Mathematical
Inequalities
& Applications
Volume 14, Number 1 (2011), 135–145

L^p–ANALOGUES OF BERNSTEIN AND MARKOV INEQUALITIES

G. A. MUÑOZ-FERNÁNDEZ, V. M. SÁNCHEZ
AND J. B. SEOANE-SEPÚLVEDA

(*Communicated by T. Erdély*)

Abstract. Let $\|\cdot\|_\infty$ denote the sup norm on $[-1,1]$. If $x\in[-1,1]$ is fixed and $\mathscr{M}_{m,n}(x)$ is the best constant in

$$|p'(x)|\leqslant \mathscr{M}_{m,n}(x)\|p\|_\infty,$$

for all trinomials p of the form $p(x)=ax^m+bx^n+c$ with $a,b,c\in\mathbb{R}$, then the exact value of $\mathscr{M}_{m,n}(x)$ is known for large families of pairs $(m,n)\in\mathbb{N}^2$. Here we consider the same problem for L^p-norms.

1. Introduction

If $n\in\mathbb{N}$ let $\mathscr{P}_n(\mathbb{R})$ denote the set of all algebraic polynomials of degree at most n with real coefficients endowed with the norm defined by $\|p\|_\infty:=\max\{|p(x)|:-1\leqslant x\leqslant 1\}$ for all $p\in\mathscr{P}_n(\mathbb{R})$. According to a well known result due to A. Markov [7, 8],

$$\|p'\|_\infty\leqslant n^2\|p\|_\infty, \tag{1}$$

for all $p\in\mathscr{P}_n(\mathbb{R})$. The constant n^2 appearing in (1) is sharp and equality is attained for the nth Chebyshev polynomial of the first kind defined by $T_n(x):=\cos(n\arccos x)$ on $[-1,1]$.

A number of results appeared in the following years (see, e.g., [4, 9]) trying to obtain generalizations of A. Markov's inequality (1) for higher derivatives and pointwise S. Bernstein's type estimates on the derivatives of a polynomial (see, e.g., [1, 4]). As far as the latter question is concerned, inequality (1) may be refined for specific values of x in the interior of $[-1,1]$. In particular, S. Bernstein [1] proved that

$$|p'(x)|\leqslant \frac{n}{\sqrt{1-x^2}}\|p\|_\infty, \tag{2}$$

for all $x\in(-1,1)$ and for all $p\in\mathscr{P}_n(\mathbb{R})$, which improves (1) for every $x\in\left(-\frac{\sqrt{n^2-1}}{n},\frac{\sqrt{n^2-1}}{n}\right)$. It is interesting to notice that inequality (2) is only sharp

Mathematics subject classification (2010): Primary 41A17; Secondary 26D05.
Keywords and phrases: Bernstein and Markov type inequality, trinomial.
The first and third authors were supported by the Spanish Ministry of Education, grant MTM2009-07848. The second author was supported by the Spanish Ministry of Education, grant MTM2008-02652.

© ELEMENT, Zagreb

for the n roots of T_n. Actually the estimate provided by (2) is not even good for the values of x close to the endpoints of $[-1,1]$.

S. Bernstein and A. Markov's original papers are not readily accessible, so for a modern exposition on this and other related topics we refer to [2] and the references therein.

Since the middle seventies a considerable volume of research has been done in order to obtain generalizations of the classical S. Bernstein and A. Markov theorems mentioned above for polynomials on a general real Banach space. In this sense we recommend the references [5, 10, 17, 18, 19, 21] and the more recent [6, 20]. The study of the L^p-analogues of the classical S. Bernstein and A. Markov inequalities for polynomials on the real line has gained some relevance in the last years. If we consider the space $(\mathscr{P}_n(\mathbb{R}), \|\cdot\|_p)$, where $1 \leqslant p < \infty$,

$$\|P\|_p := \left(\int_{-1}^{1} |P(x)|^p dx\right)^{\frac{1}{p}},$$

for every $P \in \mathscr{P}_n(\mathbb{R})$ and M_n^p is the best constant in

$$\|P'\|_p \leqslant M_n^p \|P\|_p,$$

then it is known that

$$\lim_{n\to\infty} \frac{M_n^p}{n^2} < \infty,$$

see, e.g., [16, Theorem 15.6.2]. For more background on Bernstein-Markov inequalities and other polynomial inequalities consult [3].

