right]dt$$
$$+ \sum_{l=1}^m \sigma_{il}(t,x_i(t))dw_l(t), \quad t \geqslant t_0, \tag{1}$$

where $a_i > 0$, $J_i \geqslant 0$, a_{ij}, $i,j \in \mathscr{N}$ are constants, $f_j, g_j \in C[R,R]$, $j \in \mathscr{N}$; $p_{ij}(t)$, $i,j \in \mathscr{N}$ are continuous and satisfy

$$(H): \int_0^\infty e^{\lambda_0 t}|p_{ij}(t)|dt < \infty,\ i,j \in \mathscr{N},$$

in which λ_0 is a positive constant, and $\sigma_i(t,x_i) = (\sigma_{i1}(t,x_i),\cdots,\sigma_{im}(t,x_i))$ is i-th row vector of a matrix $\sigma(t,x) = (\sigma_{il}(t,x_i))_{n\times m}$, $\sigma : R\times R^n \to R^{n\times m}$.

DEFINITION 2.1. For any given $t_0 \in R$, $\varphi \in C^b_{\mathscr{F}_{t_0}}[(-\infty,0],R^n]$, an R^n-valued stochastic process $x(t)$ is called a solution of (1) on $(-\infty,T]$ through (t_0,φ), if $x(t)$ has the following properties:

(1^0) $x : (-\infty,T]\times\Omega \to R^n$ is a measurable, sample-continuous process;

(2^0) $(x(t),t\in[t_0,T])$ is $(\mathscr{F})_{t_0\leqslant t\leqslant T}$-adapted, $x(s)$ is $\mathscr{F}_{t_0}$-measurable for all $s \leqslant t_0$;

(3^0) x satisfies, almost surely, Eq. (1) for $t \geqslant t_0$ and the initial conditions in the form

$$x(t_0+s) = \varphi(s), \quad s\in(-\infty,0]. \tag{2}$$

Throughout this paper, we assume that for any $\varphi \in C^b_{\mathscr{F}_{t_0}}[(-\infty,0],R^n]$, there exists at least one solution of (1) with the initial condition (2). For conditions guaranteeing the existence of a solution see [22].

DEFINITION 2.2. System (1) is said to be exponentially p-dissipative with exponential convergent rate λ if there is a bounded set $M \subset C^b_{\mathscr{F}_{t_0}}[(-\infty,0],R^n]$ and a pair of positive constants K,λ such that for any solution $x(t,t_0,\varphi)$ with the initial condition $\varphi \in C^b_{\mathscr{F}_{t_0}}[(-\infty,0],R^n]$,

$$\mathrm{dist}(E[x(t)]^p_+,M) \leqslant K\|\varphi\|^p_{L^p}e^{-\lambda(t-t_0)}, \quad t\geqslant t_0, \tag{3}$$

where

$$\mathrm{dist}(\psi,M) = \inf_{\phi\in M}\sup_{s\in(-\infty,0]}|\psi(t)-\phi(s)|, \text{ for } \psi\in C[R,R^n],$$

and the set M is called an exponential attracting set of system (1). Especially, system (1) is said to be exponentially dissipative in mean square when $p=2$.

DEFINITION 2.3. System (1) is said to be exponentially p-stable with exponential convergent rate λ if there is a pair of positive constants λ and K such that for any solution $x(t,t_0,\varphi)$ with the initial condition $\varphi \in C^b_{\mathscr{F}_{t_0}}[(-\infty,0],R^n]$,

$$E|x(t,t_0,\varphi)|^p \leqslant K\|\varphi\|^p_{L^p}e^{-\lambda(t-t_0)}, \quad t\geqslant t_0.$$

Especially, system (1) is said to be exponentially stable in mean square when $p=2$.

DEFINITION 2.4. [23, p. 114]. Let the matrix $D=(d_{ij})_{n\times n}$, $d_{ii}>0$ and $d_{ij}\leqslant 0$, $i\neq j$. Then D is called an M-matrix if one of the following conditions holds:

(i) all the leading principal minors of D are positive;

(ii) there exists a positive vector z such that $Dz>0$;

(iii) D is inverse positive; that is, D^{-1} exists and $D^{-1}\geqslant 0$.

