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IMPROVED HARDY–SOBOLEV INEQUALITIES FOR RADIAL DERIVATIVE

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Abstract. We prove some Hardy-Sobolev inequalities for radial derivative and obtain the corresponding sharp constant.

1. Introduction

Hardy inequality in $\mathbb{R}^N$ reads, for all $f \in C_0^\infty(\mathbb{R}^N)$ and $N \geqslant 3$,

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx \geqslant \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} dx \tag{1.1}$$

and $\frac{(N-2)^2}{4}$ is the best constant in (1.1). A similar inequality with the same best constant holds if $\mathbb{R}^N$ is replaced by an arbitrary domain $\Omega \subset \mathbb{R}^N$ and Ω contains the origin. On the other hand, the classical Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx \geqslant S_N \left(\int_{\mathbb{R}^N} |f|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}. \tag{1.2}$$

is valid for any $f \in C_0^\infty(\mathbb{R}^N)$, where $S_N = \pi N(N-2)(\Gamma(\frac{N}{2})/\Gamma(N))^{\frac{2}{N}}$ is the best constant(cf. [2, 7]). Stubbe's result ([6]) states that for $0 < \delta < \frac{(N-2)^2}{4}$,

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx - \delta \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} dx \geqslant \left(1 - \frac{\delta}{\frac{(N-2)^2}{4}}\right)^{\frac{N-1}{N}} S_N \left(\int_{\mathbb{R}^N} |f|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \tag{1.3}$$

and the constant in (1.3) is sharp (see also [4]). Recently, Adimurthi, S. Filippas and A. Tertikas established the following Hardy-Sobolev inequality: for all $f \in C_0^\infty(B_1)$,

$$\int_{B_1} |\nabla f|^2 dx - \frac{(N-2)^2}{4} \int_{B_1} \frac{f^2}{|x|^2} dx \geqslant C_{N,a} \left(\int_{B_1} X_1^{\frac{2(N-1)}{N-2}}(a,|x|) |f|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}, \tag{1.4}$$

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where $B_1 \subset \mathbb{R}^N$ is the unit ball centered at zero and

$$X_1(a,s) := (a - \ln s)^{-1}, \quad a > 0,\ 0 < s \leqslant 1.$$

The best constant $C_{N,a}$ in (1.4) satisfies

$$C_{N,a} = \begin{cases} (N-2)^{-\frac{2(N-1)}{N}} S_N,\ a \geqslant \frac{1}{N-2} \\ a^{-\frac{2(N-1)}{N}} S_N, \qquad 0 < a < \frac{1}{N-2}. \end{cases}$$

When restricted to radial functions, the best constant in (1.4) is given by

$$C_{N,a,radial} = (N-2)^{-\frac{2(N-1)}{N}} S_N, \forall a \geqslant 0.$$

Our goal in this note is to establish analogous inequalities (1.2)–(1.4) for radial derivative of f, i.e., $f_r = \nabla f \cdot \frac{x}{|x|}$ with $r = |x|$. Recall that the Hardy inequality in $\mathbb{R}^N$ for radial derivative reads, for all $f \in C_0^\infty(\mathbb{R}^N)$ and $N \geqslant 3$,

$$\int_{\mathbb{R}^N} |f_r|^2 dx \geqslant \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} dx.$$

The Sobolev inequality for radial derivative reads (cf. [3]), for all $f \in C_0^\infty(\mathbb{R}^N)$ and $N \geqslant 3$,

$$\int_{\mathbb{R}^N} |f_r|^2 dx \geqslant C_N \left(\int_{\mathbb{R}^N} |F(r)|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}, \tag{1.5}$$

where $F(r)$ is the integral mean of f over the unit sphere $\mathbb{S}^{N-1}$, that is,

$$F(r) = \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} f(r\omega) d\omega.$$

Here we use the polar co-ordinates $x = r\omega$. To this end we have:

THEOREM 1.1. *The best constant C_N in (1.5) satisfies $C_N = S_N$. The extreme function is*

$$U_\lambda(x) = c \left(\frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{N-2}{2}}, \ c \neq 0,\ \lambda > 0.$$

Furthermore, if $\Omega \subset \mathbb{R}^N$ is a bounded domain containing the origin, then for all $f \in C_0^\infty(\Omega)$ and $N \geqslant 3$,

$$\int_\Omega |f_r|^2 dx \geqslant S_N \left(\int_\Omega |F(r)|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \tag{1.6}$$

and the constant in (1.6) is sharp.