In this paper we provide sharp Bernstein and Markov estimates for some spaces of real trinomials with the norm $\|\cdot\|_2$. The same problem for the sup norm has been studied in [12] using a characterization of the unit ball of the space of real trinomials on $[-1,1]$ that appeared in [11]. Spaces of complex trinomials were considered by S. Neuwirth in [14]. In order to discuss trinomials on the real line, for every $m,n \in \mathbb{N}$ with $m > n$, let $\mathscr{P}_{m,n}(\mathbb{R})$ denote the space of trinomials $P(x) = ax^m + bx^n + c$ with $a,b,c \in \mathbb{R}$.

2. Inequalities in the space $(\mathscr{P}_{m,n}(\mathbb{R}), \|\cdot\|_2)$: Bernstein and Markov estimates

We would like to begin this section by looking at a couple of the simplest cases that one can find when studying polynomial spaces with the norms $\|\cdot\|_p$, $1 \leqslant p < \infty$.

To start with, we consider the space $(\mathscr{P}_1(\mathbb{R}), \|\cdot\|_p)$ or, in other words, the space $\mathbb{R}^2$ endowed with the norm given by

$$\|(a,b)\|_p = \left(\int_{-1}^{1} |ax+b|^p \, dx\right)^{\frac{1}{p}},$$

for all $(a,b) \in \mathbb{R}^2$.

It can be seen that the unit ball of this norm is symmetric with respect to both axes, x and y. Indeed, since it is symmetric with respect to the origin, it suffices to show that $\|(a,b)\|_p = \|(-a,b)\|_p$ for all $(a,b) \in \mathbb{R}^2$, which can be seen by means of the substitution $t = -x$.

Now, let $p \in \mathbb{R}$ with $p \geqslant 1$. If $a \neq 0$ then, performing the change of variable $t = ax + b$ and some simple calculations, one can obtain that

$$\|(a,b)\|_p^p = \frac{(a+b)|a+b|^p + (a-b)|a-b|^p}{a(p+1)}.$$

On the other hand $\|(0,b)\|_p^p = 2|b|^p$.

It can also be seen that

$$\max\{a : \|(a,b)\|_p = 1\} = \sqrt[p]{p+1},$$

from which it follows that $M_1^p = \sqrt[p]{p+1}$.

The plot thickens when studying polynomials of higher degree. Now consider the space $(\mathscr{P}_2(\mathbb{R}), \|\cdot\|_p)$ or, in other words, the space $\mathbb{R}^3$ endowed with the norm given by

$$\|(a,b,c)\|_p = \left(\int_{-1}^{1} \left| ax^2 + bx + c \right|^p dx \right)^{\frac{1}{p}},$$

for all $(a,b,c) \in \mathbb{R}^3$. A formula for $\|\cdot\|_1$ on $\mathscr{P}_2(\mathbb{R})$ is obtained below.

THEOREM 2.1. *If $(a,b,c) \in \mathbb{R}^3$, $\Delta = b^2 - 4ac$ and, when $\Delta > 0$, $r_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $r_2 = \frac{-b+\sqrt{\Delta}}{2a}$, we have*

$$\|(a,b,c)\|_1 = \begin{cases} \left| \frac{2a+6c}{3} \right| & \text{if } a = 0 \text{ or } \Delta \leqslant 0 \text{ or } \min\{|r_1|, |r_2|\} \geqslant 1, \\ \frac{\operatorname{sign}(a)(2a^3+6a^2c)+\Delta^{\frac{3}{2}}}{3a^2} & \text{if } a \neq 0,\ \Delta > 0 \text{ and } \max\{|r_1|, |r_2|\} < 1, \\ \frac{\operatorname{sign}(b)(-b^3+6a^2b+6abc)+\Delta^{\frac{3}{2}}}{6a^2} & \text{otherwise.} \end{cases}$$

Proof. If $a = 0$ the problem is trivial. Otherwise, since

$$\|(a,b,c)\|_1 = |a| \cdot \left\| \left(1, \frac{b}{a}, \frac{c}{a}\right) \right\|_1,$$

we can assume without loss of generality that $a = 1$. Thus, we will calculate the value of $\|(1,\beta,\gamma)\|_1$. In this situation one has that $\Delta = \beta^2 - 4\gamma$ and the roots of $p(x) = x^2 + \beta x + \gamma$ are

$$r_1 = \frac{-\beta - \sqrt{\Delta}}{2} \quad \text{and} \quad r_2 = \frac{-\beta + \sqrt{\Delta}}{2}$$

and, of course, if $\Delta > 0$, then $r_1 < r_2$.