LEMMA 2.1. [24] *For an M-matrix D, $\Omega_M(D) \overset{\Delta}{=} \{z \in R^n \mid Dz > 0,\ z > 0\}$ is nonempty and for any $z_1,\ z_2 \in \Omega_M(D)$, we have*

$$k_1 z_1 + k_2 z_2 \in \Omega_M(D),\ \ \forall\, k_1,\ k_2 > 0.$$

So $\Omega_M(D)$ is a cone without conical surface in R^n. We call it an"M-cone".

LEMMA 2.2.. (Arithmetic-mean–geometric-mean inequality [25]) *For $x_i \geqslant 0,\ \alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$,*

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leqslant \sum_{i=1}^{n} \alpha_i x_i,$$

the sign of equality holds if and only if $x_i = x_j$ for all $i, j \in \mathscr{N}$.

3. L-operator integro-differential inequality

In this section, we will first establish an integro-differential inequality and then introduce an L-operator integro-differential inequality.

THEOREM 3.1. *Let $P = (p_{ij})_{n\times n}$ and $p_{ij} \geqslant 0$ for $i \neq j$, $J = (J_1, \cdots, J_n)^T \geqslant 0$, $Q(t) = (q_{ij}(t))_{n\times n}$, where $q_{ij}(t) \geqslant 0$ are continuous and satisfy*

$$(H_0): \quad \int_0^\infty e^{\lambda_1} q_{ij}(t)dt < \infty,\ i, j \in \mathscr{N},$$

in which λ_1 is a positive constant. Denote $Q = (q_{ij})_{n\times n} \overset{\Delta}{=} (\int_0^\infty q_{ij}(t)dt)_{n\times n}$ and let $D = -(P+Q)$ be a nonsingular M-matrix and $u(t) = (u_1(t), \cdots, u_n(t))^T \in C[[t_0, \infty), R^n]$ be a solution of the following integro-differential inequality with the initial condition $u(s) \in C,\ -\infty < s \leqslant t_0$

$$D^+u(t) \leqslant Pu(t) + \int_0^\infty Q(s)u(t-s)ds + J,\ t \geqslant t_0. \tag{4}$$

Then

$$u(t) \leqslant ze^{-\lambda(t-t_0)} - (P+Q)^{-1}J,\ t \geqslant t_0, \tag{5}$$

provided that the initial conditions satisfy

$$u(t) \leqslant ze^{-\lambda(s-t_0)} - (P+Q)^{-1}J,\ -\infty < s \leqslant t_0, \tag{6}$$

where $z = (z_1, z_2, \cdots, z_n)^T \in \Omega_M(D)$ and the constant $\lambda \in (0, \lambda_1]$ satisfies the following inequality

$$\left[\lambda I + P + \int_0^\infty Q(s)e^{\lambda s}ds\right] z < 0. \tag{7}$$

Proof. Since D is a nonsingular M-matrix, there exists a vector $z \in \Omega_M(D)$ such that

$$Dz > 0 \text{ or } [P+Q]z < 0.$$

By using continuity and condition (H_0), we obtain that there must exist a positive constant $\lambda \leqslant \lambda_1$ satisfying the inequality (7), that is,

$$\sum_{j=1}^{n}\left[p_{ij}+\int_0^\infty q_{ij}(s)e^{\lambda s}ds\right]z_j < -\lambda z_i, \quad i \in \mathscr{N}. \tag{8}$$

Let $N \triangleq -(P+Q)^{-1}J$, $N = (N_1,...,N_n)^T$. One can get that $(P+Q)N+J=0$ or

$$\sum_{j=1}^{n} p_{ij}N_j + \sum_{j=1}^{n} q_{ij}N_j + J_i = 0,\ i \in \mathscr{N}. \tag{9}$$

