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STIRLING'S FORMULA REVISITED VIA SOME CLASSICAL AND NEW INEQUALITIES

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Abstract. The Hermite-Hadamard inequality is used to develop an approximation to the logarithm of the gamma function which is more accurate than the Stirling approximation and easier to derive. Then the concavity of the logarithm of gamma of logarithm is proved and applied to the Jensen inequality. Finally, the Wallis ratio is used to obtain the additional term in Stirling's approximation formula.

1. Introduction

Stirling's formula is one of the most interesting and intriguing formulas with theoretical and practical use in various applications. Its simplest and the best known version is the following:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{1}$$

The quality of the approximation can be seen from the following improvement (see [11, 22]):

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}. \tag{2}$$

These inequalities are connected with properties of the gamma function and can be deduced from the following asymptotic expansion (see [1]):

$$\log\Gamma(x) \approx \left(x-\frac{1}{2}\right)\log x - x + \log\sqrt{2\pi} + \frac{1}{12x} - \frac{1}{360x^3}. \tag{3}$$

Since

$$\lim_{x\to\infty}\left(x-\frac{1}{2}\right)\left(\log x - \log(x-\frac{1}{2})\right) = \frac{1}{2},$$

it is clear that

$$\log\Gamma(x) \approx \left(x-\frac{1}{2}\right)\log\left(x-\frac{1}{2}\right) - x + \frac{1}{2} + \log\sqrt{2\pi} \tag{4}$$

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gives another approximation of Stirling's type, see Burnside's result [9]:

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}. \tag{5}$$

It gives a better result than (1). Another term can be added to obtain an "n and a half" formula:

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} e^{-\frac{1}{24(n+\frac{1}{2})}}. \tag{6}$$

The quality of this approximation is slightly better than the formula (2).

There is a lot of literature about gamma function approximations, let us mention for example [1, 2] as the best known, and some recent results [3, 4, 5, 18, 19]. But, we cannot find formula (6) in published papers or books, although it must be known. The closest link is a text on web page [16] which refers to an unpublished paper of W. Smith [23], but there is no explicit mention of formula (6) therein.

In this paper we want to establish connection between formulas (5), (6) and some classical and new inequalities.

Using Hermite-Hadamard inequality, the following theorem will be proved.

THEOREM 1. *For every $x > 1$ the following inequalities hold true:*

$$\left(x-\frac{1}{2}\right)\left[\log\left(x-\frac{1}{2}\right)-1\right]+\log\sqrt{2\pi}-\frac{1}{24(x-1)} \leqslant \log\Gamma(x)$$
$$\leqslant \left(x-\frac{1}{2}\right)\left[\log\left(x-\frac{1}{2}\right)-1\right]+\log\sqrt{2\pi}-\frac{1}{24(\sqrt{x^2+x+\frac{1}{2}}-\frac{1}{2})}. \tag{7}$$

From this result, the following bound in the improvement of the formula (6) is easy to derive:

COROLLARY 1.

$$\sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} e^{-\frac{1}{24n}} < n! < \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} e^{-\frac{1}{24(n+1+\frac{1}{8n})}}$$

Using asymptotic expansion of Wallis ratio, we are able to derive a better approximation.

2. Hermite-Hadamard inequality and application to gamma function

There are several approximations of the logarithm of gamma function, various approaches lead to similar formulas. We are interested only in the simplest formulas and error terms are studied only to guarantee the accuracy of the formulas in question.

The use of Hermite-Hadamard inequality yields an approximation more accurate than the Stirling formula (1) and the derivation is very simple.

THEOREM 2. (Hermite, Hadamard) *Let a function* $f(x)$ *be convex on a finite* $[a,b]$. *Then*

$$(b-a)f(\frac{a+b}{2}) \leqslant \int_a^b f(x)dx \leqslant (b-a)\frac{f(a)+f(b)}{2}.$$

The proof can be found in [21].

The left hand side inequality is the only inequality we will need. Thus the next step is to show how to get error bounds for this inequality.