The value of $\|(1,\beta,\gamma)\|_1$ will strongly depend on the location of r_1 and r_2, i.e., we will have the following cases:

1. If $\Delta \leqslant 0$ or $\min\{|r_1|,|r_2|\} \geqslant 1$, then
$$\|(1,\beta,\gamma)\|_1 = \left|\int_{-1}^{1}(x^2+\beta x+\gamma)dx\right| = 2\left|\gamma+\frac{1}{3}\right|.$$
2. If $\Delta > 0$ and $r_1, r_2 \in (-1,1)$, then
$$\begin{aligned}\|(1,\beta,\gamma)\|_1 &= \int_{-1}^{r_1}(x^2+\beta x+\gamma)dx - \int_{r_1}^{r_2}(x^2+\beta x+\gamma)dx + \int_{r_2}^{1}(x^2+\beta x+\gamma)dx \\ &= \frac{1}{3}\left(2+6\gamma+\Delta^{\frac{3}{2}}\right).\end{aligned}$$
3. If $\Delta > 0$, $r_1 \leqslant -1$ and $r_2 \in (-1,1)$, then
$$\begin{aligned}\|(1,\beta,\gamma)\|_1 &= -\int_{-1}^{r_2}(x^2+\beta x+\gamma)dx + \int_{r_2}^{1}(x^2+\beta x+\gamma)dx \\ &= \frac{1}{6}\left(-\beta^3+6\beta(1+\gamma)+\Delta^{\frac{3}{2}}\right),\end{aligned}$$
and since $\beta > 0$, we obtain
$$\|(1,\beta,\gamma)\|_1 = \frac{1}{6}\left(\operatorname{sign}(\beta)(-\beta^3+6\beta(1+\gamma))+\Delta^{\frac{3}{2}}\right).$$
4. Finally, if $\Delta > 0$, $r_1 \in (-1,1)$ and $r_2 \geqslant 1$, then
$$\begin{aligned}\|(1,\beta,\gamma)\|_1 &= \int_{-1}^{r_1}(x^2+\beta x+\gamma)dx - \int_{r_1}^{1}(x^2+\beta x+\gamma)dx \\ &= \frac{1}{6}\left(\beta^3-6\beta(1+\gamma)+\Delta^{\frac{3}{2}}\right).\end{aligned}$$
Since now $\beta < 0$, it follows
$$\|(1,\beta,\gamma)\|_1 = \frac{1}{6}\left(\operatorname{sign}(\beta)(-\beta^3+6\beta(1+\gamma))+\Delta^{\frac{3}{2}}\right).$$

A simple substitution and some calculations lead to the conclusion of the theorem.

The complexity of the previous formula gives an idea of the difficulties involved in the study of the spaces of polynomials with the norms $\|\cdot\|_p$, $p \geqslant 1$. However a number of interesting results can be obtained dealing with the norm $\|\cdot\|_2$ as we will see right now.

If $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$ then one can obtain by direct integration that
$$\|a_0+a_1x+\ldots+a_nx^n\|_2^2 = 2\sum_{i=0}^{n}\frac{a_i^2}{2i+1} + 2\sum_{0\leqslant i<j\leqslant n}[1+(-1)^{i+j}]\frac{a_ia_j}{i+j+1}. \qquad (3)$$

If we restrict attention to the spaces of trinomials $\mathscr{P}_{m,n}(\mathbb{R})$, then:

THEOREM 2.2. *For every $m,n \in \mathbb{N}$ with $m > n$ and every $a,b,c \in \mathbb{R}$ we have that $\|ax^m + bx^n + c\|_2^2$ is given by*

$$\frac{2a^2}{2m+1} + \frac{2b^2}{2n+1} + 2c^2 + \begin{cases} \frac{4ab}{m+n+1} & \text{if } m,n \text{ are odd,} \\ \frac{4ac}{m+1} & \text{if } m \text{ is even and } n \text{ is odd,} \\ \frac{4bc}{n+1} & \text{if } m \text{ is odd and } n \text{ is even,} \\ \frac{4ab}{m+n+1} + \frac{4ac}{m+1} + \frac{4bc}{n+1} & \text{if } m,n \text{ are even.} \end{cases} \tag{4}$$

