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SINGULAR INTEGRALS WITH MIXED HOMOGENEITY IN PRODUCT SPACES

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Abstract. Let $\Omega \in L(\log L^+)^2(S^{n-1}\times S^{m-1})$ $(n,m\geqslant 2)$ satisfy some cancellation conditions. We prove the L^p boundedness ($1<p<\infty$) of the singular integral

$$Tf(x_1,x_2) = \text{p. v.} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \frac{\Omega(y_1',y_2')\,h(\rho_1(y_1),\rho_2(y_2))}{\rho_1^{\alpha}(y_1)\,\rho_2^{\beta}(y_2)} f(x_1-y_1,x_2-y_2)\,dy_1\,dy_2,$$

where ρ_1, ρ_2 are some metrics which are homogeneous with respect to certain non-isotropic dilations. We also study the above singular integral along some surfaces.

1. Introduction

Consider the elliptic differential operator with constant coefficients

$$D=\sum_{i,j=1}^{n} a_{i,j}\,\frac{\partial^2}{\partial x_i \partial x_j}.$$

As noted by E. Stein and S. Wainger [14], in order to study the existence and regularity results of D, one needs to consider singular integral operators with convolution kernels K satisfying the following conditions

$a)$ K is homogeneous of degree $-n$: $K(tx_1,\dots,tx_n)=t^{-n}K(x_1,\dots,x_n)$, $\quad t>0$,
$b)$ K is C^∞ away from the origin,
$c)$ $\displaystyle\int_{S^{n-1}} K(x)\,dx = 0.$

Similarly, to study the existence and regularity results of the heat equation

$$L(u)=\frac{\partial u}{\partial x_1}-\sum_{j=2}^{n}\frac{\partial^2 u}{\partial x_j^2},$$

one considers singular integral operators with the corresponding kernels K that satisfy

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$\tilde{a}$) $K(t^2x_1,...,tx_n) = t^{-n-1}K(x_1,...,x_n), \quad t > 0,$
$\tilde{b}$) K is C^∞ away from the origin,
$\tilde{c}$) $\int_{S^{n-1}} K(x)\,(2x'^2_1 + x'^2_2 + \cdots + x'^2_n)\,dx = 0.$

For a more general parabolic differential operator with constant coefficients, E. Fabes and N. Rivière [9] studied singular integrals with kernels K which satisfy (among some other conditions)
$\bar{a}$) $K(t^{\alpha_1}x_1,...,t^{\alpha_n}x_n) = t^{\alpha}K(x_1,...,x_n), \quad t > 0, \quad \alpha = \sum_{i=1}^n \alpha_i,$
$\bar{b}$) $\int_{S^{n-1}} K(x)J(x')\,dx = 0$, where $J(x') \in C^\infty((0,2\pi)^{n-2} \times (0,\pi))$ and without loss of generality, $1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n$.

Note that the property ($\bar{a}$) above can be expressed as $K(A_tx) = |det(A_t)|^{-1}K(x)$, where $A_t = \mathrm{diag}[t^{\alpha_1},\cdots,t^{\alpha_n}]$ is a diagonal matrix. Note also that for each nonzero $x \in \mathbb{R}^n$, the function $F(x,t) = \sum_{i=1}^n t^{-2\alpha_i}x_i^2$ is a strictly decreasing function of $t > 0$. Therefore, there exists a unique value of t which satisfies the equation $F(x,t) = 1$. If we define $\rho(x) = t$ and $\rho(0) = 0$, then it follows from [9, 14] that ρ is a metric on $\mathbb{R}^n$. It is well known that $(\mathbb{R}^n,\rho)$ is a homogeneous group which admits a family of dilations $\delta_t = \exp(A\log t)$ such that $\rho(\delta_t x) = t\rho(x),\ t > 0$. Here A is a diagonalizable linear operator with positive eigenvalues $\alpha_1,\alpha_2,...,\alpha_n$. By a change of variables to polar coordinates, each nonzero $x \in \mathbb{R}^n$ can be written as $x = \delta_\rho(x'),\ (x' = x/|x|)$. Thus, there is a unique Radon measure (see [9], [10, p. 14]) $d\tilde{\sigma}(x') = J(x')d\sigma(x')$, where $J(x') = \alpha_1x'^2_1 + ... + \alpha_nx'^2_n$ is a C^∞ function on $\mathbb{S}^{n-1}$ which is bounded below and above by α_1 and α_n respectively.

Now let $Tf(x) = \text{p. v.}K * f(x)$, where the kernel $K(x) = \Omega(x')\rho^{-\alpha}(x)$, and $\alpha = \sum_{i=1}^n \alpha_i$. The following result has been obtained by E. Fabes and N. Rivière.

THEOREM 1.1. [9] *If* $\Omega \in C^1(S^{n-1})$ *satisfies the cancellation condition*

$$\int_{S^{n-1}} \Omega(x')J(x')\,d\sigma(x') = 0,$$

then the singular integral operator T *is bounded on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

Subsequently, A. Nagel, N. Rivière, and S. Wainger [12] improved the above theorem by weakening the regularity condition on Ω as follows

THEOREM 1.2. [12] *If* $\Omega \in L\log^+ L(S^{n-1})$ *satisfies the cancellation condition above, then*

$$||Tf||_p \leqslant C||f||_p \text{ for } 1 < p < \infty.$$

Recently, the above result has been extended further by Y. Chen, Y. Ding, and D. Fan:

THEOREM 1.3. [4] *If* $\Omega \in H^1(S^{n-1})$ *(the Hardy space on the unit sphere* S^{n-1}*) satisfies the cancellation condition above, then*

$$||Tf||_p \leqslant C||f||_p \text{ for } 1 < p < \infty.$$

By the above Theorems 1.2 [12] and 1.3 [4], the L^p boundedness of singular integrals with mixed homogeneity in $\mathbb{R}^n$ has been obtained. The extension of this result to product spaces is not simple, since the singularity now becomes two lower dimensional surfaces, instead of a singular point as in the previous case. The purpose of this paper is to extend the result in [12] to product spaces (see section 2.3). In classical harmonic analysis, it is well-known that singular integrals are dominated by Hardy-Littlewood maximal functions and square functions. We will show in section 3.2 that a singular integral along surfaces in the product domain is controlled by Hardy-Littlewood maximal functions acting on each variable along surfaces. We state our theorems in sections 2.3 and 3.2, and their proofs are given in sections 2.4 and 3.3 respectively. For recent works on the topic of singular integrals, the reader may view [1-9] among many other good references that are not listed in this paper.

2. Singular integral with rough kernels

2.1. Definitions and Notations

Most recent works dealing with singular integrals follow the ideas in [7]. In this section, we will extend two lemmas in [7]. For notational convenience, throughout the rest of this paper the dimesions n_1 and n are the same, and similarly the dimensions n_2 and m are equal to each other. Let ρ_1 be the metric on $\mathbb{R}^{n_1}$ obtained from the unique solution of the equation $\sum_{i=1}^{n_1} t^{-2\alpha_i} x_i^2 = 1$, where $t > 0$, $x = (x_1, x_2, ..., x_{n_1}) \in \mathbb{R}^{n_1}$, and $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{n_1}$. Similarly, let ρ_2 be the metric on $\mathbb{R}^{n_2}$ obtained from the unique solution of the equation $\sum_{i=1}^{n_2} t^{-2\beta_i} y_i^2 = 1,\ t > 0,\ y = (y_1, y_2, ..., y_{n_2}) \in \mathbb{R}^{n_2}$, and $0 < \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_{n_2}$. Denote $A^{(1)}_{\rho_1} = \mathrm{diag}[\rho_1^{\alpha_1}, ..., \rho_1^{\alpha_{n_1}}]$ and $A^{(2)}_{\rho_2} = \mathrm{diag}[\rho_2^{\beta_1}, ..., \rho_2^{\beta_{n_2}}]$, $\rho_1, \rho_2 > 0$. Denote $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_{n_1}$, and $\beta = \beta_1 + \beta_2 + \cdots + \beta_{n_2}$. Denote $J_1(u) = \alpha_1 u_1^2 + ... + \alpha_{n_1} u_{n_1}^2$ and $J_2(v) = \beta_1 v_1^2 + ... + \beta_{n_2} v_{n_2}^2$, where $u = (u_1, ..., u_{n_1}) \in S^{n_1-1}$ and $v = (v_1, ..., v_{n_2}) \in S^{n_2-1}$.