THEOREM 3. *Let* $f(x)$ *be convex and twice differentiable on* $[a,b]$. *Let* m *and* M *be constants for which* $m \leqslant f''(x) \leqslant M$ *for all* $x \in [a,b]$. *Then*

$$m\frac{(b-a)^3}{24} \leqslant \int_a^b f(x)dx-(b-a)f(\frac{a+b}{2}) \leqslant M\frac{(b-a)^3}{24}.$$

The proof can be found in [21].

Recall that the logarithmic derivative of the gamma function

$$\psi(x) = [\log\Gamma(x)]'$$

is called the psi or digamma function. In [10] the following lemma was proved:

LEMMA 1. *Let* $s,t>0$ *and* β_0 *be defined by*

$$\beta_0 = -\frac{1}{2}+\sqrt{st+\frac{1}{4}} \tag{8}$$

Then we have

$$\frac{1}{x+r_1} < \frac{\psi(x+t)-\psi(x+s)}{t-s} < \frac{1}{x+r_2} \tag{9}$$

where

$$r_1 := \max\left\{\frac{s+t-1}{2},\beta_0\right\}, \qquad r_2 := \min\left\{\frac{s+t-1}{2},\beta_0\right\}. \tag{10}$$

The inequality

$$\frac{s+t-1}{2} \leqslant -\frac{1}{2}+\sqrt{st+\frac{1}{4}}$$

is equivalent to $|t-s|<1$. Therefore, when t approaches s we obtain the following result which we shall use in the sequel:

COROLLARY 2. *Let* $s>0$. *Then*

$$\frac{1}{x-\frac{1}{2}+\sqrt{s^2+\frac{1}{4}}} < \psi'(x+s) < \frac{1}{x+s-\frac{1}{2}}. \tag{11}$$

The right bound does not depend on the way the argument of the digamma function ψ is divided between x and s, but the left bound does. The best bound will be obtained by taking $x = 0$.

COROLLARY 3. *Let* $x > 0$. *Then*

$$\frac{1}{\sqrt{x^2+\frac{1}{4}}-\frac{1}{2}} < \psi'(x) < \frac{1}{x-\frac{1}{2}}. \tag{12}$$

LEMMA 2. *On the interval* $[a, a+1]$ *the following bounds for the second derivative of* $\log\Gamma(x)$ *are valid:*

$$\frac{1}{\sqrt{(a+1)^2+\frac{1}{4}}-\frac{1}{2}} \leqslant \psi'(x) \leqslant \frac{1}{a-\frac{1}{2}}, \qquad a \leqslant x \leqslant a+1. \tag{13}$$

Proof. Trigama function $\psi'(x)$ is decreasing. Therefore, on $[a, a+1]$ its maximum is attained at $x = a$ and by (12) it is less than $1/(a-1/2)$. The minimum on $[a, a+1]$ of the second derivative is attained at $a+1$.

LEMMA 3. (Raabe integral) *The following formula is valid*

$$\int_a^{a+1} \log\Gamma(x)dx = a(\log a - 1) + \log\sqrt{2\pi}. \tag{14}$$

The calculation of this integral is available in Nilsen (1906), Bateman (1953), and Whittaker (1927) presents it as an exercise. The clearest exposition is in the monumetal textbook Fichtengolc [12], originally in Russion, or Fichtenholz in German [13].

Proof of Theorem 1. We apply the Hermite-Hadamard inequality to $\log\Gamma(a)$ to see that

$$a(\log a - 1) + \log\sqrt{2\pi} \geqslant \log\Gamma(a+\frac{1}{2})$$

By Theorem 3 and Lemma 3, the error of the approximation is bounded by

$$\frac{1}{24(\sqrt{(a+1)^2+\frac{1}{4}}-\frac{1}{2})}$$

from below and by

$$\frac{1}{24(a-\frac{1}{2})}$$

from above. As a result we can write

$$a(\log a - 1) + \log\sqrt{2\pi} \geqslant a(\log a - 1) + \log\sqrt{2\pi} - \frac{1}{24(\sqrt{(a+1)^2+\frac{1}{4}}-\frac{1}{2})} \geqslant \log\Gamma(a+\frac{1}{2})$$

and

$$\log\Gamma(a+\frac{1}{2}) \geqslant a(\log a-1)+\log\sqrt{2\pi}-\frac{1}{24(a-\frac{1}{2})}.$$