In the special case where m and n have different parity, the above result provides a very simple parametrization of the unit sphere $\mathsf{S}_{m,n}$ of $(\mathscr{P}_n(\mathbb{R}), \|\cdot\|_2)$. In particular, if we set

$$f_{k,l}(\zeta,\eta) = \sqrt{\frac{2l+1}{2}\left(1 - \frac{2\zeta^2}{2k+1} - 2\eta^2 - \frac{4\zeta\eta}{k+1}\right)}$$

and

$$E_k = \left\{(\zeta,\eta) \in \mathbb{R}^2 : 1 - \frac{2\zeta^2}{2k+1} - 2\eta^2 - \frac{4\zeta\eta}{k+1} \geqslant 0\right\},$$

for $k,l \in \mathbb{N}$, then $E_m = \pi_{ac}(\mathsf{S}_{m,n})$ if m is even and n is odd, and $E_n = \pi_{bc}(\mathsf{S}_{m,n})$ if m is odd and n is even, where π_{ac} and π_{bc} denote the linear projections given by $\pi_{ac}(a,b,c) = (a,c)$ and $\pi_{bc}(a,b,c) = (b,c)$ respectively, for every $(a,b,c) \in \mathbb{R}^3$. Therefore

$$\mathsf{S}_{m,n} = \begin{cases} \{ax^m \pm f_{m,n}(a,c)x^n + c : (a,c) \in E_m\} & \text{if } m \text{ is even and } n \text{ is odd,} \\ \{\pm f_{n,m}(b,c)x^m + bx^n + c : (b,c) \in E_n\} & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

2.1. Markov constants in $(\mathscr{P}_{m,n}(\mathbb{R}), \|\cdot\|_2)$

Using the parametrization of $\mathsf{S}_{m,n}$ found just above, we can obtain the following sharp Markov estimate for the space $\mathscr{P}_{m,n}(\mathbb{R})$ whenever m,n have different parity.

THEOREM 2.3. *Let $m,n \in \mathbb{N}$ with $m > n$ have different parity. Also, let $M_{m,n}$ be the best constant in the inequality*

$$\|p'\|_2 \leqslant M_{m,n}\|p\|_2,$$

for every $p \in \mathscr{P}_{m,n}(\mathbb{R})$. Then

$$M_{m,n} = \begin{cases} (m+1)\sqrt{\dfrac{2m+1}{2m-1}} & \text{if } m \text{ is even and } n \text{ is odd,} \\ m\sqrt{\dfrac{2m+1}{2m-1}} & \text{if } m \text{ is odd, } n \text{ is even and } m > n+1, \\ m\sqrt{\dfrac{2m-1}{2m-3}} & \text{if } m \text{ is odd and } n = m-1. \end{cases}$$

Proof. First, when m is even and n is odd, using the parametrization of $\mathsf{S}_{m,n}$ obtained above, we have

$$\begin{aligned} M_{m,n}^2 &= \sup\{\|amx^{m-1} \pm nf_{m,n}(a,c)x^{n-1}\|_2^2 : (a,c) \in E_m\} \\ &= \sup\left\{\frac{2m^2a^2}{2m-1} + \frac{2n^2 f_{m,n}(a,c)^2}{2n-1} : (a,c) \in E_m\right\}. \end{aligned}$$

Now, since $P(a,c) := \frac{2m^2a^2}{2m-1} + \frac{2n^2 f_{m,n}(a,c)^2}{2n-1}$ is a 2-homogeneous polynomial plus a constant and E_m is the unit ball of a Banach space (a Hilbert space actually), then P attains its maximum at either $(0,0)$ or on ∂E_m. Notice that $f_{m,n} \equiv 0$ on ∂E_m. Hence

$$\begin{aligned} M_{m,n}^2 &= \max\left\{|P(0,0)|, \sup\left\{|P(a,c)| : \frac{2a^2}{2m+1} + 2c^2 + \frac{4ac}{m+1} = 1\right\}\right\} \\ &= \max\left\{n^2\frac{2n+1}{2n-1}, \frac{2m^2}{2m-1}\sup\left\{a : \frac{2a^2}{2m+1} + 2c^2 + \frac{4ac}{m+1} = 1\right\}^2\right\}. \end{aligned}$$