If f is a function defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ and $h_i \in \mathbb{R}^{n_i}$ $(i = 1, 2)$, $x_i \in \mathbb{R}^{n_i}$, we define (following the notations in [5])

$$\Delta^1_{h_1} f(x_1, x_2, x_3) = f(x_1 + h_1, x_2, x_3) - f(x_1, x_2, x_3)$$

$$\Delta^2_{h_2} f(x_1, x_2, x_3) = f(x_1, x_2 + h_2, x_3) - f(x_1, x_2, x_3)$$

$$\Delta^{1,2}_{h_1, h_2} f(x_1, x_2, x_3) = \Delta^1_{h_1}\left(\Delta^2_{h_2} f(x_1, x_2, x_3)\right).$$

Given a measure $\mu \in \mathbb{R}^{n_1+n_2+n_3}$, we define the measures $\mu^{(1)} \in \mathbb{R}^{n_2+n_3}$, $\mu^{(2)} \in \mathbb{R}^{n_1+n_3}$, $\mu^{(1,2)} \in \mathbb{R}^{n_3}$ by $\mu^{(1)}(E) = \mu(\mathbb{R}^{n_1} \times E)$, $\mu^{(2)}(F) = \mu(\mathbb{R}^{n_2} \times F)$, $\mu^{(1,2)}(G) =$

$\mu(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times G)$, where $E,\ F,\ G$ are Borel sets in $\mathbb{R}^{n_2+n_3}$, $\mathbb{R}^{n_1+n_3}$, $\mathbb{R}^{n_3}$ respectively. Finally, we write $|\sigma|$ for the total variation of the measure σ.

2.2. Preliminary Results

LEMMA 1. [10] *Let $\Phi^{(i)}$ be a function in $\mathscr{S}(\mathbb{R}^{n_i})$ $(i=1,2)$ such that supp$\hat{\Phi}^{(i)}\subset\{\zeta_i\in\mathbb{R}^{n_i}:1/2\leqslant\rho_i(\zeta_i)\leqslant 2\}$. Define $\Psi^{(i)}$ by $\hat{\Psi}^{(i)}(\zeta_i)=\hat{\Phi}^{(i)}(\rho_i(\zeta_i))$. For $j,k\in\mathbb{Z}$, $x=(x_1,x_2,...,x_{n_1})\in\mathbb{R}^{n_1}$, $y=(y_1,y_2,...,y_{n_2})\in\mathbb{R}^{n_2}$, define $\Psi_j^{(1)}(x)=2^{-j\alpha}\Psi^{(1)}(A^{(1)}_{2^{-j}}x)$ and $\Psi_k^{(2)}(y)=2^{-k\beta}\Psi^{(2)}(A^{(2)}_{2^{-k}}y)$. Set $\Psi(x,y)=\Psi^{(1)}(x)\Psi^{(2)}(y)$ and let $\Psi_{j,k}(x,y)=\Psi_j^{(1)}(x)\Psi_k^{(2)}(y)$. Let δ be the Dirac distribution on $\mathbb{R}^{n_3}$. Then for $f\in L^p(\mathbb{R}^{n_1+n_2})$, $g\in L^p(\mathbb{R}^{n_1+n_2+n_3})$, we have*

$$\left\|\left(\sum_{j,k}|\Psi_{j,k}*f|^2\right)^{1/2}\right\|_p\leqslant C\,||f||_p,$$

and

$$\left\|\left(\sum_{j,k}|(\Psi_{j,k}\otimes\delta)*g|^2\right)^{1/2}\right\|_p\leqslant C\,||g||_p \text{ for } 1<p<\infty.$$

REMARK 1. Lemma 1 is a discrete version of Theorem 7.7 [10, p. 223] which can be extended to the product setting (see [8], [13, pp. 28–47]).

LEMMA 2. [5] *Let $\sigma_{j,k}$ be Borel measures in $\mathbb{R}^{n_1+n_2+n_3}$ such that $||\sigma_{j,k}||\leqslant C$ for all $j,k\in\mathbb{Z}$. If $\sigma^*(f)=\sup_{j,k}||\sigma_{j,k}|*f|$ is bounded in $L^q(\mathbb{R}^{n_1+n_2+n_3})$ for some $q>1$, then the following vector value inquality holds*

$$\left\|\left(\sum_{j,k}|\sigma_{j,k}*g_{j,k}|^2\right)^{1/2}\right\|_{p_o}\leqslant C\left\|\left(\sum_{j,k}|g_{j,k}|^2\right)^{1/2}\right\|_{p_o}\quad \text{for}\quad \left|\frac{1}{p_o}-\frac{1}{2}\right|=\frac{1}{2q}.$$

LEMMA 3. *Let $\Omega\in L(\log L^+)^2(S^{n-1}\times S^{m-1})$, $(n,m\geqslant 2)$ satisfy*

a) $\int_{S^{n-1}}\Omega(u,v)J_1(u)\,d\mu_1(u)=0\ \forall\ v\in S^{m-1}$, *and*

b) $\int_{S^{m-1}}\Omega(u,v)J_2(v)\,d\mu_2(v)=0\ \forall\ u\in S^{n-1}$, *where $J_1(u)=\alpha_1u_1^2+\cdots+\alpha_nu_n^2$ and $J_2(v)=\beta_1v_1^2+\cdots+\beta_mv_m^2$, $u=(u_1,...,u_n)\in S^{n-1}$, and $v=(v_1,...,v_m)\in S^{m-1}$. Here μ_1, μ_2 are normalized measures on S^{n-1} and S^{m-1} respectively.*

Let $E_o=\{(u,v)\in S^{n-1}\times S^{m-1}:|\Omega(u,v)|\leqslant 2\}$, and for $l\in\mathbb{N}$, let $E_l=\{(u,v)\in S^{n-1}\times S^{m-1}:2^l<|\Omega(u,v)|\leqslant 2^{l+1}\}$. Let $A(\Omega)=\{l\in\mathbb{N}:\mu(E_l)>2^{-4l}\}$, where $\mu=\mu_1\times\mu_2$ is the product measure on $S^{n-1}\times S^{m-1}$. Then Ω has a decomposition

$$\Omega=\Omega_o+\sum_{l\in A(\Omega)}\Omega_l,$$

where $\Omega_o,\ \Omega_l\ \ (l \in A(\Omega))$ *all satisfy the cancellation conditions above and*

$$||\Omega_o||_{L^2(S^{n-1}\times S^{m-1})} \leqslant C,\ ||\Omega_o||_{L^1(S^{n-1}\times S^{m-1})} \leqslant C,$$

$$||\Omega_l||_{L^2(S^{n-1}\times S^{m-1})} \leqslant C2^{2l}||\Omega||_{L^1(E_l)},\ ||\Omega_l||_{L^1(S^{n-1}\times S^{m-1})} \leqslant C||\Omega||_{L^1(E_l)} \text{ for all } l \in A(\Omega).$$

REMARK 2. The proof of this Lemma 3 will be given in section 2.5.

THEOREM A. [5] *Let* $\mu_{j,k}$ *be uniformly bounded positive measures in* $\mathbb{R}^{n_1+n_2+n_3}$. *Suppose that*

$$|\hat{\mu}_{j,k}(\zeta)| \leqslant C|A^{(1)}_{2^{lj}}\zeta_1|^{-a/l}|A^{(2)}_{2^{lk}}\zeta_2|^{-b/l} \tag{1}$$

$$|\Delta^1_{\zeta_1}\hat{\mu}_{j,k}(0,\zeta_2,\zeta_3)| \leqslant C|A^{(1)}_{2^{l(j+1)}}\zeta_1|^{a/l}|A^{(2)}_{2^{lk}}\zeta_2|^{-b/l} \tag{2}$$

$$|\Delta^2_{\zeta_2}\hat{\mu}_{j,k}(\zeta_1,0,\zeta_3)| \leqslant C|A^{(1)}_{2^{lj}}\zeta_1|^{-a/l}|A^{(2)}_{2^{l(k+1)}}\zeta_2|^{b/l} \tag{3}$$

$$|\Delta^{1,2}_{\zeta_1,\zeta_2}\hat{\mu}_{j,k}(0,0,\zeta_3)| \leqslant C|A^{(1)}_{2^{l(j+1)}}\zeta_1|^{a/l}|A^{(2)}_{2^{l(k+1)}}\zeta_2|^{b/l} \tag{4}$$

for some $a,b>0$ *and some arbitrary positive integer* $l \geqslant 1$, *and for all* $j,k \in \mathbb{Z}$. *Suppose also that the maximal functions*

$$\tilde{M}^{(i)}g_i = \sup_{j,k}|\mu^{(i)}_{j,k} * g_i|\ (i=1,2),\ \tilde{M}^{(1,2)}g = \sup_{j,k}|\mu^{(1,2)}_{j,k} * g|$$

are bounded in L^p *for all* $p>1$. *Then* $Mf(x) = \sup_{j,k}|\mu_{j,k} * f(x)|$ *is bounded in* L^p *for all* $p>1$, *and the bound is independent of* l.