To prove (5), we first prove, for any given $\varepsilon > 0$, when $u(s) \leqslant ze^{-\lambda(s-t_0)} - (P+Q)^{-1}J$, $-\infty < s \leqslant t_0$,

$$u_i(t) < (1+\varepsilon)[z_ie^{-\lambda(t-t_0)}+N_i] \triangleq y_i(t),\ t \geqslant t_0,\ i \in \mathscr{N}. \tag{10}$$

If (10) is not true, then there must be a $t_1 > t_0$ and some integer m such that

$$u_m(t_1) = y_m(t_1), \quad D^+u_m(t_1) \geqslant y_m'(t_1), \tag{11}$$

$$u_i(t) \leqslant y_i(t), \quad -\infty < t \leqslant t_1,\ i \in \mathscr{N}. \tag{12}$$

By using (4), (8)–(12) and $p_{ij} \geqslant 0 (i \neq j)$, $q_{ij}(t) \geqslant 0$, we obtain that

$$\begin{aligned}
D^+u_m(t_1) &\leqslant \sum_{j=1}^{n} p_{mj}u_j(t_1) + \sum_{j=1}^{n}\int_0^\infty q_{mj}(s)u_j(t_1-s)ds + J_m \\
&\leqslant \sum_{j=1}^{n} p_{mj}(1+\varepsilon)(z_je^{-\lambda(t_1-t_0)}+N_j) \\
&\quad + \sum_{j=1}^{n}\int_0^\infty q_{mj}(s)(1+\varepsilon)(z_je^{-\lambda(t_1-s-t_0)}+N_j)ds + J_m \\
&= \sum_{j=1}^{n}\left[p_{mj}+\int_0^\infty q_{mj}(s)e^{\lambda s}ds\right](1+\varepsilon)z_je^{-\lambda(t_1-t_0)} \\
&\quad + \sum_{j=1}^{n}\left[p_{mj}+\int_0^\infty q_{mj}(s)ds\right]N_j(1+\varepsilon)+J_m \\
&= \sum_{j=1}^{n}\left[p_{mj}+\int_0^\infty q_{mj}(s)e^{\lambda s}ds\right](1+\varepsilon)z_je^{-\lambda(t_1-t_0)} - \varepsilon J_m \\
&\leqslant \sum_{j=1}^{n}\left[p_{mj}+\int_0^\infty q_{mj}(s)e^{\lambda s}ds\right](1+\varepsilon)z_je^{-\lambda(t_1-t_0)} \\
&< -\lambda z_m(1+\varepsilon)z_je^{-\lambda(t_1-t_0)} \\
&= y_m'(t_1),
\end{aligned}$$

This contradicts the inequality (11), and so (10) holds. Letting $\varepsilon \to 0$, then (5) holds, and the proof is completed. □

REMARK 3.1. Suppose that $J = 0$ in Theorem 3.1, then we get Theorem 1 in [26].

Let $C^{2,1}[R_+ \times R^n, R_+]$ denote the family of all nonnegative functions $V(t,x)$ on $R_+ \times R^n$ which are twice continuously differentiable in x and once in t. For each $V(t,x) \in C^{2,1}[R_+ \times R^n, R_+]$, we define an operator LV from $C^{2,1}[R_+ \times R^n, R_+]$ to $C[R,R]$, associated with the system (1), by

$$LV(t,x) = V_t(t,x) + V_x(t,x)F(t,x) + \frac{1}{2}\mathrm{trace}[\sigma^T(t,x)V_{xx}\sigma(t,x)],$$
$$V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \quad V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \cdots, \frac{\partial V(t,x)}{\partial x_n}\right), \; V_{xx} = \left(\frac{\partial V^2(t,x)}{\partial x_i \partial x_j}\right)_{n\times n},$$

where

$$F(t,x) = col\{-a_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} \int_{-\infty}^{t} p_{ij}(t-s) g_j(x_j(s))ds + J_i\}_n.$$

