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STABILITY OF THE BARON—VOLKMANN FUNCTIONAL EQUATIONS

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Abstract. In this paper we prove the stability of the equations $\sup_{\lambda\in T} f(x+\lambda y) = f(x)+f(y)$ and $\inf_{\lambda\in T} f(x+\lambda y) = |f(x)-f(y)|$. Here, f is a real-valued function on V, where V is a complex vector space and $T = \{z\in\mathbb{C} : |z| = 1\}$. Each of these equations characterizes the absolute value of complex linear functionals.

1. Introduction

Denote by $\mathbb{R}$ the space of real numbers, by $\mathbb{C}$ the space of complex numbers, and by T the unit circle in $\mathbb{C}$, that is,

$$T = \{z\in\mathbb{C} : |z| = 1\}.$$

Throughout this paper let V be a complex vector space. Consider the functional equations

$$\sup_{\lambda\in T} f(x+\lambda y) = f(x)+f(y), \quad x,y\in V \tag{1.1}$$

and

$$\inf_{\lambda\in T} f(x+\lambda y) = |f(x)-f(y)|, \quad x,y\in V. \tag{1.2}$$

These equations were introduced by K. Baron and P. Volkmann in [1], where the following result was proven:

THEOREM 1.1. ([1], Theorem 1) *For $f : V\to\mathbb{R}$ the following are equivalent:*
(a) f *satisfies* (1.1).
(b) f *satisfies* (1.2).
(c) f *is of the form* $f(x) = |\phi(x)|$, $x\in V$, *where* $\phi : V\to\mathbb{C}$ *is a linear functional.*

In this paper we investigate the stability of equations (1.1) and (1.2); that is, for each of these equations we show that if g is an approximate solution then there exists an exact solution which is close to g.

Our main results read as follows:

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THEOREM 1.2. *For* $\delta \geqslant 0$, *let* $g\colon V \to \mathbb{R}$ *be an approximate solution of* (1.1), *i.e.,*

$$|\sup_{\lambda \in T} g(x+\lambda y) - g(x) - g(y)| \leqslant \delta, \quad x, y \in V. \tag{1.3}$$

Then, there exists a solution of (1.1) *of the form* $f = |\phi|$, *where* $\phi\colon V \to \mathbb{C}$ *is a linear functional, such that*

$$|f(x) - g(x)| \leqslant 17\delta, \quad x \in V. \tag{1.4}$$

THEOREM 1.3. *For* $\delta \geqslant 0$, *let* $g\colon V \to \mathbb{R}$ *be an approximate solution of* (1.2), *i.e.,*

$$|\inf_{\lambda \in T} g(x+\lambda y) - |g(x) - g(y)|| \leqslant \delta, \quad x, y \in V. \tag{1.5}$$

Then, there exists a solution of (1.2) *of the form* $f = |\phi|$, *where* $\phi\colon V \to \mathbb{C}$ *is a linear functional, such that*

$$|f(x) - g(x)| \leqslant 49\delta, \quad x \in V.$$

2. Proof of Theorem 1.2

Given real numbers a, b and $\varepsilon \geqslant 0$ we shall often use the notation $a \overset{\varepsilon}{\sim} b$ to mean $|a - b| \leqslant \varepsilon$. Notice that the following implications hold:

$$a \overset{\varepsilon_1}{\sim} b \overset{\varepsilon_2}{\sim} c \Rightarrow a \overset{\varepsilon_1+\varepsilon_2}{\sim} c,$$

$$a \leqslant b \overset{\varepsilon}{\sim} c \Rightarrow a \leqslant c + \varepsilon,$$

$$a \overset{\varepsilon}{\sim} b \leqslant c \Rightarrow a \leqslant c + \varepsilon.$$

Further, if I is a set of indices, then we have

$$a_i \overset{\varepsilon}{\sim} b_i,\, i \in I \Rightarrow \sup_{i \in I} a_i \overset{\varepsilon}{\sim} \sup_{i \in I} b_i \text{ and } \inf_{i \in I} a_i \overset{\varepsilon}{\sim} \inf_{i \in I} b_i,$$

provided the involved suprema and infima are finite.

We proceed with the following simple lemma.

LEMMA 2.1. *Let* $g\colon V \to \mathbb{R}$ *satisfy* (1.3). *Then:*
(i) $|g(0)| \leqslant \delta$,
(ii) $g(x) \geqslant -\delta, \quad x \in V$,
(iii) $g(x+y) \leqslant g(x) + g(y) + \delta, \quad x, y \in V$.