THEOREM B. [5] *Let* $\sigma_{j,k}$ *be Borel measures in* $\mathbb{R}^{n_1+n_2+n_3}$ *such that* $||\sigma_{j,k}|| \leqslant C$, *and*

$$|\hat{\sigma}_{j,k}(\zeta_1,\zeta_2,\zeta_3)| \leqslant C|A^{(1)}_{2^{l(j+1)}}\zeta_1|^{a/l}|A^{(2)}_{2^{l(k+1)}}\zeta_2|^{b/l} \tag{5}$$

$$\leqslant C|A^{(1)}_{2^{l(j+1)}}\zeta_1|^{a/l}|A^{(2)}_{2^{lk}}\zeta_2|^{-b/l} \tag{6}$$

$$\leqslant C|A^{(1)}_{2^{lj}}\zeta_1|^{-a/l}|A^{(2)}_{2^{l(k+1)}}\zeta_2|^{b/l} \tag{7}$$

$$\leqslant C|A^{(1)}_{2^{lj}}\zeta_1|^{-a/l}|A^{(2)}_{2^{lk}}\zeta_2|^{-b/l} \tag{8}$$

for some $a,b>0$ *and some arbitrary positive integer* $l \geqslant 1$, *and for all* $j,k \in \mathbb{Z}$. *If* $\sigma^*(f) = \sup_{j,k}||\sigma_{j,k}| * f|$ *is bounded in* $L^q(\mathbb{R}^{n_1+n_2+n_3})$ *for some* $q>1$, *then* $Tf(x) = \sum_{j,k}\sigma_{j,k} * f(x)$ *and* $g(f)(x) = \left(\sum_{j,k}|\sigma_{j,k} * f(x)|^2\right)^{1/2}$ *are bounded in* $L^q(\mathbb{R}^{n_1+n_2+n_3})$ *for* $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{1}{2q}$ *and the bound does not depend on* l.

REMARK 3. Theorems A and B are the modified versions of Theorems 1 and 2 [5] respectively. Note that when $l=1$, $\alpha_1=\alpha_2=\cdots=\alpha_n=1$ and $\beta_1=\beta_2=\cdots=\beta_n=1$, Theorems A and B recover Theorems 1 and 2 [5]. The proofs of these theorems will be given in section 2.6. Also for the rest of this paper, we denote the letter C as a constant which may vary at different occurences. However, it does not depend on any essential variable.

2.3. Main Theorem

Let $h \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$, where $\mathbb{R}^+ = [0,\infty)$. Let $\Omega \in L(\log L^+)^2(S^{n-1} \times S^{m-1})$, $(n,m \geqslant 2)$ satisfy

a) $\int_{S^{n-1}} \Omega(u,v) J_1(u)\, d\mu_1(u) = 0 \ \forall \ v \in S^{m-1}$, and

b) $\int_{S^{m-1}} \Omega(u,v) J_2(v)\, d\mu_2(v) = 0 \ \forall \ u \in S^{n-1}$, where $J_1(u) = \alpha_1 u_1^2 + \cdots + \alpha_n u_n^2$ and $J_2(v) = \beta_1 v_1^2 + \cdots + \beta_m v_m^2$, $u = (u_1,...,u_n) \in S^{n-1}$, and $v = (v_1,...,v_m) \in S^{m-1}$.

For $f \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^m)$ $(n,m \geqslant 2)$, define the singular integral Tf by

$$\begin{aligned} Tf(x_1,x_2) &= \text{p. v. } K * f(x_1,x_2) \\ &= \text{p. v.} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega(y_1', y_2')\, h(\rho_1(y_1), \rho_2(y_2))}{\rho_1^\alpha(y_1)\, \rho_2^\beta(y_2)} f(x_1 - y_1, x_2 - y_2)\, dy_1\, dy_2 \end{aligned}$$

Recall that $\alpha = \alpha_1 + \cdots + \alpha_n$ and $\beta = \beta_1 + \cdots + \beta_m$.

THEOREM 1. *Let $h \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ and let $\Omega \in L(\log L^+)^2(S^{n-1} \times S^{m-1})$, $(n,m \geqslant 2)$ satisfy the cancellation conditions above. Then the singular integral Tf initially defined on the Schwartz space $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^m)$ has a bounded extension from $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$, provided that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_m$. The above results also hold when $0 < \alpha_1 = \alpha_2 = \cdots = \alpha_n$, or $0 < \beta_1 = \beta_2 = \cdots = \beta_m$.*

It should be remarked that $\Omega \in L(\log L^+)^2(S^{n-1} \times S^{m-1})$ $(n,m \geqslant 2)$ is best possible, since it has been shown in [3] that $\forall \varepsilon > 0$, there is an $\Omega \in L(\log L^+)^{2-\varepsilon}(S^{n-1} \times S^{m-1})$ (satisfying certain cancellation conditions) such that the singular integral

$$T_\Omega f(x_1,x_2) = \text{p. v.} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^m} \frac{\Omega({y'}_1, {y'}_2) f(x_1 - y_1, x_2 - y_2)}{|y_1|^n |y_2|^m}\, dy_1\, dy_2$$

is not bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for any $p \in (1,\infty)$.

2.4. Proof of Theorem 1

The decomposition of Ω in Lemma 3 induces the decomposition of the corresponding operators

$$T = T_o + \sum_{l \in A(\Omega)} T_l, \tag{9}$$

where $T_o f = \text{p. v.} K_o * f$ and $T_l f = \text{p. v.} K_l * f$. Here K_o and K_l are defined the same way as the kernel K with Ω being replaced by Ω_o and Ω_l respectively. We will show that for $1 < p < \infty$,

$$||T_o f||_p \leqslant C||f||_p \quad \text{and} \tag{10}$$

$$||T_l f||_p \leqslant C l^2 ||\Omega||_{L^1(E_l)} ||f||_p \quad , l \in A(\Omega). \tag{11}$$

Therefore we will obtain

$$\begin{aligned}||Tf||_p &\leqslant ||T_o f||_p + \sum_{l\in A(\Omega)} ||T_l f||_p \\ &\leqslant C\left(1+\sum_{l\in A(\Omega)} l^2\,||\Omega||_{L^1(E_l)}\right)||f||_p \\ &\leqslant C\left(1+||\Omega||_{L(\log L^+)^2(S^{n-1}\times S^{m-1})}\right)||f||_p \text{ for } 1<p<\infty.\end{aligned}$$

Since the proofs of inequalities (10) and (11) are basically the same, we will only prove (11). We will apply Theorems A and B (without the third variable) to prove (11). For $(x_1,x_2),\ (y_1,y_2),\ (\xi_1,\xi_2)\ \in\ \mathbb{R}^n\times\mathbb{R}^m$, we write

$$\begin{aligned}T_l f(x_1,x_2) &= \text{p. v.} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \frac{\Omega_l(y_1',y_2')h(\rho_1(y_1),\rho_2(y_2))}{\rho_1^{\alpha}(y_1)\,\rho_2^{\beta}(y_2)} f(x_1-y_1,x_2-y_2)\,dy_1\,dy_2 \\ &= \sum_{j,k} \int_{\rho_1(y_1)\cong 2^{lj}} \int_{\rho_2(y_2)\cong 2^{lk}} \frac{\Omega_l(y_1',y_2')h(\rho_1(y_1),\rho_2(y_2))}{\rho_1^{\alpha}(y_1)\rho_2^{\beta}(y_2)} f(x_1-y_1,x_2-y_2)dy_2dy_1 \\ &\equiv \sum_{j,k}(\sigma_{j,k}*f)(x_1,x_2). \qquad (12)\end{aligned}$$

(Here $\rho_1(y_1)\cong 2^{lj}$ means $2^{lj}\leqslant\rho_1(y_1)\leqslant 2^{l(j+1)}$ and similar definition for $\rho_2(y_2)\cong 2^{lk}$).

The Fourier transform of $\sigma_{j,k}$ is

$$\hat{\sigma}_{j,k}(\xi_1,\xi_2) = \int_{\rho_1(y_1)\cong 2^{lj}} \int_{\rho_2(y_2)\cong 2^{lk}} \frac{e^{i(\xi_1\cdot y_1+\xi_2\cdot y_2)}\Omega_l(y_1',y_2')h(\rho_1(y_1),\rho_2(y_2))}{\rho_1^{\alpha}(y_1)\,\rho_2^{\beta}(y_2)}\,dy_2\,dy_1. \qquad (13)$$

Let $\mu_{j,k}$ denote the total variations of $\sigma_{j,k}$ for all $j,k\in\mathbb{Z}$. In order to apply Theorems A and B, we must show that

$$||\sigma_{j,k}||,\ ||\mu_{j,k}||\leqslant Cl^2||\Omega||_{L^1(E_l)}, \text{ and}$$