THEOREM 3.2. *Let matrices P, J, $Q(t)$ and the condition (H_0) be defined as in Theorem 3.1. Assume that there exist functions $V_i(x) \in C^2[R^n, R_+]$ such that for the operator LV which is associated with the system (1), such that*

$$LV_i(x) \leqslant \sum_{j=1}^{n}\left(p_{ij}V_j(x) + \int_0^{\infty} q_{ij}(s)V_j(x(t-s))ds\right) + J_i, \; t \geqslant t_0, \; i \in \mathcal{N}. \tag{13}$$

Then

$$EV_i(x(t)) \leqslant z_i e^{-\lambda(t-t_0)} + N_i, \; t \geqslant t_0, \; i \in \mathcal{N}, \tag{14}$$

provided that the initial conditions satisfy

$$EV_i(x(t)) \leqslant z_i e^{-\lambda(t-t_0)} + N_i, \; -\infty < t \leqslant t_0, \; i \in \mathcal{N}, \tag{15}$$

where $N = (N_1, \cdots, N_n)^T = -(P+Q)^{-1}J$, $z = (z_1, z_2, \cdots, z_n)^T \in \Omega_M(D)$ and the constant $\lambda \in (0, \lambda_1]$ satisfies the following inequality

$$\left[\lambda I + P + \int_0^{\infty} Q(s)e^{\lambda s}ds\right] z < 0. \tag{16}$$

Proof. Since $x(t)$ is the solution of the equation (1) and $V_i(x) \in C^2[R^n, R_+]$, by the *Itô* formula, we can get

$$V_i(x(t)) = V_i(x(t_0)) + \int_{t_0}^{t} LV_i(x(s))ds + \int_{t_0}^{t} \frac{\partial V_i(x(s))}{\partial x}\sigma(s,x(s))dw(s), \; t \geqslant t_0, \; i \in \mathcal{N}.$$

Then we have

$$EV_i(x(t)) = EV_i(x(t_0)) + \int_{t_0}^{t} ELV_i(x(s))ds, \quad t \geqslant t_0,\ i \in \mathcal{N}. \tag{17}$$

So, for small enough $\Delta t > 0$, we have

$$EV_i(x(t+\Delta t)) = EV_i(x(t_0)) + \int_{t_0}^{t+\Delta t} ELV_i(x(s))ds, \quad t \geqslant t_0,\ i \in \mathcal{N}. \tag{18}$$

Thus from $(13), (17)$ and (18), we have

$$\begin{aligned} EV_i(x(t+\Delta t)) - EV_i(x(t)) &= \int_{t}^{t+\Delta t} ELV_i(x(s))ds \\ &\leqslant \int_{t}^{t+\Delta t} \left\{ \sum_{j=1}^{n} \left[p_{ij} EV_j(x(r)) + \int_0^{\infty} q_{ij}(s) EV_j(x(r-s))ds \right] + J_i \right\} dr, \quad t \geqslant t_0,\ i \in \mathcal{N}. \end{aligned} \tag{19}$$

From (19), we obtain that

$$D^+ EV_i(x(t)) \leqslant \sum_{j=1}^{n} \left(p_{ij} EV_j(x(t)) + \int_0^{\infty} q_{ij}(s) EV_j(x(t-s))ds \right) + J_i,\ t \geqslant t_0,\ i \in \mathcal{N}. \tag{20}$$

Since D is a nonsingular M-matrix, there exists a vector $z \in \Omega_M(D)$ such that

$$Dz > 0 \quad or \quad [P+Q]z < 0.$$

By using continuity, we obtain that (16) has at least one positive solution λ. Thus from Theorem 3.1, we know Theorem 3.2 is true. □

4. Exponential P-dissipativity

In this section, we will obtain several sufficient conditions ensuring the exponential p-dissipativity of stochastic integro-differential equations (1) by employing Theorem 3.2. Here, we first introduce the following assumptions.