Proof. Part (i) is obtained by putting $x = y = 0$ in (1.3). Part (ii) is a consequence of (1.3) and part (i), since

$$2g(x) \overset{\delta}{\sim} \sup_{\lambda \in T} g(x + \lambda x) \geqslant g(0) \geqslant -\delta.$$

Part (iii) follows from

$$g(x+y) \leqslant \sup_{\lambda \in T} g(x+\lambda y) \overset{\delta}{\sim} g(x)+g(y). \qquad \square$$

We shall next deal with the stability of equation (1.1) in the one-dimensional case $V=\mathbb{C}$. In this case every linear functional $\phi\colon \mathbb{C} \to \mathbb{C}$ is, of course, of the form $\phi(z) = cz$ where c is a complex constant.

PROPOSITION 2.1. *Let $g\colon \mathbb{C} \to \mathbb{R}$ satisfy* (1.3). *Then there exists a solution of equation* (1.1) *of the form $f(z) = c|z|$ where c is a nonnegative constant, such that*

$$|g(z)-f(z)| \leqslant 7\delta, \quad z \in \mathbb{C}.$$

Proof. For convenience denote $S_r := \sup_{\lambda \in T} g(\lambda r)$ for $r \in [0,\infty)$. By (1.3) with $x=0$ and by Lemma 2.1(i), we get

$$S_{|y|} = \sup_{\lambda \in T} g(\lambda y) \overset{2\delta}{\sim} g(y), \quad y \in \mathbb{C}. \tag{2.1}$$

More precisely, since

$$g(y) \leqslant \sup_{\lambda \in T} g(\lambda y) = \sup_{\lambda \in T} g(\lambda |y|) = S_{|y|},$$

we have

$$S_{|y|} - 2\delta \leqslant g(y) \leqslant S_{|y|}, \tag{2.2}$$

and

$$S_{|y|} - 2\delta \leqslant g(|y|) \leqslant S_{|y|}.$$

Therefore,

$$g(y) \overset{2\delta}{\sim} g(|y|), \quad y \in \mathbb{C}. \tag{2.3}$$

We will show that the restriction $g|_{[0,\infty)}$ is 5δ–approximately additive, that is,

$$g(x+y) \overset{5\delta}{\sim} g(x)+g(y), \quad x,y \in [0,\infty). \tag{2.4}$$

By Lemma 2.1(iii), we have $g(x+y) \leqslant g(x)+g(y)+\delta$. Hence, to prove (2.4) it suffices to show that $g(x)+g(y) \leqslant g(x+y)+5\delta$. Let $0 \leqslant r \leqslant t$. From (1.3), (2.2) and (2.1) we obtain

$$\begin{aligned} 2g(r) &\overset{\delta}{\sim} \sup_{\lambda \in T} g(r+\lambda r) \leqslant \sup_{\lambda \in T} S_{r|1+\lambda|} = \sup_{|z| \leqslant 2r} g(z) \\ &\leqslant \sup_{|z| \leqslant 2t} g(z) = \sup_{\lambda \in T} S_{t|1+\lambda|} \overset{2\delta}{\sim} \sup_{\lambda \in T} g(t+\lambda t) \overset{\delta}{\sim} 2g(t). \end{aligned}$$

Hence,

$$g(r) \leqslant g(t)+2\delta. \tag{2.5}$$

Fix $0 \leqslant y \leqslant x$. By (1.3), (2.3) and (2.5) we get

$$g(x)+g(y) \overset{\delta}{\sim} \sup_{\lambda\in T} g(x+\lambda y) \overset{2\delta}{\sim} \sup_{\lambda\in T} g(|x+\lambda y|) \leqslant g(x+y)+2\delta,$$

so (2.4) follows.

Since (2.4) holds, it is well known ([2], Ch. XVII, §1, Theorem 1) that there exists a unique additive function $h\colon [0,\infty) \to \mathbb{R}$ such that

$$|g(x)-h(x)| \leqslant 5\delta, \quad x \in [0,\infty). \tag{2.6}$$

Thus, combining (2.6) and Lemma 2.1(ii), we obtain $h(x) \geqslant -6\delta$ for all $x \in [0,\infty)$. Consequently, h is continuous (e.g. [2], Ch. XII, §1, Theorem 3), so h must be of the form $h(x) = cx$, $x \in [0,\infty)$, where c is a real constant. Of course, $c \geqslant 0$, as $cx \geqslant -6\delta$, $x \in [0,\infty)$. Finally, for an arbitrary $z \in \mathbb{C}$, we use (2.3) and (2.6), to obtain

$$g(z) \overset{2\delta}{\sim} g(|z|) \overset{5\delta}{\sim} c|z|. \qquad \square$$