$\hat{\sigma}_{j,k}$ (resp. $\hat{\mu}_{j,k}$) satisfy the estimates (5)–(8) (resp. (1)–(4)) for all $j,k\in\mathbb{Z},\ l\in A(\Omega)$ (with the bound $Cl^2||\Omega||_{L^1(E_l)}$ instead of C). Moreover, we need to show that the partial maximal functions

$$\tilde{M}^{(i)}g_i = \sup_{j,k}|\mu_{j,k}^{(i)}*g_i|\ (i=1,2)$$

are bounded in L^p for all $p>1$. By a change of variables to polar coordinates,

$$\begin{aligned}\hat{\sigma}_{j,k}(\xi_1,\xi_2) = &\int_{2^{lj}}^{2^{l(j+1)}}\int_{2^{lk}}^{2^{l(k+1)}}\iint_{S^{n-1}\times S^{m-1}} e^{i(A^{(1)}_{\rho_1}\xi_1\cdot u+A^{(2)}_{\rho_2}\xi_2\cdot v)} \\ &\times\Omega_l(u,v)h(\rho_1,\rho_2)J_1(u)J_2(v)\,d\sigma(u)\,d\sigma(v)\frac{d\rho_2}{\rho_2}\frac{d\rho_1}{\rho_1} \qquad (14)\end{aligned}$$

and note that

$$\hat{\mu}_{j,k}(\xi_1,\xi_2) = \int_{2^{lj}}^{2^{l(j+1)}} \int_{2^{lk}}^{2^{l(k+1)}} \iint_{S^{n-1}\times S^{m-1}} e^{i(A^{(1)}_{\rho_1}\xi_1\cdot u + A^{(2)}_{\rho_2}\xi_2\cdot v)} \times |\Omega_l(u,v)h(\rho_1,\rho_2)|J_1(u)J_2(v)\,d\sigma(u)\,d\sigma(v)\frac{d\rho_2}{\rho_2}\frac{d\rho_1}{\rho_1}. \tag{15}$$

It is clear that $||\sigma_{j,k}|| = ||\mu_{j,k}|| = \hat{\mu}_{j,k}(0,0) \leqslant Cl^2||h||_\infty||\Omega||_{L^1(E_l)}$ for all $j,k \in \mathbb{Z}$. By the cancellation conditions of Ω_l, we have

$$\begin{aligned}
|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| &= |\Delta^{1,2}_{\xi_1,\xi_2}\hat{\sigma}_{j,k}(0,0)| \\
&\leqslant \int_{2^{lj}}^{2^{l(j+1)}} \int_{2^{lk}}^{2^{l(k+1)}} \iint_{S^{n-1}\times S^{m-1}} |\Omega_l(u,v)h(\rho_1,\rho_2)| \\
&\quad \times |(e^{iA^{(1)}_{\rho_1}\xi_1\cdot u}-1)(e^{iA^{(2)}_{\rho_2}\xi_2\cdot v}-1)|J_1(u)J_2(v)d\sigma(u)d\sigma(v)\frac{d\rho_2}{\rho_2}\frac{d\rho_1}{\rho_1} \\
&\leqslant Cl^2||\Omega||_{L^1(E_l)}\,|A^{(1)}_{2^{l(j+1)}}\xi_1|\,|A^{(2)}_{2^{l(k+1)}}\xi_2|.
\end{aligned}$$

Also,

$$\begin{aligned}
|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| &= |\Delta^{1}_{\xi_1}\hat{\sigma}_{j,k}(0,\xi_2)| \\
&\leqslant \int_{2^{lj}}^{2^{l(j+1)}} \int_{S^{n-1}} |I(\rho_1,u)||e^{iA^{(1)}_{\rho_1}\xi_1\cdot u}-1|J_1(u)d\sigma(u)\frac{d\rho_1}{\rho_1},
\end{aligned} \tag{16}$$

where $I(\rho_1,u) = \int_{2^{lk}}^{2^{l(k+1)}} \int_{S^{m-1}} \Omega_l(u,v)h(\rho_1,\rho_2)e^{iA^{(2)}_{\rho_2}\xi_2\cdot v}J_2(v)d\sigma(v)\frac{d\rho_2}{\rho_2}$. Note that

$$|I(\rho_1,u)|^2 \leqslant Cl||h||^2_\infty \iint_{S^{m-1}\times S^{m-1}} \Omega_l(u,v)\bar{\Omega}_l(u,\tilde{v})|K(\xi_2;v,\tilde{v})|J_2(v)J_2(\tilde{v})d\sigma(v)d\sigma(\tilde{v}),$$

where

$$K(\xi_2;v,\tilde{v}) = \int_{2^{lk}}^{2^{l(k+1)}} e^{iA^{(2)}_{\rho_2}\xi_2\cdot(v-\tilde{v})}\frac{d\rho_2}{\rho_2}. \tag{17}$$

It is clear that $|K(\xi_2;v,\tilde{v})| \leqslant Cl$. On the other hand,

$$\begin{aligned}
|K(\xi_2;v,\tilde{v})| &\leqslant \sum_{j=0}^{l-1}\left|\int_{2^{lk+j}}^{2^{lk+j+1}} e^{iA^{(2)}_{\rho_2}\xi_2\cdot(v-\tilde{v})}\frac{d\rho_2}{\rho_2}\right| \\
&= \sum_{j=0}^{l-1}\left|\int_1^2 e^{iA^{(2)}_{2^{lk+j}t}\xi_2\cdot(v-\tilde{v})}\frac{dt}{t}\right| \\
&\leqslant C\sum_{j=0}^{l-1} 2^{-j\beta_1/m}|A^{(2)}_{2^{lk}}\xi_2|^{-1/m}|\eta\cdot(v-\tilde{v})|^{-1/m} \\
&\leqslant Cl|A^{(2)}_{2^{lk}}\xi_2|^{-1/m}|\eta\cdot(v-\tilde{v})|^{-1/m}
\end{aligned}$$

where $\eta = \dfrac{A^{(2)}_{2^{lk}}\xi_2}{|A^{(2)}_{2^{lk}}\xi_2|}$, and the next to last inequality follows from Theorem 1 [12]. Thus

$$|K(\xi_2; v, \tilde{v})| \leqslant C l \min\left\{1,\ |A^{(2)}_{2^{lk}}\xi_2|^{-1/m}|\eta.(v-\tilde{v})|^{-1/m}\right\}$$
$$\leqslant C l |A^{(2)}_{2^{lk}}\xi_2|^{-1/2m}|\eta\cdot(v-\tilde{v})|^{-1/2m},$$

and therefore by Hölder's inequality,

$$|I(\rho_1, u)|^2 \leqslant C l^2 |A^{(2)}_{2^{lk}}\xi_2|^{-1/2m}\int_{S^{m-1}}|\Omega_l(u,v)|^2 d\sigma(v).$$

An application of Hölder's inequality to (16) produces

$$|\hat{\sigma}_{j,k}(\xi_1,\xi_2)|^2 \leqslant C l |A^{(1)}_{2^{l(j+1)}}\xi_1|^2 \int_{2^{lj}}^{2^{l(j+1)}}\int_{S^{n-1}}|I(\rho_1,u)|^2 d\sigma(u)\frac{d\rho_1}{\rho_1}$$
$$\leqslant C l^4 |A^{(1)}_{2^{l(j+1)}}\xi_1|^2 |A^{(2)}_{2^{lk}}\xi_2|^{-1/2m}||\Omega_l||_2^2.$$

Thus $|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| \leqslant C l^2\, |A^{(1)}_{2^{l(j+1)}}\xi_1||A^{(2)}_{2^{lk}}\xi_2|^{-1/4m}2^{2l}||\Omega||_{L^1(E_l)}$. On the other hand, a direct integration of inequality (16) yields $|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| \leqslant C l^2 |A^{(1)}_{2^{l(j+1)}}\xi_1|||\Omega||_{L^1(E_l)}$, which together with the above inequality implies that

$$|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| \leqslant C l^2 ||\Omega||_{L^1(E_l)}|A^{(1)}_{2^{l(j+1)}}\xi_1||A^{(2)}_{2^{lk}}\xi_2|^{-1/4ml}.$$

By symmetry, we also obtain

$$|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| = |\Delta^2_{\xi_2}\hat{\sigma}_{j,k}(\xi_1,0)| \leqslant C l^2 ||\Omega||_{L^1(E_l)}|A^{(1)}_{2^{lj}}\xi_1|^{-1/4nl}|A^{(2)}_{2^{l(k+1)}}\xi_2|.$$