We generalize Stubbe's result to the radial derivative.

THEOREM 1.2. *Let $f \in C_0^\infty(\mathbb{R}^N)$ and $N \geqslant 3$. There holds, for $0 < \delta < \frac{(N-2)^2}{4}$,*

$$\int_{\mathbb{R}^N} |f_r|^2 dx - \delta \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} dx \geqslant \left(1 - \frac{\delta}{\frac{(N-2)^2}{4}}\right)^{\frac{N-1}{N}} S_N \left(\int_{\mathbb{R}^N} |F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \tag{1.7}$$

and the constant in (1.7) is sharp.

The following corollary generalizes a result of A. Balinsky et al (cf. [3], inequality (4.12)).

COROLLARY 1.3. *Let $f \in C_0^\infty(\mathbb{R}^N)$ and $N \geqslant 3$. There holds, for $(N-1) < \delta < \frac{N^2}{4}$,*

$$\int_{\mathbb{R}^N} |\langle x, \nabla f\rangle|^2 dx - \delta \int_{\mathbb{R}^N} f^2 dx \geqslant \frac{\left(\frac{N^2}{4} - \delta\right)^{\frac{N-1}{N}}}{\left(\frac{(N-2)^2}{4}\right)^{\frac{N-1}{N}}} S_N \left(\int_{\mathbb{R}^N} |rF(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \tag{1.8}$$

and the constant in (1.8) is sharp.

Finally, we generalize Adimurthi et al's result to the radial derivative.

THEOREM 1.4. *Let $f \in C_0^\infty(B_1)$ and $N \geqslant 3$. There holds, for all $a > 0$,*

$$\int_{B_1} |f_r|^2 dx - \frac{(N-2)^2}{4} \int_{B_1} \frac{f^2}{|x|^2} dx \geqslant (N-2)^{-\frac{2(N-1)}{N}} S_N \left(\int_{B_1} X_1^{\frac{2(N-1)}{N-2}}(a, |x|) |F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \tag{1.9}$$

and the constant in (1.9) is sharp. Furthermore, if $\Omega \subset \mathbb{R}^N$ is a bounded domain containing the origin, then

$$\int_{\Omega} |f_r|^2 dx - \frac{(N-2)^2}{4} \int_{\Omega} \frac{f^2}{|x|^2} dx \geqslant (N-2)^{-\frac{2(N-1)}{N}} S_N \left(\int_{\Omega} X_1^{\frac{2(N-1)}{N-2}}\left(a, \frac{|x|}{D}\right) |F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \tag{1.10}$$

with $D = \sup_{x\in\Omega} |x|$ and the constant in (1.10) is sharp.

COROLLARY 1.5. *Let $f \in C_0^\infty(\Omega)$ and $N \geqslant 3$. There holds, for all $a > 0$,*

$$\int_{\Omega} |\langle x, \nabla f\rangle|^2 dx - \frac{N^2}{4} \int_{\Omega} f^2 dx \geqslant (N-2)^{-\frac{2(N-1)}{N}} S_N \left(\int_{\Omega} X_1^{\frac{2(N-1)}{N-2}}\left(a, \frac{|x|}{D}\right) |rF(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \tag{1.11}$$

and the constant in (1.11) is sharp.

REMARK 1.6. If f is supported in the annulus $A_R := \{x \in \mathbb{R}^N : R^{-1} < |x| < R\}$, then

$$X_1^{\frac{2(N-1)}{N-2}}\left(a, \frac{|x|}{D}\right) = \left(\frac{1}{a - \ln\frac{|x|}{R}}\right)^{\frac{2(N-1)}{N-2}} \geqslant \left(\frac{1}{a + 2\ln R}\right)^{\frac{2(N-1)}{N-2}}.$$

By Theorem 1.4 and Corollary 1.5 and letting $a \to 0+$, we have

$$\int_{A_R} |f_r|^2 dx - \frac{(N-2)^2}{4} \int_{A_R} \frac{f^2}{|x|^2} dx \geqslant [2(N-2)\ln R]^{-\frac{2(N-1)}{N}} S_N \left(\int_{A_R} |F(r)|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}$$

and

$$\int_{A_R} |\langle x, \nabla f \rangle|^2 dx - \frac{N^2}{4} \int_{A_R} f^2 dx \geqslant [2(N-2)\ln R]^{-\frac{2(N-1)}{N}} S_N \left(\int_{A_R} |rF(r)|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}},$$

which generalize the results of Balinsky et al (cf. [3], Corollary 4.6, Corollary 4.7).