We now pass to the general case:

Proof of Theorem 1.2. In view of Proposition 2.1 we may assume that the complex vector space V is at least two-dimensional. Define

$$W = \{x \in V : \sup_{\alpha\in\mathbb{C}} g(\alpha x) < \infty\}. \tag{2.7}$$

Considering Lemma 2.1(iii), we see that W is a subspace of V.

For $x \in V$, let $g_x\colon \mathbb{C} \to \mathbb{R}$ be the function defined by

$$g_x(\alpha) := g(\alpha x). \tag{2.8}$$

It is not hard to see that for every $x \in V$,

$$\sup_{\lambda\in T} g_x(\alpha+\lambda\beta) \overset{\delta}{\sim} g_x(\alpha)+g_x(\beta), \quad \alpha,\beta \in \mathbb{C}.$$

Hence, in view of Proposition 2.1, we get

$$g(\alpha x) = g_x(\alpha) \overset{7\delta}{\sim} c_x|\alpha|, \quad \alpha \in \mathbb{C}, \tag{2.9}$$

where c_x is a nonnegative constant. If $x \in W$ we thus infer that $c_x = 0$. So consequently, $g(\alpha x) \leqslant 7\delta$ for all $\alpha \in \mathbb{C}$, $x \in W$, and in particular,

$$g(x) \leqslant 7\delta, \quad x \in W. \tag{2.10}$$

If $W = V$, the above inequality ensures that $f = 0$ satisfies (1.4). So we may assume that W is a proper subspace of V, in which case we will show that $\operatorname{codim} W = 1$. As in [1], this will be done by proving that every two-dimensional subspace of V has a nonzero element which belongs to W. To this end fix arbitrary linearly independent elements $\hat{x},\hat{y}\in V$. Consider the function $p\colon \mathbb{C}^2\to\mathbb{R}$ defined by $p(\alpha,\beta)=g(\alpha\hat{x}+\beta\hat{y})$. Further, for each pair $(\alpha,\beta)\in\mathbb{C}^2$ define the function $q_{(\alpha,\beta)}\colon \mathbb{C}\to\mathbb{R}$ by $q_{(\alpha,\beta)}(\xi)=p(\xi\alpha,\xi\beta)$. It is easily verified that p, as well as the functions $q_{(\alpha,\beta)}$, satisfy the functional inequality in (1.3). From Proposition 2.1 we conclude that for every $(\alpha,\beta)\in\mathbb{C}^2$ there is a nonnegative value $k(\alpha,\beta)$ such that

$$q_{(\alpha,\beta)}(\xi)\overset{7\delta}{\sim}k(\alpha,\beta)|\xi|,\quad \xi\in\mathbb{C}. \tag{2.11}$$

Now, by (2.11), and since p satisfies (1.3), we find that for every $z,w,u,v,\xi\in\mathbb{C}$,

$$\begin{aligned}\sup_{\lambda\in T}k((z,w)+\lambda(u,v))|\xi| &\overset{7\delta}{\sim}\sup_{\lambda\in T}q_{(z,w)+\lambda(u,v)}(\xi)\\ &=\sup_{\lambda\in T}p(\xi(z,w)+\lambda\xi(u,v))\overset{\delta}{\sim}p(\xi(z,w))+p(\xi(u,v))\\ &=q_{(z,w)}(\xi)+q_{(u,v)}(\xi)\overset{7\delta+7\delta}{\sim}k(z,w)|\xi|+k(u,v)|\xi|.\end{aligned}$$

Thereby,

$$\sup_{\lambda\in T}k((z,w)+\lambda(u,v))|\xi|\overset{22\delta}{\sim}(k(z,w)+k(u,v))|\xi|. \tag{2.12}$$