Some technical (but simple) calculations, lead to

$$M_{m,n} = \max\left\{(m+1)\sqrt{\frac{2m+1}{2m-1}}, n\sqrt{\frac{2n+1}{2n-1}}\right\}.$$

Similarly, for m odd and n even we have

$$M_{m,n} = \max\left\{m\sqrt{\frac{2m+1}{2m-1}}, (n+1)\sqrt{\frac{2n+1}{2n-1}}\right\}.$$

Now, if m is even and n is odd, then $m\sqrt{\frac{2m+1}{2m-1}} > n\sqrt{\frac{2n+1}{2n-1}}$, and if m is odd and n is even, then $m\sqrt{\frac{2m+1}{2m-1}} < (n+1)\sqrt{\frac{2n+1}{2n-1}}$ if and only if $n = m-1$. This proves the result.

REMARK 2.4. Notice that in the previous theorem we have used that

$$\sup\left\{a : \frac{2a^2}{2m+1} + 2c^2 + \frac{4ac}{m+1} = 1\right\}$$

is attained at the values

$$a = (m+1)\frac{\sqrt{m+1/2}}{m} \text{ and } c = -\frac{\sqrt{m+1/2}}{m}.$$

This fact will be again used in Theorem 2.6.

REMARK 2.5. If $m,n \in \mathbb{N}$ have the same parity, the idea used to prove Theorem 2.3 yields an extraordinary long, complicated and not-easy-to-handle formula for $M_{m,n}$. In the authors' opinion, neither the description of that proof nor the inclusion of an explicit formula for $M_{m,n}$ whenever $m,n \in \mathbb{N}$ have the same parity, would improve the present paper, for which reason the details are spared to the interested reader.

2.2. Bernstein functions in $(\mathscr{P}_{m,n}(\mathbb{R}), \|\cdot\|_2)$

Given a fixed $x \in [-1,1]$, in this section we determine the maximum of $p'(x)$ when p ranges through all the elements of $\mathsf{B}_{m,n}$. This maximum is the solution to the L^2-version of the classical Bernstein problem for polynomials in $\mathscr{P}_{m,n}(\mathbb{R})$.

THEOREM 2.6. *If for every $m,n \in \mathbb{N}$ with different parity and every $x \in [-1,1]$ we define*

$$\mathscr{M}_{m,n}(x) = \begin{cases} \sqrt{\frac{n^2(2n+1)x^{2(n-1)}+(m+1)^2(2m+1)x^{2(m-1)}}{2}} & \text{if } m \text{ is even and } n \text{ is odd,} \\ \sqrt{\frac{m^2(2m+1)x^{2(m-1)}+(n+1)^2(2n+1)x^{2(n-1)}}{2}} & \text{if } m \text{ is odd and } n \text{ is even,} \end{cases}$$

then

$$|p'(x)| \leqslant \mathscr{M}_{m,n}(x)\|p\|_2,$$

for all $p \in \mathscr{P}_{m,n}(\mathbb{R})$. Moreover, the estimate $\mathscr{M}_{m,n}(x)$ is sharp in the previous inequality.

Proof. For a fixed $x \in [-1,1]$, notice that if $\mathscr{M}_{m,n}(x)$ represents the best constant in

$$|p'(x)| \leqslant \mathscr{M}_{m,n}(x)\|p\|_2,$$

for every $p \in \mathscr{P}_{m,n}(\mathbb{R})$, then

$$\mathscr{M}_{m,n}(x) = \sup\{|p'(x)| : p \in \mathsf{S}_{m,n}\}.$$

Recall that $\mathsf{S}_{m,n}$ is the unit sphere of $(\mathscr{P}_n(\mathbb{R}), \|\cdot\|_2)$. From now on we will assume that m is even and n is odd. The other case is similar. Then using the parametrization of $\mathsf{S}_{m,n}$ obtained as a consequence of Theorem 2.2, we have