An application of Hölder's inequality to (14) yields

$$|\hat{\sigma}_{j,k}(\xi_1,\xi_2)|^2 \leqslant C l^2 ||h||^2_\infty \int_{2^{lj}}^{2^{l(j+1)}}\int_{2^{lk}}^{2^{l(k+1)}}\left|\iint_{S^{n-1}\times S^{m-1}}\Omega_l(u,v)J_1(u)J_2(v)\right.$$
$$\left.\times e^{i(A^{(1)}_{\rho_1}\xi_1\cdot u + A^{(2)}_{\rho_2}\xi_2\cdot v)}d\sigma(u)d\sigma(v)\right|^2\frac{d\rho_2}{\rho_2}\frac{d\rho_1}{\rho_1}$$
$$\leqslant C l^2 \iint_{(S^{n-1}\times S^{m-1})^2}\Omega_l(u,v)\bar{\Omega}_l(\tilde{u},\tilde{v})\left|\left(\int_{2^{lj}}^{2^{l(j+1)}} e^{iA^{(1)}_{\rho_1}\xi_1\cdot(u-\tilde{u})}\frac{d\rho_1}{\rho_1}\right)\right.$$
$$\left.\times\left(\int_{2^{lk}}^{2^{l(k+1)}} e^{iA^{(2)}_{\rho_2}\xi_2\cdot(v-\tilde{v})}\frac{d\rho_2}{\rho_2}\right)\right|d\sigma(u)d\sigma(v)d\sigma(\tilde{u})d\sigma(\tilde{v})$$
$$\leqslant C l^4 |A^{(1)}_{2^{lj}}\xi_1|^{-1/2n}|A^{(2)}_{2^{lk}}\xi_2|^{-1/2m}||\Omega_l||_2^2.$$

Thus $|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| \leqslant Cl^2|A^{(1)}_{2^{lj}}\xi_1|^{-1/4n}|A^{(2)}_{2^{lk}}\xi_2|^{-1/4m}2^{2l}||\Omega||_{L^1(E_l)}$. Also a direct integration of (14) leads to $|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| \leqslant Cl^2||\Omega||_{L^1(E_l)}$, which together with the above inequality yields

$$|\hat{\sigma}_{j,k}(\xi_1,\xi_2)| \leqslant Cl^2||\Omega||_{L^1(E_l)}|A^{(1)}_{2^{lj}}\xi_1|^{-1/4nl}|A^{(2)}_{2^{lk}}\xi_2|^{-1/4ml}.$$

Consequently, $\hat{\sigma}_{j,k}$ satisfy the estimates (5)–(8) in Theorem B for all $j,k \in \mathbb{Z}$. By inspecting equations (14) and (15), we infer that $\hat{\mu}_{j,k}$ also satisfy the estimates (1)–(4) in Theorem A for all $j,k \in \mathbb{Z}$.

It remains to prove the L^p boundedness of the partial maximal functions $\tilde{M}^{(i)}g_i = \sup_{j,k}|\mu^{(i)}_{j,k} * g_i|$ $(i=1,2)$ for all $p>1$. By symmetry it is enough to consider the case $i=1$. We may assume $g_1 \geqslant 0$. For $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
\tilde{M}^{(1)}g_1(x) &= \sup_{j,k}|\mu^{(1)}_{j,k} * g_1(x)| \\
&\leqslant \sup_{j,k}\int_{2^{lj}}^{2^{l(j+1)}}\int_{2^{lk}}^{2^{l(k+1)}}\iint_{S^{n-1}\times S^{m-1}}|\Omega_l(u,v)h(\rho_1,\rho_2)|g_1(x-A^{(1)}_{\rho_1}u) \\
&\quad \times J_1(u)J_2(v)d\sigma(u)d\sigma(v)\frac{d\rho_2}{\rho_2}\frac{d\rho_1}{\rho_1} \\
&\leqslant Cl||h||_\infty \iint_{S^{n-1}\times S^{m-1}}|\Omega_l(u,v)|\left\{\sup_j\int_{2^{lj}}^{2^{l(j+1)}} g_1(x-A^{(1)}_{\rho_1}u)\frac{d\rho_1}{\rho_1}\right\}d\sigma(u)d\sigma(v).
\end{aligned}$$

Note that

$$\sup_j\int_{2^{lj}}^{2^{l(j+1)}} g_1(x-A^{(1)}_{\rho_1}u)\frac{d\rho_1}{\rho_1} \leqslant 2\sum_{i=0}^{l-1}\sup_j\left\{\frac{1}{2^{lj+i+1}}\int_{2^{lj+i}}^{2^{lj+i+1}} g_1(x-A^{(1)}_t u)dt\right\},$$

where each summand in the sum above is bounded in $L^p(\mathbb{R}^n)$ for all $p>1$ and the bound is independent of the direction vector $u \in S^{n-1}$ (see [7, Corollary 5.1]). It follows from Minskowski's inequality that $||\tilde{M}^{(1)}g_1||_p \leqslant Cl^2||\Omega||_{L^1(E_l)}||g_1||_p$ for $1<p\leqslant\infty$. Consequently, we have proved that

$$||T_l f||_p \leqslant Cl^2||\Omega||_{L^1(E_l)}||f||_p,\ l \in A(\Omega).$$

Finally observe that the estimate of the function $K(\xi_2;v,\tilde{v})$ in equation (17) and Corollary 5.1 [7] both rely on Theorem 1 [14], which in turn require all coefficients α_i (and similarly all β_j) to be distinct. However, the estimate of the function $K(\xi_2;v,\tilde{v})$ and Corollary 5.1 [7] still hold in case $0<\alpha_1=\alpha_2=\cdots=\alpha_n$, or $0<\beta_1=\beta_2=\cdots=\beta_m$. The proof of Theorem 1 is complete.

2.5. Proof of Lemma 3

The decomposition of $\Omega \in L(\log L^+)(S^{n-1})$ can be found in [2]. We generalize this idea to the product setting. Let μ_1 and μ_2 be the normalized measures on S^{n-1}

and S^{m-1} respectively. Let $\mu=\mu_1\times\mu_2$ be defined on $S^{n-1}\times S^{m-1}$. For $l\in A(\Omega)$, define

$$\Omega_l^{(1)}(u)=\int_{S^{m-1}}\Omega(u,v)\chi_{E_l}(u,v)J_2(v)\,d\mu_2(v)$$

and

$$\Omega_l^{(2)}(v)=\int_{S^{n-1}}\Omega(u,v)\chi_{E_l}(u,v)J_1(u)\,d\mu_1(u).$$

Now define a_l $(l\in A(\Omega))$ on $S^{n-1}\times S^{m-1}$ by

$$\begin{aligned}a_l(u,v)\,=\,&\frac{1}{||J_1||_\infty||J_2||_\infty||\Omega||_{L^1(E_l)}}\Big\{||J_1||_1||J_2||_1\Omega(u,v)\chi_{E_l}(u,v)-||J_1||_1\Omega_l^{(1)}(u)\\&-||J_2||_1\Omega_l^{(2)}(v)\Big\}+\frac{1}{||J_1||_\infty||J_2||_\infty||\Omega||_{L^1(E_l)}}\iint_{S^{n-1}\times S^{m-1}}\Omega(u,v)\chi_{E_l}(u,v)\\&\times J_1(u)J_2(v)\,d\mu_1(u)\,d\mu_2(v).\end{aligned}$$

Then each a_l satisfies the cancellation conditions as Ω does. Moreover, $||a_l||_{L^1(S^{n-1}\times S^{m-1})}\leqslant 4$ and $||a_l||_{L^2(S^{n-1}\times S^{m-1})}\leqslant 2^{2l+3}$.

Define $\tilde{\Omega}_o$ as

$$\begin{aligned}\tilde{\Omega}_o\,&=\,||J_1||_1||J_2||_1\Omega-\sum_{l\in A(\Omega)}||J_1||_\infty||J_2||_\infty||\Omega||_{L^1(E_l)}a_l\\&\equiv||J_1||_1||J_2||_1\Omega-\sum_{l\in A(\Omega)}\tilde{\Omega}_l,\ \text{ where }\tilde{\Omega}_l=||J_1||_\infty||J_2||_\infty||\Omega||_{L^1(E_l)}a_l.\end{aligned}$$

Observe that $\tilde{\Omega}_o$ also satisfies the cancellation conditions. Moreover, by the cancellation condtions of Ω, we have

$$\begin{aligned}\tilde{\Omega}_o(u,v)\,=\,&||J_1||_1||J_2||_1\Omega(u,v)-||J_1||_1\int_{S^{m-1}}\Omega(u,v)J_2(v)d\sigma(v)\\&-||J_2||_1\int_{S^{n-1}}\Omega(u,v)J_1(u)d\sigma(u)\\&+\iint_{S^{n-1}\times S^{m-1}}\Omega(u,v)J_1(u)J_2(v)d\sigma(u)d\sigma(v)-\sum_{l\in A(\Omega)}\tilde{\Omega}_l(u,v)\\=\,&||J_1||_1||J_2||_1\Omega(u,v)-\sum_{l=0}^{\infty}||J_1||_1\Omega_l^{(1)}(u)-\sum_{l=0}^{\infty}||J_2||_1\Omega_l^{(2)}(v)\\&+\sum_{l=0}^{\infty}\iint_{S^{n-1}\times S^{m-1}}\Omega(u,v)\chi_{E_l}(u,v)J_1(u)J_2(v)d\sigma(u)d\sigma(v)-\sum_{l\in A(\Omega)}\tilde{\Omega}_l(u,v)\\=\,&||J_1||_1||J_2||_1\Omega(u,v)\sum_{l\notin A(\Omega)}\chi_{E_l}(u,v)-\sum_{l\notin A(\Omega)}||J_1||_1\Omega_l^{(1)}(u)-\sum_{l\notin A(\Omega)}||J_2||_1\Omega_l^{(2)}(v)\\&+\sum_{l\notin A(\Omega)}\iint_{S^{n-1}\times S^{m-1}}\Omega(u,v)\chi_{E_l}(u,v)J_1(u)J_2(v)d\sigma(u)d\sigma(v).\end{aligned}$$

Now if $l=0$, then $|\Omega(u,v)\chi_{E_o}(u,v)|\leqslant 2$, and $\displaystyle\int\int_{S^{n-1}\times S^{m-1}} |\Omega(u,v)|^2\chi_{E_o}(u,v)d\sigma(u)d\sigma(v)$ $\leqslant C$.