Next, dividing (2.12) by $|\xi|$ and letting $|\xi|\to\infty$, we get

$$\sup_{\lambda\in T}k((z,w)+\lambda(u,v))=k(z,w)+k(u,v),$$

i.e., the function $k\colon \mathbb{C}^2\to\mathbb{R}$ satisfies (1.1). Therefore, by Theorem 1.1, k is the absolute value of a linear functional from $\mathbb{C}^2$ to $\mathbb{C}$; hence of the form $k(\alpha,\beta)=|c\alpha+d\beta|$, where c and d are complex constants. We conclude that there exist $\hat{\alpha},\hat{\beta}\in\mathbb{C}$, not both zero, for which $k(\hat{\alpha},\hat{\beta})=0$. By (2.11), it follows that

$$g(\xi(\hat{\alpha}\hat{x}+\hat{\beta}\hat{y}))=p(\xi\hat{\alpha},\xi\hat{\beta})=q_{(\hat{\alpha},\hat{\beta})}(\xi)\overset{7\delta}{\sim}k(\hat{\alpha},\hat{\beta})|\xi|=0,\quad \xi\in\mathbb{C}.$$

This implies that $\hat{\alpha}\hat{x}+\hat{\beta}\hat{y}\in W$, showing that $\operatorname{codim}W=1$.

We now claim that

$$g(x+y)\overset{10\delta}{\sim}g(x),\quad x\in V, y\in W. \tag{2.13}$$

Indeed, by (1.3),

$$g(x)+g(y)\overset{\delta}{\sim}\sup_{\lambda\in T}g(x+\lambda y)\overset{\delta}{\sim}g(x)+g(-y),$$

which implies

$$g(y)\overset{2\delta}{\sim}g(-y). \tag{2.14}$$

By Lemma 2.1(iii) and by (2.10) and (2.14), for every $x\in V$ and $y\in W$,

$$\begin{aligned}g(x+y)&\leqslant g(x)+g(y)+\delta\leqslant g(x)+7\delta+\delta\\ &=g(x+y-y)+8\delta\leqslant g(x+y)+g(-y)+\delta+8\delta\\ &\overset{2\delta}{\sim}g(x+y)+g(y)+9\delta\leqslant g(x+y)+7\delta+9\delta,\end{aligned} \tag{2.15}$$

which proves (2.13).

Finally, since $\operatorname{codim} W = 1$, we have $V = \mathbb{C}x_0 \oplus W$, where x_0 is some fixed vector in V and $\mathbb{C}x_0$ is the one-dimensional subspace of all complex multiples of x_0. It follows that every $x \in V$ can be written as $x = \alpha_x x_0 + y_x$, where $\alpha_x \in \mathbb{C}$ and $y_x \in W$ are uniquely determined. By (2.13) and (2.9) we obtain

$$g(x) = g(\alpha_x x_0 + y_x) \overset{10\delta}{\sim} g(\alpha_x x_0) = g_{x_0}(\alpha_x) \overset{7\delta}{\sim} c_{x_0}|\alpha_x|,$$

so

$$|g(x) - c_{x_0}|\alpha_x|| \leqslant 17\delta.$$

Hence, introducing the linear functional $\phi(x) := c_{x_0}\alpha_x$ we see that $f(x) = |\phi(x)|$ satisfies the inequality in (1.4), and the proof is complete. $\square$

3. Proof of Theorem 1.3

LEMMA 3.1. *Let* $g\colon V \to \mathbb{R}$ *satisfy* (1.5). *Then*
(i) $|g(0)| \leqslant \delta$,
(ii) $g(x) \geqslant -\delta$, $x \in V$,
(iii) $g(x+y) \leqslant g(x) + g(y) + \delta$, $x, y \in V$.

Proof. We obtain part (i) by putting $x = y = 0$ in (1.5). Part (ii) is an immediate consequence of (1.5). Part (iii) follows from

$$g(x+y) - g(y) \leqslant |g(x+y) - g(y)| \overset{\delta}{\sim} \inf_{\lambda \in T} g(x+y+\lambda y) \leqslant g(x+y-y). \qquad \square$$

As in previous section, we first deal with the one-dimensional case of Theorem 1.3, that is, $V = \mathbb{C}$.

PROPOSITION 3.1. *Let* $g\colon \mathbb{C} \to \mathbb{R}$ *satisfy* (1.5). *Then there exists a solution of equation* (1.2) *of the form* $f(z) = c|z|$ *where* c *is a nonnegative constant, such that*

$$|f(z) - g(z)| \leqslant 23\delta, \quad z \in \mathbb{C}. \tag{3.1}$$

Proof. We claim that

$$\inf_{\lambda \in T} g(\lambda y) \leqslant g(y) \leqslant \inf_{\lambda \in T} g(\lambda y) + 2\delta, \quad y \in \mathbb{C}. \tag{3.2}$$