To get bounds for $\log\Gamma(x)$ we substitute $a = x-\frac{1}{2}$

$$\left(x-\frac{1}{2}\right)\left[\log\left(x-\frac{1}{2}\right)-1\right]+\log\sqrt{2\pi}-\frac{1}{24(\sqrt{x^2+x+\frac{1}{2}}-\frac{1}{2})}$$
$$\geqslant \log\Gamma(x) \geqslant \left(x-\frac{1}{2}\right)\left[\log\left(x-\frac{1}{2}\right)-1\right]+\log\sqrt{2\pi}-\frac{1}{24(x-1)}.$$

The theorem is proved. □

Now, comparing with (3) it is clear that the approximation

$$\log\Gamma(x) \approx (x-\frac{1}{2})(\log(x-\frac{1}{2})-1)+\log\sqrt{2\pi}$$

is more accurate than the Stirling formula. While $(x-\frac{1}{2})\log x - x + \log\sqrt{2\pi}$ approximates $\log\Gamma(x)$ from below, this approximation is from above.

Proof of Corollary 1. The right bound follows from this inequality:

$$\sqrt{(n+1)^2+n+1+\frac{1}{2}}-\frac{1}{2} < n+1+\frac{1}{8n}.$$

3. Application of Jensen inequality

In the sequel we shall show connection of Stirling's formula and another classical inequality.

THEOREM 4. (Jensen) *Let $f(x)$ be concave on $[A,B]$, and let ϕ maps the interval $[a,b]$ in $[A,B]$ continuously. Then*

$$f\left(\frac{1}{b-a}\int_a^b \phi(x)dx\right) \geqslant \frac{1}{b-a}\int_a^b f(\phi(x))dx.$$

The proof can be found in [21].

We shall aply this theorem to the function $f(x) = \log\Gamma(\log x)$.

THEOREM 5. *There exists an x_0 such that $f(x) = \log\Gamma(\log x)$ is concave for $x > x_0$.*

Proof. From the second derivative

$$f''(x) = \frac{(\Gamma''(\log x)-\Gamma'(\log x))\Gamma(\log x)-(\Gamma'(\log x))^2}{(x\Gamma(\log(x)))^2},$$

using substitution $z = \log(x)$ we have

$$f''(x) = \frac{(\Gamma''(z) - \Gamma'(z))\Gamma(z) - (\Gamma'(z))^2}{(e^z\Gamma(z))^2}.$$

Since the denominator is always positive, we are only interested in the sign of

$$(\Gamma''(z) - \Gamma'(z))\Gamma(z) - (\Gamma'(z))^2.$$

It can be written as

$$\Gamma(z)^2[\psi'(z) - \psi(z)].$$

We shall use the following inequality ([10, Corollary 2]):

$$\psi'(x) < e^{-\psi(x)}, \qquad x > 0, \tag{15}$$

to estimate upper bound of x_0. For second derivative of $f(x)$ to be negative, it is sufficient that the following holds true

$$e^{-\psi(z)} < \psi(z).$$

The approximative solution of the equation $e^{-t} = t$ is $t \approx 0.567$. Numerical solution of $\psi(z) = 0.567$ is $z = 2.240$. Hence, from $\log x_0 = 2.240$ we obtain $x_0 = 9.393$ and this is the upper bound for x_0.