If $l\in\mathbb{N}\backslash A(\Omega)$, then $|\Omega(u,v)\chi_{E_l}(u,v)|\leqslant 2^{l+1}$, $\mu(E_l)\leqslant 2^{-4l}$, and thus $\displaystyle\int\int_{S^{n-1}\times S^{m-1}} |\Omega(u,v)|^2\chi_{E_l}(u,v)d\sigma(u)d\sigma(v)\leqslant 2^{-2l+2}$. It follows that $||\tilde{\Omega}_o||_{L^2(S^{n-1}\times S^{m-1})}\leqslant C$, which implies $||\tilde{\Omega}_o||_{L^1(S^{n-1}\times S^{m-1})}\leqslant C$.

Now let $\Omega_o=||J_1||_1^{-1}||J_2||_1^{-1}\tilde{\Omega}_o$ and $\Omega_l=||J_1||_1^{-1}||J_2||_1^{-1}\tilde{\Omega}_l$. We then have the decomposition

$$\Omega=\Omega_o+\sum_{l\in A(\Omega)}\Omega_l,$$

where Ω_o and Ω_l $(l\in A(\Omega))$ satisfy the desired conclusions stated in the lemma. Lemma 3 is proved.

2.6. Proofs of Theorems A and B

The proofs of these theorems are essentially similar to the proofs of Theorems 1 and 2 [5] respectively, except for some modifications. However for clarity, we will prove Theorem B in detail, and omit the proof of Theorem A since the proof of this Theorem A (unlike the proof of Theorem B) only requires some minor adjustments (see Remark 4 at the end of this section). We now proceed to prove Theorem B.

Let $\phi^{(i)}$ be a function in $\mathscr{S}(\mathbb{R}^{n_i})$ $(i=1,2)$ such that $\text{supp}\hat{\phi}^{(i)}\subset\{\zeta_i\in\mathbb{R}^{n_i}:1/2\leqslant\rho_i(\zeta_i)\leqslant 2\}$, $0\leqslant\hat{\phi}^{(i)}\leqslant 1$. Moreover, we require $\displaystyle\sum_{j=-\infty}^{\infty}\left\{\hat{\phi}^{(i)}(2^{lj}\rho_i(\zeta_i))\right\}^2=1$ for all $\zeta_i\neq 0$. For $i=1,2$ and $j\in\mathbb{Z}$, define $\psi^{(i)}$ and $\psi_j^{(i)}$ by $\hat{\psi}^{(i)}(\zeta_i)=\hat{\phi}^{(i)}(\rho_i(\zeta_i))$ and $\hat{\psi}_j^{(i)}(\zeta_i)=\hat{\phi}^{(i)}(2^{lj}\rho_i(\zeta_i))$ respectively. Now set $\psi(x_1,x_2)=\psi^{(1)}(x_1)\psi^{(2)}(x_2)$ $(x_i\in\mathbb{R}^{n_i},i=1,2)$ and let $\psi_{j,k}(x_1,x_2)=\psi_j^{(1)}(x_1)\psi_k^{(2)}(x_2)$ for $j,k\in\mathbb{Z}$. Note that $\hat{\psi}_{j,k}(\zeta_1,\zeta_2)$ $=\hat{\psi}_j^{(1)}(\zeta_1)\hat{\psi}_k^{(2)}(\zeta_2)$ and $\displaystyle\sum_{j,k}\hat{\psi}_{j,k}^2(\zeta_1,\zeta_2)=1$ for all $(\zeta_1,\zeta_2)\neq\mathbf{0}\in\mathbb{R}^{\mathbf{n_1}}\times\mathbb{R}^{\mathbf{n_2}}$. We write

$$\begin{aligned}Tf&=\sum_{j,k}\sigma_{j,k}*f=\sum_{j,k}\sigma_{j,k}*\left(\sum_{m,n}(\psi_{j+m,k+n}\otimes\delta)*(\psi_{j+m,k+n}\otimes\delta)*f\right)\\&=\sum_{m,n}\sum_{j,k}(\psi_{j+m,k+n}\otimes\delta)*(\sigma_{j,k}*(\psi_{j+m,k+n}\otimes\delta)*f)\equiv\sum_{m,n}T_{m,n}f.\end{aligned}$$

Then $||Tf||_p\leqslant\sum_{m,n}||T_{m,n}f||_p$. We estimate each of the terms of this sum by interpolating between the L^2 norm and the L^{p_o} norm (where p_o is as in Lemma 2). Observe that

$$\begin{aligned}||T_{m,n}f||_{p_o} &\leqslant C\|\left(\sum_{j,k}|\sigma_{j,k}*(\psi_{j+m,k+n}\otimes\delta)*f|^2\right)^{1/2}\|_{p_o}\\ &\leqslant C\|\left(\sum_{j,k}|(\psi_{j+m,k+n}\otimes\delta)*f|^2\right)^{1/2}\|_{p_o}\leqslant C||f||_{p_o},\end{aligned}$$

where the first and the last inequalities follow from Lemma 1, and the second inequality follows from Lemma 2. We now calculate the L^2 estimates of $T_{m,n}f$. Notice that

$$||T_{m,n}f||_2^2\leqslant\sum_{j,k}\int_{\mathbb{R}^{n_3}}\iint_{R^{m,n}_{j,k}}|\hat{\sigma}_{j,k}(\zeta_1,\zeta_2,\zeta_3)\hat{f}(\zeta_1,\zeta_2,\zeta_3)|^2d\zeta_1d\zeta_2d\zeta_3,$$

where

$$R^{m,n}_{j,k}=\left\{(\zeta_1,\zeta_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}:1/2\leqslant 2^{l(j+m)}\rho_1(\zeta_1)\leqslant 2,\ 1/2\leqslant 2^{l(k+n)}\rho_2(\zeta_2)\leqslant 2\right\}.$$

By a change of the first two variables to polar coordinates, we obtain

$$\begin{aligned}||T_{m,n}f||_2^2\leqslant\sum_{j,k}\int_{\mathbb{R}^{n_3}}\iint_{S^{n_1-1}\times S^{n_2-1}}\iint_{D^{m,n}_{j,k}}&|\hat{\sigma}_{j,k}(A^{(1)}_{\rho_1}\zeta_1',A^{(2)}_{\rho_2}\zeta_2',\zeta_3)\hat{f}(A^{(1)}_{\rho_1}\zeta_1',A^{(2)}_{\rho_2}\zeta_2',\zeta_3)|^2\\ &\times\rho_1^{\alpha-1}\rho_2^{\beta-1}J_1(\zeta_1')J_2(\zeta_2')d\rho_1d\rho_2d\sigma(\zeta_1')d\sigma(\zeta_2')d\zeta_3,\end{aligned}$$

where $D^{m,n}_{j,k}=\left\{(\rho_1,\rho_2)\in\mathbb{R}^+\times\mathbb{R}^+:1/2\leqslant 2^{l(j+m)}\rho_1\leqslant 2,\ 1/2\leqslant 2^{l(k+n)}\rho_2\leqslant 2\right\}$. For $(\rho_1,\rho_2)\in D^{m,n}_{j,k}$, we consider four cases: $m,n>0$, $m>0$ and $n\leqslant 0$, $m\leqslant 0$ and $n>0$, and finally $m,n\leqslant 0$.

Case 1. $m,n>0$. By inequality (5) we have

$$\begin{aligned}|\hat{\sigma}_{j,k}(A^{(1)}_{\rho_1}\zeta_1',A^{(2)}_{\rho_2}\zeta_2',\zeta_3)| &\leqslant C|A^{(1)}_{2^{l(j+1)}}A^{(1)}_{\rho_1}\zeta_1'|^{a/l}|A^{(2)}_{2^{l(k+1)}}A^{(2)}_{\rho_2}\zeta_2'|^{b/l}\\ &\leqslant C2^{-(m-1)\alpha_1 a}2^{-(n-1)\beta_1 b}.\end{aligned}$$

Thus

$$\begin{aligned}||T_{m,n}f||_2^2\leqslant C2^{-2(m-1)\alpha_1 a}2^{-2(n-1)\beta_1 b}\sum_{j,k}\int_{\mathbb{R}^{n_3}}\iint_{S^{n_1-1}\times S^{n_2-1}}\iint_{D^{m,n}_{j,k}}&|\hat{f}(A^{(1)}_{\rho_1}\zeta_1',A^{(2)}_{\rho_2}\zeta_2',\zeta_3)|^2\\ \times\rho_1^{\alpha-1}\rho_2^{\beta-1}J_1(\zeta_1')J_2(\zeta_2')d\rho_1d\rho_2d\sigma(\zeta_1')d\sigma(\zeta_2')d\zeta_3,&\end{aligned}$$

and hence $||T_{m,n}f||_2\leqslant C2^{-(m-1)\alpha_1 a}2^{-(n-1)\beta_1 b}||f||_2,\ m,n>0$.