Indeed, the first inequality is obvious. To obtain the second, we consider two cases. If $g(y) \geqslant g(0)$, then by (1.5) with $x = 0$ and Lemma 3.1(i),

$$\inf_{\lambda \in T} g(\lambda y) \overset{\delta}{\sim} |g(0) - g(y)| = g(y) - g(0) \overset{\delta}{\sim} g(y).$$

Otherwise, by parts (i) and (ii) of Lemma 3.1, we have

$$-\delta \leqslant \inf_{\lambda \in T} g(\lambda y) \leqslant g(y) < g(0) \leqslant \delta.$$

So, (3.2) is proved, from which we obtain

$$g(y) \overset{2\delta}{\sim} g(|y|), \quad y \in \mathbb{C}. \tag{3.3}$$

Let $x \geqslant 0$. Using (3.3) and Lemma 3.1(iii) we get

$$\begin{aligned} g(|1+\lambda|x) &\overset{2\delta}{\sim} g(x+\lambda x) \leqslant g(x)+g(\lambda x)+\delta \\ &\overset{2\delta}{\sim} g(x)+g(x)+\delta = 2g(x)+\delta \end{aligned}$$

for every $\lambda \in T$. Therefore,

$$\sup g([0,2x]) = \sup_{\lambda \in T} g(|1+\lambda|x) \leqslant 2g(x)+5\delta, \quad x \geqslant 0. \tag{3.4}$$

Taking again advantage of (3.3), we have

$$\begin{aligned} \inf g([t-x,t+x]) &= \inf_{\lambda\in T} g(|t+\lambda x|) \\ &\overset{2\delta}{\sim} \inf_{\lambda\in T} g(t+\lambda x) \overset{\delta}{\sim} |g(t)-g(x)|, \quad 0 \leqslant x \leqslant t. \end{aligned} \tag{3.5}$$

Thereby, for $0 \leqslant x \leqslant y \leqslant t$, we obtain

$$|g(t)-g(y)| \overset{3\delta}{\sim} \inf g([t-y,t+y]) \leqslant \inf g([t-x,t+x]) \overset{3\delta}{\sim} |g(t)-g(x)|.$$

Hence,

$$|g(t)-g(y)| \leqslant |g(t)-g(x)|+6\delta, \quad 0 \leqslant x \leqslant y \leqslant t. \tag{3.6}$$

Notice that if $g(z) \leqslant 15\delta$ for every $z \in \mathbb{C}$, then with $f=0$ we get (3.1). So, from now on we can assume that there is $z \in \mathbb{C}$ with $g(z) > 15\delta$. By (3.3), therefore,

$$g(|z|) \overset{2\delta}{\sim} g(z) > 15\delta. \tag{3.7}$$

We will next show that

$$\lim_{\mathbb{R}\ni x\to\infty} g(x) = \infty. \tag{3.8}$$

Suppose, on the contrary, that there is a real sequence $\{a_n\}_{n\in\mathbb{N}}$ tending to infinity, such that $\{g(a_n)\}_{n\in\mathbb{N}}$ converges to some real limit. Since $\{g(a_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that $|g(a_n)-g(a_N)| < \delta$ for every $n \geqslant N$. By this and by (3.5) we infer that $\inf g([a_n-a_N, a_n+a_N]) < 4\delta$ for all $n \geqslant N$. Hence, there exists a real sequence $\{b_n\}_n$ tending to infinity, such that $g(b_n) < 4\delta$, $n \in \mathbb{N}$. Substituting b_n for x in (3.4) we obtain $\sup g([0,\infty)) \leqslant 13\delta$. But this contradicts (3.7); so we proved (3.8). Thereby, for any $0 \leqslant x \leqslant y$, we can choose t, $t \geqslant y$, with $g(t) \geqslant \max\{g(x),g(y)\}$. For this t, (3.6) implies

$$g(x) \leqslant g(y)+6\delta, \quad 0 \leqslant x \leqslant y. \tag{3.9}$$

Fix $0 \leqslant x \leqslant y$ and notice that if $g(y) < g(x)$ then, by (3.9), $|g(y)-g(x)| = g(x) - g(y) \leqslant 6\delta$. Therefore, $|g(y)-g(x)|+g(x)-g(y) \leqslant 12\delta$. Hence,

$$|g(y)-g(x)| \leqslant g(y)-g(x)+12\delta.$$

Of course, if $g(y) \geqslant g(x)$, then the above inequality is also true. Moreover, by (3.9), $g(y-x) \leqslant \inf g([y-x,y+x]) + 6\delta$. By this inequality, and by (3.3) and (1.5), we get

$$\begin{aligned} g(y-x) - 6\delta &\leqslant \inf g([y-x,y+x]) = \inf_{\lambda \in T} g(|y+\lambda x|) \\ &\overset{2\delta}{\sim} \inf_{\lambda \in T} g(y+\lambda x) \overset{\delta}{\sim} |g(y)-g(x)| \leqslant g(y)-g(x)+12\delta, \quad 0 \leqslant x \leqslant y. \end{aligned}$$