(A_1) There exist nonnegative constants u_j, v_j such that for $x_j \in R$,

$$|f_j(x_j)| \leqslant u_j |x_j|, \qquad |g_j(x_j)| \leqslant v_j |x_j|,\ j \in \mathcal{N}. \tag{21}$$

(A_2) There exist nonnegative constants $\hat{c}_i$ such that for $x_i \in R$,

$$|(\sigma_i(t,x_i))^T(\sigma_i(t,x_i))| \leqslant \hat{c}_i |x_i|^2,\ i \in \mathcal{N}.$$

(A_3) Let $\bar{D} = -(\hat{A}+\hat{B})$ be an M-matrix, where

$$\begin{aligned}\hat{A} =&(\hat{a}_{ij})_{n\times n},\ \hat{a}_{ij} = |a_{ij}u_j|,\ i\neq j,\\ \hat{a}_{ii} =&-pa_i+\sum_{j=1}^{n}|a_{ij}u_j|(p-1)+\sum_{j=1}^{n}\int_0^{\infty}|p_{ij}(s)|v_j(p-1)ds+(p-1)\\ &+\frac{1}{2}p(p-1)\hat{c}_i+|a_{ii}u_i|,\\ \hat{B} =&(\hat{b}_{ij})_{n\times n}\triangleq\left(\int_0^{\infty}\hat{b}_{ij}(t)dt\right)_{n\times n},\ \hat{b}_{ij}(t) = |p_{ij}(t)|v_j,\ i,j\in\mathscr{N}.\end{aligned}$$

THEOREM 4.1. *Assume that the hypothesis* (H) *and the conditions* (A_1)–(A_3) *hold. Then the system* (1) *is exponentially* p*-dissipative with the exponential convergent rate* λ *which is determined by* (22)*, and the exponential attracting set*

$$M = \left\{\phi\in C^b_{\mathscr{F}_{t_0}}[(-\infty,0],R^n]\,\big|\,[\phi]^{\infty}_{+}\leqslant(-\hat{A}-\hat{B})^{-1}\hat{J}\right\},$$

where $\hat{J} = (\hat{J}_1,\cdots,\hat{J}_n)^T$ *and* $\hat{J}_i = J_i^p, i\in\mathscr{N}$.

Proof. Since $\bar{D} = -(\hat{A}+\hat{B})$ is a nonsingular M-matrix, there exists a vector $z\in\Omega_M(\bar{D})$ such that

$$\bar{D}z>0\quad or\quad (\hat{A}+\hat{B})z<0.$$

By using continuity and hypothesis (H), we obtain that there must exist a positive constant $\lambda\leqslant\lambda_0$ satisfying the following inequality

$$\left[\lambda I+\hat{A}+\int_0^{\infty}\hat{B}(s)e^{\lambda s}ds\right]z<0. \tag{22}$$

Let $\hat{N}\triangleq-(\hat{A}+\hat{B})^{-1}\hat{J}$, $\hat{N} = (\hat{N}_1,\cdots,\hat{N}_n)^T$. Then from the definition of M-matrix, we have

$$\hat{N}\geqslant 0\quad or\quad \hat{N}_i\geqslant 0,\quad i\in\mathscr{N}. \tag{23}$$

Let $V_i(x(t)) = |x_i(t)|^p$, $i\in\mathscr{N}$, where $x(t) = (x_1(t),\cdots,x_n(t))^T$ is the solution of (1). Then

$$\frac{\partial V_i(x)}{\partial x_i} = p|x_i|^{p-1}\text{sgn}(x_i) = p|x_i|^{(p-2)}x_i,\quad \frac{\partial V_i^2(x)}{\partial x_i^2} = p(p-1)|x_i|^{(p-2)}\text{sgn}(x_i),$$

where $\mathrm{sgn}(\cdot)$ is the sign function. Thus, by the conditions $(A_1),(A_2)$ and Lemma 2.2, we have