Case 2. $m>0$ and $n\leqslant 0$. It follows from (6) that

$$|\hat{\sigma}_{j,k}(A^{(1)}_{\rho_1}\zeta_1',A^{(2)}_{\rho_2}\zeta_2',\zeta_3)|\leqslant C|A^{(1)}_{2^{l(j+1)}}A^{(1)}_{\rho_1}\zeta_1'|^{a/l}|A^{(2)}_{2^{lk}}A^{(2)}_{\rho_2}\zeta_2'|^{-b/l}\leqslant C2^{-(m-1)\alpha_1 a}2^{n\beta_1 b},$$

which leads to the inquality $||T_{m,n}f||_2 \leqslant C2^{-(m-1)\alpha_1 a}2^{n\beta_1 b}||f||_2,\ m>0$ and $n \leqslant 0$.

Case 3. $m \leqslant 0$ and $n > 0$. By using inequality (7) we obtain

$$|\hat{\sigma}_{j,k}(A^{(1)}_{\rho_1}\zeta_1', A^{(2)}_{\rho_2}\zeta_2', \zeta_3)| \leqslant C|A^{(1)}_{2^{lj}}A^{(1)}_{\rho_1}\zeta_1'|^{-a/l}|A^{(2)}_{2^{l(k+1)}}A^{(2)}_{\rho_2}\zeta_2'|^{b/l} \leqslant C2^{m\alpha_1 a}2^{-(n-1)\beta_1 b},$$

so that $||T_{m,n}f||_2 \leqslant C2^{m\alpha_1 a}2^{-(n-1)\beta_1 b}||f||_2,\ m \leqslant 0$ and $n > 0$.

Case 4. $m,n \leqslant 0$. From (8) we have

$$|\hat{\sigma}_{j,k}(A^{(1)}_{\rho_1}\zeta_1', A^{(2)}_{\rho_2}\zeta_2', \zeta_3)| \leqslant C|A^{(1)}_{2^{lj}}A^{(1)}_{\rho_1}\zeta_1'|^{-a/l}|A^{(2)}_{2^{lk}}A^{(2)}_{\rho_2}\zeta_2'|^{-b/l} \leqslant C2^{m\alpha_1 a}2^{n\beta_1 b},$$

and thus $||T_{m,n}f||_2 \leqslant C2^{m\alpha_1 a}2^{n\beta_1 b}||f||_2,\ m,n \leqslant 0$.

Consequently, $||T_{m,n}f||_2 \leqslant C2^{-\varepsilon_1|m|}2^{-\varepsilon_2|n|}||f||_2$ for some $\varepsilon_1, \varepsilon_2 > 0$ and for all $m,n \in \mathbb{Z}$.

Now if $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2q}$, then by interpolation we have $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_o}$ for some $0 < \theta \leqslant 1$. Therefore, $||Tf||_p \leqslant \sum_{m,n} ||T_{m,n}f||_p \leqslant C\sum_{m,n} 2^{-\theta\varepsilon_1|m|}2^{-\theta\varepsilon_2|n|}||f||_p \leqslant C||f||_p$.

The proof of the L^p bounds of $g(f)$ is similar. Theorem B is proved.

REMARK 4. To prove Theorem A, we choose positive Schwartz functions $\phi^{(i)}$ in $\mathscr{S}(\mathbb{R}^{n_i})$ $(i=1,2)$ such that $\hat{\phi}^{(1)}(0) = 1 = \hat{\phi}^{(2)}(0)$, and define $\hat{\phi}^{(1)}_j(\zeta_1) = \hat{\phi}^{(1)}(A^{(1)}_{2^{l(j+1)}}\zeta_1)$ and $\hat{\phi}^{(2)}_k(\zeta_2) = \hat{\phi}^{(2)}(A^{(2)}_{2^{l(k+1)}}\zeta_2)$ for all $j,k \in \mathbb{Z}$. Observe that for each $f_i \in L^p(\mathbb{R}^{n_i})$ $(i=1,2)$, $\sup_{j\in\mathbb{Z}}|\phi^{(i)}_j * f_i(x_i)| \leqslant CM_1 \circ \cdots \circ M_{n_i}f_i(x_i)$, where each M_j $(j=1,....,n_i)$ is the Hardy-Littlewood maximal operator acting on the j-th component of the variable x_i. The rest of the proof of this theorem is identical to the proof of Theorem 1 [5].

3. Singular integrals along surfaces

3.1. Notations, Definitions and Background

Let $\mathbb{R}^+$ stand for $[0,\infty)$. A function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is called a type I function if ϕ is strictly increasing on $[0,\infty)$ and ϕ' is increasing on $(0,\infty)$.

A function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ or $\phi : (0,\infty) \longrightarrow \mathbb{R}$ is a type II function if ϕ is strictly decreasing on its domain and ϕ' is increasing on $(0,\infty)$.

A function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a type III function if

i) $\phi(0) = 0$ and ϕ is strictly increasing on $[0,\infty)$,

ii) ϕ' is decreasing on $(0,\infty)$ and

iii) $t\,\phi'(t) \geqslant c\,\phi(t)$ for all $t \in (0,\infty)$ and for some fixed $c > 0$.

For $f \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ $(n,m \geqslant 2)$, consider the following singular integral along surface

$$\Im f(x_1,x_2,x_3) = \text{p. v.} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \frac{\Omega(y_1',y_2')\,h(\rho_1(y_1),\rho_2(y_2))}{\rho_1^{\alpha}(y_1)\,\rho_2^{\beta}(y_2)} \times f(x_1-y_1, x_2-y_2, x_3-\Gamma(\rho_1(y_1),\rho_2(y_2)))\,dy_1\,dy_2,$$

where $x_1, y_1 \in \mathbb{R}^n,\ x_2, y_2 \in \mathbb{R}^m,\ x_3 \in \mathbb{R},\ y_i' = y_i/|y_i|\ \ (i = 1,2)$, and h and Γ are measurable real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$.

Denote $\gamma_t(s) = \Gamma(s,t)$ for every fixed $t \geqslant 0$. Similarly, denote $\gamma_s(t) = \Gamma(s,t)$ for every fixed $s \geqslant 0$.

THEOREM 2. [11] *Suppose $\Gamma(s,t)$ has continuous first order partial derivatives for all $s,t > 0$. If $\gamma_t(s)$ and $\gamma_s(t)$ are either type I, type II or type III functions (with the constant c in the definition of type III function independent of both variables s and t) for each fixed $t > 0$ and for each fixed $s > 0$ respectively, then the maximal functions*

$$M_{1,2}g(x_3) = \sup_{r_1,r_2>0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} |g(x_3 - \Gamma(s,t))|\, ds\, dt \right\},$$

$$M_1 g_1(x_1,x_3) = \sup_{r_1,r_2>0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} |g_1(x_1 - s, x_3 - \Gamma(s,t))|\, ds\, dt \right\}$$

and

$$M_2 g_2(x_2,x_3) = \sup_{r_1,r_2>0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} |g_2(x_2 - t, x_3 - \Gamma(s,t))|\, ds\, dt \right\}$$

($x_1,x_2,x_3 \in \mathbb{R},\ g \in L^p(\mathbb{R})$, and $g_1, g_2 \in L^p(\mathbb{R}^2)$) are bounded in L^p for all $p > 1$.

3.2. Theorem 2

Let Ω and h be given as in Theorem 1. Suppose $\Gamma(s,t)$ has continuous first order partial derivatives for all $s,t > 0$. If $\gamma_t(s)$ and $\gamma_s(t)$ are either type I, type II or type III functions (with the constant c in the definition of type III function independent of both variables s and t) for each fixed $t > 0$ and for each fixed $s > 0$ respectively, then the singular integral along surface $\Im f$ initially defined on the Schwartz space $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ has a bounded extension from $L^p((\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ to $L^p((\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for $1 < p < \infty$.

EXAMPLES.