Therefore,

$$g(x)+g(y) \leqslant g(x+y)+21\delta, \quad x,y \geqslant 0. \tag{3.10}$$

Lemma 3.1(iii) and (3.10) imply that the restriction $g|_{[0,\infty)}$ is 21δ–approximately additive. Now, as in Proposition 2.1 we conclude that there exists an additive continuous function of the form $h(x) = cx$, $x \in [0,\infty)$, where c is a nonnegative constant, satisfying $g(x) \overset{21\delta}{\sim} h(x)$. Finally,

$$g(z) \overset{2\delta}{\sim} g(|z|) \overset{21\delta}{\sim} c|z|, \quad z \in \mathbb{C}.$$

Hence, (3.1) holds with $f(z) := c|z|$. □

We are now ready to prove the general case of Theorem 1.3.

Proof. [Proof of Theorem 1.3] As in the proof of Theorem 1.2, we may assume that $\dim V \geqslant 2$. Let W be the subspace of V defined in (2.7). As before, it can be shown that $\operatorname{codim} W = 1$, provided $W \subsetneq V$.

Now, for every $x \in V$ we recall the function g_x in (2.8). Since g_x is a solution of the functional inequality (1.5), we use Proposition 3.1 to obtain

$$g(\alpha x) = g_x(\alpha) \overset{23\delta}{\sim} c_x|\alpha|, \quad \alpha \in \mathbb{C}, \tag{3.11}$$

where c_x is a nonnegative constant. We see that $c_x = 0$ for $x \in W$; whence

$$g(x) \leqslant 23\delta, \quad x \in W. \tag{3.12}$$

Moreover, we claim that

$$g(x) \overset{2\delta}{\sim} g(-x), \quad x \in V. \tag{3.13}$$

Indeed, notice that

$$|g(0)-g(x)| \overset{\delta}{\sim} \inf_{\lambda \in T} g(\lambda x) = \inf_{\lambda \in T} g(\lambda(-x)) \overset{\delta}{\sim} |g(0)-g(-x)|.$$

Thereby,

$$|g(0)-g(x)| \overset{2\delta}{\sim} |g(0)-g(-x)|.$$

If the signs of the differences $g(0)-g(x)$ and $g(0)-g(-x)$ are identical, then (3.13) is a consequence of the last formula. Otherwise, we can assume without loss of generality that $g(-x) < g(0) < g(x)$. By Lemma 3.1(i) and (1.5), we get

$$\begin{aligned} g(-x) &< g(x) = (g(x)-g(0)) + g(0) \\ &\overset{\delta}{\sim} \inf_{\lambda \in T} g(\lambda x) + g(0) \overset{\delta}{\sim} \inf_{\lambda \in T} g(\lambda x) \leqslant g(-x). \end{aligned}$$

This ends the proof of (3.13).

Using Lemma 3.1(iii), (3.12) and (3.13), we repeat the calculations in (2.15) to find that

$$g(x+y) \overset{26\delta}{\sim} g(x), \quad x \in V, y \in W. \tag{3.14}$$

Again, we have two possibilities: either $W = V$, so in view of (3.12), $f = 0$ satisfies (1.3); or $V = \mathbb{C}x_0 \oplus W$ for an $x_0 \in V$. In this latter case, for every $x \in V$ there are unique $\alpha_x \in \mathbb{C}$ and $y_x \in W$ such that $x = \alpha_x x_0 + y_x$. The function $\phi\colon V \to \mathbb{C}$ defined by $\phi(x) := c_{x_0}\alpha_x$ is a linear functional. By (3.14) and (3.11), we have

$$g(x) = g(\alpha_x x_0 + y_x) \overset{26\delta}{\sim} g(\alpha_x x_0) = g_{x_0}(\alpha_x) \overset{23\delta}{\sim} c_{x_0}|\alpha_x| = |\phi(x)| =: f(x),$$

and the proof is at hand. □

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