2. The proofs

To prove the main result, we first need the following useful lemma.

LEMMA 2.1. *Let $f \in C_0^\infty(\mathbb{R}^N)$ be real-valued and $p > 1$. There holds*

$$|F(r)|^p = \left| \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} f(r\omega) d\omega \right|^p \leqslant \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} |f(r\omega)|^p d\omega; \tag{2.1}$$

$$|F'(r)|^p = \left| \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} f_r(r\omega) d\omega \right|^p \leqslant \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} |f_r(r\omega)|^p d\omega. \tag{2.2}$$

Inequalities (2.1) *and* (2.2) *becomes equalities if and only if $f(x)$ is radial, i.e., $f(r\omega) = f(r)$.*

Proof. By Hölder's inequality,

$$\begin{aligned} \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} |f(r\omega)| d\omega &\leqslant \frac{1}{|\mathbb{S}^{N-1}|} \left(\int_{\mathbb{S}^{N-1}} |f(r\omega)|^p d\omega \right)^{\frac{1}{p}} \cdot \left(\int_{\mathbb{S}^{N-1}} d\omega \right)^{\frac{1}{p'}} \\ &\qquad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right) \\ &= \frac{1}{|\mathbb{S}^{N-1}|^{1-\frac{1}{p'}}} \left(\int_{\mathbb{S}^{N-1}} |f(r\omega)|^p d\omega \right)^{\frac{1}{p}} \\ &= \frac{1}{|\mathbb{S}^{N-1}|^{\frac{1}{p}}} \left(\int_{\mathbb{S}^{N-1}} |f(r\omega)|^p d\omega \right)^{\frac{1}{p}}. \end{aligned} \tag{2.3}$$

Inequalities (2.3) becomes equalities if and only if $|f(r\omega)|$ is constant for all $\omega \in \mathbb{S}^{N-1}$, i.e. $f(r\omega)$ is constant for all $\omega \in \mathbb{S}^{N-1}$ since $\mathbb{S}^{N-1}$ is a connected set. Therefore, we obtain, by (2.3),

$$|F(r)|^p \leqslant \left(\frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} |f(r\omega)| d\omega \right)^p \leqslant \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} |f(r\omega)|^p d\omega.$$

Similarly, we have

$$|F'(r)|^p \leqslant \frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}} |f_r(r\omega)|^p d\omega.$$

The inequality above becomes an equality if and only if $f(x)$ is radial.

Using lemma 2.1 and polar co-ordinates on $\mathbb{R}^N$, we have the following:

COROLLARY 2.2. *Let $f \in C_0^\infty(\mathbb{R}^N)$ and $p > 1$. There holds, for $\alpha < N$,*

$$\int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^\alpha}dx \geqslant \int_{\mathbb{R}^N} \frac{|F(r)|^p}{|x|^\alpha}dx; \quad \int_{\mathbb{R}^N} \frac{|f_r(x)|^p}{|x|^\alpha}dx \geqslant \int_{\mathbb{R}^N} \frac{|F'(r)|^p}{|x|^\alpha}dx. \tag{2.4}$$

Inequalities (2.4) becomes equalities if and only if $f(x)$ is radial.

Proof of Theorem 1.1. By corollary 2.2,

$$\int_{\mathbb{R}^N} |f_r|^2 dx \geqslant \int_{\mathbb{R}^N} |F'(r)|^2 dx = \int_{\mathbb{R}^N} |\nabla F(r)|^2 dx \geqslant S_N \left(\int_{\mathbb{R}^N} |F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}. \tag{2.5}$$

Inequalities (2.5) becomes equalities if and only if $f(x)$ is radial and

$$F(r) = U_\lambda(x) = c\left(\frac{\lambda}{\lambda^2 + |x|^2}\right)^{\frac{N-2}{2}}$$

for some $c \neq 0$ and $\lambda > 0$ (cf. [2, 7]). To finish the proof of theorem 1.1, it is enough to show the constant in (1.6) is sharp since $f \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^N)$. Consider the sequence of functions

$$V_\lambda = U_\lambda \cdot \phi_\delta(|x|)$$

where $\phi_\delta(t)$ is a C_0^∞ cutoff function which is zero for $t > \delta$ and equal to one for $t < \delta/2$; δ is small enough so that $B_\delta := \{x \in \mathbb{R}^N | |x| < \delta\} \subset \Omega$. Then $V_\lambda \in C_0^\infty(B_\delta) \subset C_0^\infty(\Omega)$. It is well known that (cf. [5])

$$S_N = \lim_{\lambda \to 0+} \frac{\int_\Omega |\nabla V_\lambda|^2 dx}{\left(\int_\Omega |V_\lambda|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}}.$$