1) Consider the surface $\Gamma(s,t) = s^a t^b,\ a,b \neq 0$ and $s,t \geqslant 0$ ($s > 0$ if $a < 0$ and similarly $t > 0$ if $b < 0$). For each fixed $t > 0$, the function $\gamma_t(s) = \Gamma(s,t)$ is a type I function if $a \geqslant 1$, a type III function if $0 < a < 1$ and a type II function if $a < 0$. Similar conclusion holds for the function $\gamma_s(t) = \Gamma(s,t)$. By Theorem 2, the singular integral along surface $\Im f$ is bounded in $L^p(\mathbb{R}^{n+m+1})$ for $1 < p < \infty$.

2) Let $\Gamma(s,t) = s^{a_1} e^{a_2 s} t^{b_1} e^{b_2 t},\ a_1, b_1 \geqslant 1$ and $a_2, b_2 \geqslant 0$. Then $\gamma_t(s) = \Gamma(s,t)$ is a type I function for each fixed $t > 0$. Similarly, $\gamma_s(t) = \Gamma(s,t)$ is also a type I function for each $s > 0$. Thus, $\Im f$ is bounded in $L^p(\mathbb{R}^{n+m+1})$ for $1 < p < \infty$.

3) Consider the surface with a contact of infinite order at the origin, $\Gamma(s,t) = s^2 t^2 (e^{-1/s} + e^{-1/t})\ \ (s,t > 0)$. The functions $\gamma_t(s) = \Gamma(s,t)$ and $\gamma_s(t) = \Gamma(s,t)$ are type I functions for each fixed $s > 0$ and each fixed $t > 0$ respectively. Hence, $\Im f$ is bounded in $L^p(\mathbb{R}^{n+m+1})$ for $1 < p < \infty$.

3.3. Proof of Theorem 2

The proof of this theorem is essentially similar to the proof of Theorem 1. It is enough to prove the L^p boundedness of the operator $\Im_l$, where $\Im_l f$ is obtained from $\Im f$ by replacing the kernel Ω by Ω_l. As in the proof of Theorem 1, we write $\Im_l f = \sum_{j,k} \sigma_{j,k} * f$, where

$$\hat{\sigma}_{j,k}(\xi_1,\xi_2,\xi_3) = \int_{\rho_1(y_1)\cong 2^{lj}} \int_{\rho_2(y_2)\cong 2^{lk}} \frac{e^{i(\xi_1\cdot y_1+\xi_2\cdot y_2)}\Omega_l(y_1',y_2')h(\rho_1(y_1),\rho_2(y_2))e^{i\xi_3\Gamma(\rho_1(y_1),\rho_2(y_2))}}{\rho_1^{\alpha}(y_1)\,\rho_2^{\beta}(y_2)}\,dy_2\,dy_1.$$

Let $\mu_{j,k}$ denote the total variations of $\sigma_{j,k}$ for all $j,k\in\mathbb{Z}$. In view of Theorems A and B, we need to show that

$$||\sigma_{j,k}||,\ ||\mu_{j,k}|| \leqslant Cl^2||\Omega||_{L^1(E_l)},\ \text{and}$$

$\hat{\sigma}_{j,k}$ (resp. $\hat{\mu}_{j,k}$) satisfy the estimates (5)–(8) (resp. (1)–(4)) for all $j,k\in\mathbb{Z}$, $l\in A(\Omega)$ (with the bound $Cl^2||\Omega||_{L^1(E_l)}$). Moreover, we need to prove that the partial maximal functions $\tilde{M}^{(1)}g_1$, $\tilde{M}^{(2)}g_2$, and $\tilde{M}^{(1,2)}g$ are bounded in L^p for all $p>1$.

It is clear that the measures $\sigma_{j,k}$ and $\mu_{j,k}$ are uniformly bounded by $Cl^2||\Omega||_{L^1(E_l)}$. The proof for the estimates of $\hat{\sigma}_{j,k}$ and $\hat{\mu}_{j,k}$ is the same as those in the proof of Theorem 1 (simply replace $h(\rho_1(y_1),\rho_2(y_2))$ in the proof of Theorem 1 by $e^{i\xi_3\Gamma(\rho_1(y_1),\rho_2(y_2))}h(\rho_1(y_1),\rho_2(y_2))$).

Now observe that for $g\in L^p(\mathbb{R})$, $g\geqslant 0$, we have

$$\begin{aligned}
\tilde{M}^{(1,2)}g(x_3) &= \sup_{j,k}|\mu_{j,k}^{(1,2)} * g(x_3)| \\
&\leqslant \sup_{j,k}\int_{2^{lj}}^{2^{l(j+1)}}\int_{2^{lk}}^{2^{l(k+1)}}\iint_{S^{n-1}\times S^{m-1}} |\Omega_l(u,v)h(\rho_1,\rho_2)g(x_3-\Gamma(\rho_1,\rho_2))| \\
&\quad \times J_1(u)J_2(v)d\sigma(u)d\sigma(v)\frac{d\rho_2}{\rho_2}\frac{d\rho_1}{\rho_1} \\
&\leqslant Cl^2||h||_\infty \iint_{S^{n-1}\times S^{m-1}} |\Omega_l(u,v)| \\
&\quad \times \sup_{r_1,r_2>0}\left\{\frac{1}{r_1r_2}\int_0^{r_2}\int_0^{r_1} g(x_3-\Gamma(s,t))\,ds\,dt\right\}d\sigma(u)d\sigma(v). \\
&\leqslant Cl^2||h||_\infty \iint_{S^{n-1}\times S^{m-1}} |\Omega_l(u,v)|M^H g(x_3)d\sigma(u)d\sigma(v),
\end{aligned}$$

where the last inequality follows from Theorem 2 [11] and $M^H g$ denotes the Hardy-Littlewood maximal function. Therefore, $||\tilde{M}^{(1,2)}g||_p \leqslant Cl^2||\Omega||_{L^1(E_l)}||M^H g||_p \leqslant Cl^2||\Omega||_{L^1(E_l)}||g||_p$ for $1<p\leqslant\infty$.

For $g_1 \in L^p(\mathbb{R}^n \times \mathbb{R}),\ g_1 \geqslant 0,$ we have

$$\begin{aligned}
\tilde{M}^{(1)} g_1(x_1,x_3) &= \sup_{j,k} |\mu^{(1)}_{j,k} * g_1(x_1,x_3)| \\
&\leqslant \sup_{j,k} \int_{2^{lj}}^{2^{l(j+1)}} \int_{2^{lk}}^{2^{l(k+1)}} \iint_{S^{n-1}\times S^{m-1}} |\Omega_l(u,v) h(\rho_1,\rho_2) \\
&\quad \times g_1(x_1 - A^{(1)}_{\rho_1} u, x_3 - \Gamma(\rho_1,\rho_2))| J_1(u) J_2(v) d\sigma(u) d\sigma(v) \frac{d\rho_2}{\rho_2}\frac{d\rho_1}{\rho_1} \\
&\leqslant C l^2 ||h||_\infty \iint_{S^{n-1}\times S^{m-1}} |\Omega_l(u,v)| \\
&\quad \times \sup_{r_1,r_2>0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_1}\int_0^{r_2} g_1(x_1 - A^{(1)}_s u, x_3 - \gamma_s(t))\, dt\, ds \right\} d\sigma(u) d\sigma(v).
\end{aligned}$$

Note that $\dfrac{1}{r_2}\displaystyle\int_0^{r_2} g_1(x_1 - A^{(1)}_s u, x_3 - \gamma_s(t)) dt \leqslant M^H_3 g_1(x_1 - A^{(1)}_s u, x_3)$ for all $r_2 > 0$ and for each fixed $s > 0$ (please see the proof of Theorem 2 [11]). Here $M^H_3 g_1$ is the Hardy-Littlewood maximal function acting on the variable x_3. Thus

$$\begin{aligned}
&\sup_{r_1,r_2>0} \left\{ \frac{1}{r_1 r_2} \int_0^{r_1}\int_0^{r_2} g_1(x_1 - A^{(1)}_s u, x_3 - \gamma_s(t))\, dt\, ds \right\} \\
&\qquad \leqslant \sup_{r_1>0} \left\{ \frac{1}{r_1} \int_0^{r_1} M^H_3 g_1(x_1 - A^{(1)}_s u, x_3)\, ds \right\},
\end{aligned}$$

which is bounded in L^p for all $p > 1$ (see [7, Corollary 5.1]) and the bound is independent of $u \in S^{n-1}$. Consequently,

$$||\tilde{M}^{(1)} g_1||_{L^p(\mathbb{R}^{n+1})} \leqslant C l^2 ||\Omega||_{L^1(E_l)} ||g_1||_{L^p(\mathbb{R}^{n+1})} \text{ for } 1 < p \leqslant \infty.$$

By symmetry we also obtain

$$||\tilde{M}^{(2)} g_2||_{L^p(\mathbb{R}^{m+1})} \leqslant C l^2 ||\Omega||_{L^1(E_l)} ||g_2||_{L^p(\mathbb{R}^{m+1})} \text{ for } 1 < p \leqslant \infty.$$

The proof of Theorem 2 is complete.

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