$$\begin{aligned} \mathscr{M}_{m,n}(x) &= \sup\{|amx^{m-1} \pm n f_{m,n}(a,c)x^{n-1}| : (a,c) \in E_m\} \\ &= \sup\{m|a||x|^{m-1} + n f_{m,n}(a,c)|x|^{n-1} : (a,c) \in E_m\}, \end{aligned}$$

where

$$f_{m,n}(a,c) = \sqrt{\frac{2n+1}{2}\left(1 - \frac{2a^2}{2m+1} - 2c^2 - \frac{4ac}{m+1}\right)},$$

and

$$E_m = \left\{(a,c) \in \mathbb{R}^2 : 1 - \frac{2a^2}{2m+1} - 2c^2 - \frac{4ac}{m+1} \geqslant 0\right\}.$$

Let us define

$$F_{x,m,n}(a,c) := m|a||x|^{m-1} + nf_{m,n}(a,c)|x|^{n-1},$$

for every $(a,c) \in E_m$. Then $F_{x,m,n}$ attains its maximum either at an interior point of E_m or at a point in ∂E_m. In order to maximize $F_{x,m,n}$ over ∂E_m, just notice that $F_{x,m,n}(a,c) = m|a||x|^{m-1}$ for every $(a,c) \in \partial E_m$ and that the maximum value of $|a|$ for the points (a,c) in ∂E_m can be easily proved to be $a_{\max} = \frac{m+1}{m}\sqrt{\frac{2m+1}{2}}$. On the other hand, if $F_{x,m,n}$ attains it maximum over E_m at an interior point of E_m, it has to be either at a critical point of $F_{x,m,n}$ or at a point where $F_{x,m,n}$ is not differentiable, namely the points $(0,c)$. The reader can check that $\frac{\partial F_{x,m,n}}{\partial c}(a,c) = 0$ is equivalent to $a = -(m+1)c$. Now, if we assume that a is positive, replacing the previous value of a in the equation $\frac{\partial F_{x,m,n}}{\partial a}(a,c) = 0$, after some algebraic calculations we arrive at the following value for c:

$$c_0 = -\frac{(2m+1)(m+1)|x|^{m-n}}{m\sqrt{2(2n+1)n^2 + 2(m+1)^2(2m+1)x^{2(m-n)}}}. \tag{5}$$

A similar conclusion is derived if a is negative. Thus we have proved that $F_{x,m,n}$ has a pair of symmetrical critical points, namely $(-(m+1)c_0, c_0)$ and $((m+1)c_0, -c_0)$, with c_0 given by (5) and that these critical points turn out to be in the interior of E_m as the reader can easily check. Moreover, at those critical points we have that

$$\begin{aligned} F_{x,m,n}(-(m+1)c_0, c_0) &= F_{x,m,n}((m+1)c_0, -c_0) \\ &= \sqrt{\frac{n^2(2n+1)x^{2(n-1)} + (m+1)^2(2m+1)x^{2(m-1)}}{2}}, \end{aligned}$$

and that, clearly,

$$(m+1)\sqrt{\frac{2m+1}{2}}|x|^{m-1} \leqslant \sqrt{\frac{n^2(2n+1)x^{2(n-1)} + (m+1)^2(2m+1)x^{2(m-1)}}{2}},$$

for every $x \in [-1,1]$. Also, notice that if $a = 0$,

$$\max\{F_{x,m,n}(0,c) : (0,c) \in E_m\} = n\sqrt{\frac{2n+1}{2}}|x|^{n-1},$$

and that, clearly,

$$n\sqrt{\frac{2n+1}{2}}|x|^{n-1} \leqslant \sqrt{\frac{n^2(2n+1)x^{2(n-1)} + (m+1)^2(2m+1)x^{2(m-1)}}{2}}.$$

This concludes the proof.