Thus, the constant in (1.6) is sharp and this completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By corollary 2.2 we have, for $0 < \alpha < (N-2)/2$,

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|f_r(x)|^2}{|x|^{2\alpha}}dx &\geqslant \int_{\mathbb{R}^N} \frac{|F'(r)|^2}{|x|^{2\alpha}}dx = |\mathbb{S}^{N-1}| \int_0^\infty r^{N-1-2\alpha}|F'(r)|^2 dr \\ &\geqslant \left(1 - \frac{\alpha(N-2-\alpha)}{\frac{(N-2)^2}{4}}\right)^{\frac{N-1}{N}} S_N \left(\int_{\mathbb{R}^N} \left|\frac{F(r)}{|x|^\alpha}\right|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}\end{aligned} \tag{2.6}$$

(cf. [4], the proof of theorem 2) and the constant $\left(1-\frac{\alpha(N-2-\alpha)}{\frac{(N-2)^2}{4}}\right)^{\frac{N-1}{N}} S_N$ is sharp. Set $g(x)=\frac{f(x)}{|x|^\alpha}$ and $G(r)=\frac{F(r)}{|x|^\alpha}$. Through integration by parts, we have, by (2.6),

$$\begin{aligned} \int_{\mathbb{R}^N} |g_r(x)|^2 dx - \alpha(n-2-\alpha)\int_{\mathbb{R}^N} \frac{g^2(x)}{|x|^2}dx &= \int_{\mathbb{R}^N} \frac{|f_r(x)|^2}{|x|^{2\alpha}}dx \\ &\geqslant \left(1-\frac{\alpha(N-2-\alpha)}{\frac{(N-2)^2}{4}}\right)^{\frac{N-1}{N}} S_N \left(\int_{\mathbb{R}^N} \left|\frac{F(r)}{|x|^\alpha}\right|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \\ &= \left(1-\frac{\alpha(N-2-\alpha)}{\frac{(N-2)^2}{4}}\right)^{\frac{N-1}{N}} S_N \left(\int_{\mathbb{R}^N} |G(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}. \end{aligned} \tag{2.7}$$

The desired result follows. □

Proof of Corollary 1.3. On substituting $f(x)=|x|g(x)$ in Theorem 1.2, we have, through integration by parts,

$$\begin{aligned} \int_{\mathbb{R}^N} |\langle x,\nabla g\rangle|^2 dx - \delta\int_{\mathbb{R}^N} g^2 dx &= \int_{\mathbb{R}^N} |f_r|^2 dx - (\delta-N+1)\int_{\mathbb{R}^N} \frac{f^2}{|x|^2}dx \\ &\geqslant \frac{\left(\frac{N^2}{4}-\delta\right)^{\frac{N-1}{N}}}{\left(\frac{(N-2)^2}{4}\right)^{\frac{N-1}{N}}} S_N \left(\int_{\mathbb{R}^N} |F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \\ &= \frac{\left(\frac{N^2}{4}-\delta\right)^{\frac{N-1}{N}}}{\left(\frac{(N-2)^2}{4}\right)^{\frac{N-1}{N}}} S_N \left(\int_{\mathbb{R}^N} |rG(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}. \end{aligned} \tag{2.8}$$

The desired result follows. □

Proof of Theorem 1.4. Using the change of variables $f(x)=|x|^{-\frac{N-2}{2}}g(x)$ and $F(r)=|x|^{-\frac{N-2}{2}}G(r)$ inequality (1.9) is seen to be equivalent to

$$\int_{B_1} |x|^{-(N-2)}|g_r(x)|^2 dx \geqslant (N-2)^{-\frac{2(N-1)}{N}} S_N \left(\int_{B_1} |x|^{-n} X_1^{\frac{2(N-1)}{N-2}}(a,|x|)|G(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}.$$