Now we can apply Jensen inequality. Let $f(x) = \log\Gamma(\log x)$, $b = a+1$, $\phi(x) = \exp(x)$. Then, since $f(x)$ is concave for large x, we have

$$\log\Gamma\left(\log\int_a^{a+1} e^x dx\right) \geqslant \int_a^{a+1} \log\Gamma(\log e^x)dx = \int_a^{a+1} \log\Gamma(x)dx.$$

The last integral is the Raabe integral (Lemma 3) so we get

$$a\log a - a + \log\sqrt{2\pi} \leqslant \log\Gamma(\log(e^{a+1} - e^a)) = \log\Gamma(a + \log(e-1)).$$

By substitution $a = x - \log(e-1)$ it follows

$$\log\Gamma(x) \geqslant (x - \log(e-1))\log(x - \log(e-1)) - (x - \log(e-1)) + \log\sqrt{2\pi}. \tag{16}$$

REMARK 1. Function

$$h(x,a) = (x+a)[\log(x+a) - 1)] + \log\sqrt{2\pi}$$

is increasing in a. From (7) and (16) it holds

$$h(x, -\log(e-1)) < \log\Gamma(x) < h(x, -1/2)$$

at least for $x > x_0$. It is open question to find continued fraction expansion of a, as a function of x, such that

$$\log\Gamma(x) = h(x, a(x))$$

holds, or to find asymptotic expansion of a such that we have

$$\log\Gamma(x) \sim h(x, a(x)), \qquad \text{as } x \to \infty.$$

4. Wallis ratio

The Wallis quotient and corresponding inequalities have been investigated in various papers, see [10, 14, 15, 20] and the literature cited therein.

In a recent paper [20], Mortici studied classical Wallis ratio and obtained the following approximation:

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \approx \sqrt{n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}-\frac{5}{2048n^3}-\frac{23}{8192n^3}}.$$

We shall use his idea to obtain a similar approximation in a general case. The method of calculation is covered by the following lemma, see [17]:

LEMMA 4. *If (ω_n) is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = p \in \mathbf{R},$$

with $k > 1$, then there exists the limit:

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{p}{k-1}.$$

THEOREM 6. *It holds*

$$\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} \approx x+\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}+\frac{\delta}{x^3}+\frac{\varepsilon}{x^4}, \tag{17}$$

where

$$\alpha = \frac{1}{2}[s+t-1], \tag{18}$$

$$\beta = \frac{1}{24}[1-(t-s)^2], \tag{19}$$

$$\gamma = -\alpha\beta, \tag{20}$$

$$\delta = \frac{\beta}{10}\Big(10\alpha^2 - 13\beta - 1\Big), \tag{21}$$

$$\varepsilon = -\frac{\alpha\beta}{10}\Big(10\alpha^2 - 39\beta - 3\Big). \tag{22}$$

Proof. We shall use the method from Lemma 4. Let us denote

$$w(x) = \log\Gamma(x+t) - \log\Gamma(x+s) - (t-s)\log\left(x+\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}+\frac{\delta}{x^3}+\frac{\varepsilon}{x^4}\right).$$

Then, expanding the difference $w(x) - w(x+1)$ into an asymptotic series, we can choose the coefficients α, β, γ, δ, ε in such a way that the first five terms of the series vanish.

Let us write a few steps of the calculation. We restrict overselves to writing the first four terms and give only a result for the next one. It holds

$$\log\Gamma(x+1+t)=\log(x+t)\Gamma(x+t)=\log(x+t)+\log\Gamma(x+t).$$

Therefore, we have, keeping the expansions to the few necessary terms,

$$\begin{aligned}
w(x)-w(x+1)\approx{}&\log(x+s)-\log(x+t)\\
&-(t-s)\log\left(x+\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}+\frac{\delta}{x^3}\right)\\
&+(t-s)\log\Big[x+1+\alpha+\beta\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}\right)\\
&\qquad+\gamma\left(\frac{1}{x^2}-\frac{2}{x^3}+\frac{3}{x^4}\right)+\delta\left(\frac{1}{x^3}-\frac{3}{x^4}\right)\Big]\\
\approx{}&\log\left(1+\frac{s}{x}\right)-\log\left(1+\frac{t}{x}\right)\\
&-(t-s)\log\left(1+\frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma}{x^3}+\frac{\delta}{x^4}\right)\\
&+(t-s)\log\left(1+\frac{1+\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma-\beta}{x^3}+\frac{\beta-2\gamma+\delta}{x^4}+\frac{-\beta+3\gamma-3\delta}{x^5}\right)
\end{aligned}$$