To finish we will consider another interesting question related to the study of polynomial inequalities when considering constraint families of polynomials. Given a mapping $\phi : [-1,1] \to [0,+\infty)$ (a majorant in the sequel), the set $\mathscr{P}_n^{\phi}(\mathbb{R})$ stands for the set of polynomials of degree at most n on the real line satisfying $|p(x)| \leqslant \phi(x)$ for all

$x \in [-1,1]$. The Markov problem for polynomials with a curved majorant ϕ consists of finding the best constant M_n^{ϕ} in the inequality

$$\|p'\| \leqslant M_n^{\phi} \|p\|,$$

for $p \in \mathscr{P}_n^{\phi}(\mathbb{R})$. The cases where $\phi(x) = c(x) = \sqrt{1-x^2}$ (circular majorant) and $\phi(x) = |x|$ (linear majorant) were studied by Rahman in [15], where sharp Markov constants are given, and in [13] where the authors provide sharp Markov constants and Bernstein estimates in both cases. In the following remark we give the Markov constants in $\mathscr{P}_3^{c}(\mathbb{R})$ (polynomials with circular majorant) for the L^1 and L^2 norms. Notice that those constants are the same as the Markov constants of the polynomials in $\mathscr{P}_3(\mathbb{R})$ with roots at ± 1 with the L^1 and L^2 norms.

REMARK 2.7. If $p_{a,b}(x) = (1-x^2)(ax+b)$ then

$$\|p_{a,b}\|_1 = \begin{cases} \dfrac{4|b|}{3} & \text{if } a=0 \text{ or } \left(a \neq 0 \text{ and } \left|\dfrac{b}{a}\right| \geqslant 1\right), \\ \left|\dfrac{b^4 - 6a^2b^2 - 3a^4}{6a^3}\right| & \text{if } a \neq 0 \text{ and } \left|\dfrac{b}{a}\right| \leqslant 1. \end{cases}$$

This, together with Theorem 2.1, let us obtain

$$\|p'_{a,b}\|_1 = \begin{cases} 2|b| & \text{if } a=0, \\ \dfrac{(12a^2+4b^2)^{\frac{3}{2}}}{27a^2} & \text{if } a \neq 0 \text{ and } \left|\dfrac{b}{a}\right| \leqslant 1, \\ \dfrac{\operatorname{sign}(a)4|b|(9a^2-b^2) + 4(3a^2+b^2)^{\frac{3}{2}}}{27a^2} & \text{if } a \neq 0 \text{ and } \left|\dfrac{b}{a}\right| \geqslant 1. \end{cases}$$

Thus, if we call $\alpha = \frac{b}{a}$, one has

$$\frac{\|p'_{a,b}\|_1}{\|p_{a,b}\|_1} = \begin{cases} 3/2 & \text{if } a=0, \\ \dfrac{16(\alpha^2+3)^{\frac{3}{2}}}{9|\alpha^4 - 6\alpha^2 - 3|} & \text{if } |\alpha| \leqslant 1, \\ \dfrac{9|\alpha| - |\alpha|^3 + (\alpha^2+3)^{\frac{3}{2}}}{9|\alpha|} & \text{if } |\alpha| \geqslant 1. \end{cases}$$

Now, since the maximum of the previous mapping is equal to $\frac{16\sqrt{3}}{9}$, which is attained at $\alpha = 0$, we have that

$$\|p'_{a,b}\|_1 \leqslant \frac{16\sqrt{3}}{9} \|p_{a,b}\|_1,$$

and equality is attained for the polynomials $p_{a,0}$, with $a \neq 0$.

Similarly, if we now work with the L^2 norm, one arrives at

$$\frac{\|p'_{a,b}\|_2}{\|p_{a,b}\|_2} = \begin{cases} \sqrt{5/2} & \text{if } a = 0, \\ \sqrt{\dfrac{21+35\alpha^2}{2+14\alpha^2}} & \text{otherwise.} \end{cases}$$

Using the latter we obtain

$$\|p'_{a,b}\|_2 \leqslant \sqrt{\frac{21}{2}}\|p_{a,b}\|_2,$$

and equality is attained for the polynomials $p_{a,0}$, with $a \neq 0$. We spare the details of the technical calculations to the interested reader.

REFERENCES

[1] S. BERNSTEIN, *Sur L'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Memoires de l'Académie Royale de Belgique, **4** (1912), 1–103.

[2] R. P. BOAS, *Inequalities for the derivatives of polynomials*, Math. Mag., **42** (1969), 165–174.

[3] P. BORWEIN AND T. ERDÉLYI, *Polynomials and polynomial inequalities*, Graduate Texts in Mathematics, **161**, Springer-Verlag, New York, 1995.