Set

$$C_{HS}(a)=\inf_{g\in C_0^\infty(B_1)} \frac{\int_{B_1} |x|^{-(N-2)}|g_r(x)|^2 dx}{\left(\int_{B_1} |x|^{-n} X_1^{\frac{2(N-1)}{N-2}}(a,|x|)|G(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}}.$$

Following ([1]), we change variables by

$$v(x) = y(\tau,\theta), \ \ \tau = \frac{1}{X_1(a,r)} = a - \ln r, \ \ \ \theta = \frac{x}{|x|}.$$

Then

$$C_{HS}(a) = \inf_{y(a,\theta)=0} \frac{\int_a^\infty \int_{\mathbb{S}^{N-1}} y_\tau^2 d\omega d\tau}{\left(\int_a^\infty \int_{\mathbb{S}^{N-1}} \tau^{-\frac{2(N-1)}{N-2}} |Y(\tau)|^{\frac{2N}{N-2}} d\omega d\tau\right)^{\frac{N-2}{N}}} \tag{2.9}$$

with $Y(\tau) = G(\tau)$. By theorem 1.1, for any $R > 0$, we have

$$S_N = \inf_{g \in C_0^\infty(B_R)} \frac{\int_{B_R} |g_r|^2 dx}{S_N \left(\int_{B_R} |G(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}}. \tag{2.10}$$

Changing variables in (2.10) by

$$g(x) = z(t,\theta), \quad t = |x|^{-(N-2)}, \quad \theta = \frac{x}{|x|},$$

it follows that for all $R > 0$,

$$(N-2)^{-\frac{2(N-1)}{N}} S_N = \inf_{z(R^{-(N-2)},\theta)=0} \frac{\int_{R^{-(N-2)}}^\infty \int_{\mathbb{S}^{N-1}} z_t^2 d\omega dt}{\left(\int_{R^{-(N-2)}}^\infty \int_{\mathbb{S}^{N-1}} t^{-\frac{2(N-1)}{N-2}} |Z(t)|^{\frac{2N}{N-2}} d\omega dt\right)^{\frac{N-2}{N}}} \tag{2.11}$$

with $Z(t) = G(t)$. Combining this with (2.9) we conclude our claim (1.9).

We now prove inequality (1.10). The lower bound on the best constant follows from (1.9) with a simple scaling argument. To establish the upper bound, we set, for $a > 0$ and $\rho > 0$ small enough such that $B_\rho \subset \Omega$,

$$D_{HS}(a,\rho) = \inf_{f \in C_0^\infty(B_\rho)} \frac{\int_{B_\rho} |f_r|^2 dx - \frac{(N-2)^2}{4} \int_{B_\rho} \frac{f^2}{|x|^2} dx}{\left(\int_{B_\rho} X_1^{\frac{2(N-1)}{N-2}}(a,|x|) |F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}}.$$

Through a scaling argument, we have

$$D_{HS}(a,\rho) = \inf_{f \in C_0^\infty(B_1)} \frac{\int_{B_1} |f_r|^2 dx - \frac{(N-2)^2}{4} \int_{B_1} \frac{f^2}{|x|^2} dx}{\left(\int_{B_1} X_1^{\frac{2(N-1)}{N-2}}(a - \ln\rho,|x|) |F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}} = C_{HS}(a - \ln\rho).$$

By (1.9), we obtain

$$D_{HS}(a,\rho) = (N-2)^{-\frac{2(N-1)}{N}} S_N$$

and the upper bound follows. These complete the proof of Theorem 1.4. □

Proof of Corollary 1.5. On substituting $f(x) = |x|g(x)$ and $F(r) = rG(r)$ in Theorem 1.4, we have, through integration by parts,

$$\begin{aligned}\int_\Omega |\langle x, \nabla g\rangle|^2 dx - \frac{N^2}{4}\int_\Omega g^2 dx &= \int_\Omega |f_r|^2 dx - \frac{(N-2)^2}{4}\int_\Omega \frac{f^2}{|x|^2} dx \\ &\geqslant (N-2)^{-\frac{2(N-1)}{N}} S_N \left(\int_\Omega X_1^{\frac{2(N-1)}{N-2}}(a, \frac{|x|}{D})|F(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \qquad (2.12)\\ &= (N-2)^{-\frac{2(N-1)}{N}} S_N \left(\int_\Omega X_1^{\frac{2(N-1)}{N-2}}(a, \frac{|x|}{D})|rG(r)|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}.\end{aligned}$$

The desired result follows. □

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