$$
\begin{aligned}
&LV_i(x(t))\\
=&p|x_i(t)|^{(p-2)}x_i(t)\left[-a_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}\int_{-\infty}^{t}p_{ij}(t-s)g_j(x_j(s))ds+J_i\right]\\
&+\frac{1}{2}p(p-1)|x_i(t)|^{p-2}sgn(x_i(t))\sigma_i^T(t,x_i(t))\sigma_i(t,x_i(t))\\
\leqslant&-pa_i|x_i(t)|^p+p|x_i(t)|^{(p-1)}\left[\sum_{j=1}^{n}|a_{ij}u_j||x_j(t))|+\sum_{j=1}^{n}\int_{-\infty}^{t}|p_{ij}(t-s)|v_j|x_j(s)|ds+J_i\right]\\
&+\frac{1}{2}p(p-1)|x_i(t)|^{p-2}\left|\sigma_i^T(t,x_i(t))\sigma_i(t,x_i(t))\right|\\
\leqslant&-pa_i|x_i(t)|^p+p|x_i(t)|^{(p-1)}\left[\sum_{j=1}^{n}|a_{ij}u_j||(x_j(t))|+\sum_{j=1}^{n}\int_{0}^{\infty}|p_{ij}(s)|v_j|(x_j(t-s)|ds+J_i\right]\\
&+\frac{1}{2}p(p-1)\hat{c}_i|x_i(t)|^p\\
\leqslant&-pa_i|x_i(t)|^p+\sum_{j=1}^{n}|a_{ij}u_j|\left[(p-1)|x_i(t)|^p+|x_j(t)|^p\right]\\
&+\sum_{j=1}^{n}\int_{0}^{\infty}|p_{ij}(s)|v_j[(p-1)|x_i(t)|^p+|x_j(t-s)|^p]ds\\
&+(p-1)|x_i(t)|^p+J_i^p+\frac{1}{2}p(p-1)\hat{c}_i|x_i(t)|^p\\
=&\left[-pa_i+\sum_{j=1}^{n}|a_{ij}u_j|(p-1)+\sum_{j=1}^{n}\int_{0}^{\infty}|p_{ij}(s)|v_j(p-1)ds+(p-1)+\frac{1}{2}p(p-1)\hat{c}_i\right]|x_i(t)|^p\\
&+\sum_{j=1}^{n}|a_{ij}u_j||x_j(t)|^p+\sum_{j=1}^{n}\int_{0}^{\infty}|p_{ij}(s)|v_j|x_j(t-s)|^pds+J_i^P\\
=&\sum_{j=1}^{n}\hat{a}_{ij}V_j(x)+\sum_{j=1}^{n}\int_{0}^{\infty}\hat{b}_{ij}(s)V_j(x(t-s))ds+\hat{J}_i. \qquad (24)
\end{aligned}
$$

So from the condition (A_3), we know that the inequality (13) holds. For the initial condition $\varphi\in C^b_{\mathscr{F}_{t_0}}[(-\infty,0],\ R^n]$, we can get

$$EV_i(x(t))\leqslant hz_i\|\varphi\|^p_{L^p}e^{-\lambda(t-t_0)}\leqslant hz_i\|\varphi\|^p_{L^p}e^{-\lambda(t-t_0)}+\hat{N}_i,\ t\in(-\infty,t_0],\ i\in\mathscr{N}, \quad (25)$$

where, $h=\frac{1}{min_{1\leqslant i\leqslant n}\{z_i\}}$, $z=(z_1,\cdots,z_n)^T\in\Omega_M(\bar{D})$ and λ satisfies (22).

From Lemma 2.1 and $z=(z_1,\cdots,z_n)^T\in\Omega_M(\bar{D})$, we have $h\|\varphi\|^p_{L^p}z\in\Omega_M(\bar{D})$. Then, all conditions of Theorem 3.2 are satisfied by $(24),(25)$ and (A_3), so

$$EV_i(x(t))\leqslant hz_i\|\varphi\|^p_{L^p}e^{-\lambda(t-t_0)}+\hat{N}_i,\ t\in[t_0,\infty),\ i\in\mathscr{N}, \quad (26)$$

that is

$$E[x(t)]_+^p \leqslant h\|\varphi\|_{L^p}^p z e^{-\lambda(t-t_0)} - (\hat{A}+\hat{B})^{-1}\hat{J},\ t\in[t_0,\infty). \tag{27}$$

This implies that the conclusion holds and the proof is completed. □

COROLLARY 4.1. *Assume that the hypothesis (H) and the conditions (A_1)–(A_3) hold. Then the system (1) with $J=0$ is exponentially p-stable with exponential convergent rate λ.*

5. Example

The following illustrative example will demonstrate the effectiveness of our results.