[4] R. J. DUFFIN AND A. C. SCHAEFFER, *On some inequalities of S. Bernstein and W. Markoff*, Bull. Amer. Math. Soc., **44** (1938), 289–297.

[5] L. A. HARRIS, *Bounds on the derivatives of holomorphic functions of vectors*, Colloque D'Analyse (Rio de Janeiro, 1972), 145–163, ed. L. Nachbin, Act. Sc. et Ind., 1367, Herman, Paris, 1975.

[6] L. A. HARRIS, *Multivariate Markov polynomial inequalities and Chebyshev nodes*, J. Math. Anal. Appl., **338** (2008), 350–357.

[7] A. A. MARKOV, *On a problem of D. I. Mendeleev* (Russian), Zap. Im. Akad. Nauk., **62** (1889), 1–24.

[8] A. A. MARKOV, *On a question by D. I. Mendeleev*, Electronic article to be downloaded from http://www.math.technion.ac.il/hat/papers.html.

[9] V. MARKOV, *Über Polynome, die in einen gegebenen Intervalle möglichst wenig von Null abweichen*, Math. Ann. **77** (1916), 213–258.

[10] G. A. MUÑOZ-FERNÁNDEZ AND Y. SARANTOPOULOS, *Bernstein and Markov-type inequalities for polynomials on real Banach spaces*, Math. Proc. Camb. Phil. Soc., **133** (2002), 515–530.

[11] G. A. MUÑOZ-FERNÁNDEZ AND J.B. SEOANE-SEPÚLVEDA, *Geometry of Banach spaces of Trinomials*, J. Math. Anal. Appl., **340** (2008), 1069–1087.

[12] G. A. MUÑOZ-FERNÁNDEZ, Y. SARANTOPOULOS AND J.B. SEOANE-SEPÚLVEDA, *An application of the Krein-Milman Theorem to Bernstein and Markov inequalities*, J. Convex Anal., **15** (2008), 299–312.

[13] G. A. MUÑOZ-FERNÁNDEZ, V.M. SÁNCHEZ AND J.B. SEOANE-SEPÚLVEDA, *Estimates on the derivative of a polynomial with a curved majorant using convex techniques*, J. Convex Anal., **17** (2010) 241–252.

[14] S. NEUWIRTH, *The maximum modulus of a trigonometric trinomial*, J. Anal. Math., **104** (2008), 371–396.

[15] Q. I. RAHMAN, *On a problem of Turán about polynomials with curved majorants*, Trans. Amer. Math. Soc., **163** (1972), 447–455.

[16] Q. I. RAHMAN AND G. SCHMEISSER, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, **26**. The Clarendon Press, Oxford University Press, Oxford, 2002.

[17] Y. SARANTOPOULOS, *Bounds on the derivatives of polynomials on Banach spaces*, Math. Proc. Camb. Phil. Soc., **110** (1991), 307–312.

[18] V. I. SKALYGA, *Analogues of the Markov and Bernstein inequalities for polynomials in Banach spaces*, Izv. Math., **61** (1998), 143–159.
[19] V. I. SKALYGA, *Analogues of the Markov and Benstein inequalities on convex bodies in Banach spaces*, Izv. Math., **62** (1998), 375–397.
[20] V. I. SKALYGA, *Bounds on the derivatives of polynomials on entrally symmetric convex bodies* (Russian), Izv. Ross. Akad. Nauk Ser. Mat., **69** (2005), 179–192; translation in Izv. Math., **69** (2005), 607–621.
[21] D. R. WILHELMSEN, *A Markov inequality in several dimensions*, J. Approx. Theory, **11** (1974), 216–220.

(Received November 26, 2008)

G. A. Muñoz-Fernández
Departamento de Análisis Matemático
Universidad Complutense de Madrid
Plaza de Ciencias 3
28040 Madrid
Spain
e-mail: gustavo_fernandez@mat.ucm.es

V. M. Sánchez
Departamento de Análisis Matemático
Universidad Complutense de Madrid
Plaza de Ciencias 3
28040 Madrid
Spain
e-mail: victorms@mat.ucm.es

J. B. Seoane-Sepúlveda
Departamento de Análisis Matemático
Universidad Complutense de Madrid
Plaza de Ciencias 3
28040 Madrid
Spain
e-mail: jseoane@mat.ucm.es