EXAMPLE 5.1. Consider the following stochastic integro-differential equations:

$$\begin{cases} dx_1(t) = \Big[-10x_1(t) + (|x_1(t)+1| - |x_1(t)-1|) - \int_{-\infty}^t e^{-(t-s)}|x_1(s)|ds \\ \qquad\qquad + \int_{-\infty}^t 4e^{-2(t-s)}|x_2(s)|ds + 7\Big]dt + 2x_1(t)dw_1(t), \\ dx_2(t) = \Big[-8x_2(t) + (|x_2(t)+1| - |x_2(t)-1|) - \int_{-\infty}^t 6e^{-2(t-s)}|x_1(s)|ds \\ \qquad\qquad + \int_{-\infty}^t e^{-(t-s)}|x_2(s)|ds + 8\Big]dt + x_2(t)dw_2(t). \end{cases} \tag{28}$$

For system (28), we have

$$\begin{aligned} &f_i(x_i) = |x_i(t)+1| - |x_i(t)-1|,\ g_i(x_i) = |x_i(t)|,\ i=1,2, \\ &\sigma(t,x) = \operatorname{diag}\{2x_1(t), x_2(t)\}. \end{aligned}$$

and

$$\begin{aligned} &|f_i(x_i)| \leqslant 2|x_i|,\ |g_i(x_i)| \leqslant |x_i|,\ i=1,2, \\ &\left|\sigma_1(t,x_1)\sigma_1(t,x_1)^T\right| \leqslant 4|x_1(t)|^2,\ \left|\sigma_2(t,x_2)\sigma_2(t,x_2)^T\right| \leqslant |x_2(t)|^2. \end{aligned}$$

So, it is easy to check that the conditions (A_1) - (A_3) are satisfied by taking $a_1 = 10, a_2 = 8, p = 2, J_1 = 7, J_2 = 8, a_{11} = a_{22} = 1, a_{12} = a_{21} = 0, u_1 = u_2 = 2, v_1 = v_2 = 1, \hat{c}_1 = 4, \hat{c}_2 = 1, p_{11}(s) = -e^{-s}, p_{12}(s) = 4e^{-2s}, p_{21}(s) = -6e^{-2s}, p_{22}(s) = e^{-s}$.

Then

$$\hat{A} = \begin{pmatrix} -8 & 0 \\ 0 & -6 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \qquad \bar{D} = -(\hat{A}+\hat{B}) = \begin{pmatrix} 7 & -2 \\ -3 & 5 \end{pmatrix},$$

and (H) is satisfied with $0<\lambda_0<1$. In this example, we may let $\lambda_0 = 0.8$. It is easy to prove that $\bar{D}$ is an M-matrix and

$$\Omega_M(\bar{D}) = \left\{(z_1,z_2)^T > 0 \mid \frac{2}{7}z_2 < z_1 < \frac{5}{3}z_2\right\}.$$

Clearly, all the conditions of the Theorem 4.1 are satisfied, so system (28) is exponentially dissipative in mean square.

In order to determine the exponential convergent rate λ, we choose $z^* = (1,2)^T \in \Omega_M(\bar{D})$. From (22), that is,

$$\left[\lambda I + \hat{A} + \int_0^\infty \hat{B}(s)e^{\lambda s}ds\right] z^* < 0,$$

we obtain $\lambda = 0.5 < 0.8$.

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Liguang Xu
Department of Applied Mathematics
Zhejiang University of Technology
Hangzhou, 310023, PR China
e-mail: xlg132@126.com

Fajin Qin
Department of Mathematics and Computer Science
Liuzhou Teachers College
Liuzhou, 545004, PR China
e-mail: qinfajin@126.com