After expanding these functions, we obtain

$$\begin{aligned}
w(x)-w(x+1)\approx{}&\frac{s}{x}-\frac{s^2}{2x^2}+\frac{s^3}{3x^3}-\frac{s^4}{4x^4}+\frac{s^5}{4x^5}\\
&-\frac{t}{x}+\frac{t^2}{2x^2}-\frac{t^3}{3x^3}+\frac{t^4}{4x^4}-\frac{t^5}{4x^5}\\
&-(t-s)\Big[\left(\frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma}{x^3}+\frac{\delta}{x^4}\right)\\
&\qquad-\frac{1}{2}\left(\frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma}{x^3}+\frac{\delta}{x^4}\right)^2\\
&\qquad+\frac{1}{3}\left(\frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma}{x^3}\right)^3-\frac{1}{4}\left(\frac{\alpha}{x}+\frac{\beta}{x^2}\right)^4+\frac{1}{5}\left(\frac{\alpha}{x}\right)^5\Big]\\
&+(t-s)\Big[\left(\frac{1+\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma-\beta}{x^3}+\frac{\beta-2\gamma+\delta}{x^4}+\frac{3\gamma-\beta-3\delta}{x^5}\right)\\
&\qquad-\frac{1}{2}\left(\frac{1+\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma-\beta}{x^3}+\frac{\beta-2\gamma+\delta}{x^4}\right)^2\\
&\qquad+\frac{1}{3}\left(\frac{1+\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma-\beta}{x^3}\right)^3-\frac{1}{4}\left(\frac{1+\alpha}{x}+\frac{\beta}{x^2}\right)^4+\frac{1}{5}\left(\frac{1+\alpha}{x}\right)^5\Big]
\end{aligned}$$

It is easy to see that the coefficient with x^{-1} is equal to zero. After some computing one can find that the next four coefficients vanish if we choose α, β, γ, δ as stated in the theorem. Adding another term, the value of the coefficient ε can be calculated.

REMARK 2. The choice $t=1$, $s=\frac{1}{2}$ gives $\alpha=\frac{1}{4}$, $\beta=\frac{1}{32}$, $\gamma=-\frac{1}{128}$, $\delta=-\frac{5}{2048}$, $\varepsilon=-\frac{23}{8192}$, as stated in the Mortici result, [20].

If $t-s=1$ or $t-s=-1$, the whole expansion collapses to the term $x+s$, which indicates that β is a factor in each of the following terms.

In the computation of the classical Wallis ratio when $t-s=\frac{1}{2}$, it seems that the best choice is $t=\frac{3}{4}$, $s=\frac{1}{4}$. In this case we have $\alpha=0$, $\beta=\frac{1}{32}$, $\gamma=0$, $\delta=-\frac{9}{2048}$, $\varepsilon=0$. Therefore one obtains the following "n and a quarter" formula

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \approx \sqrt{\left(n+\frac{1}{4}\right)+\frac{1}{32(n+\frac{1}{4})}-\frac{9}{2048(n+\frac{1}{4})^3}}.$$

Finally, we shall use (17) to obtain improved Stirling's formula. Let us denote again

$$\begin{aligned}\frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)} &\approx \log\left(x+\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}+\frac{\delta}{x^4}+\frac{\varepsilon}{x^5}\right)\\ &\approx \log x+\frac{\alpha}{x}+\frac{\beta-\alpha^2/2}{x^2}+\frac{\gamma-\alpha\beta+\alpha^3/3}{x^3}+\frac{\delta-\alpha\gamma-\beta^2/2+\alpha^2\beta-\alpha^4/4}{x^4}\\ &\quad+\frac{\varepsilon-\alpha\delta-\beta\gamma+\alpha^2\gamma+\alpha\beta^2+\alpha^5/5}{x^5}\end{aligned}$$

Since it holds

$$\lim_{t\to s}\frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)}=\psi(x+s),$$

we have

$$\begin{aligned}\log\Gamma(x+s)=\int\psi(x+s)dx &\approx x\log x-x+\alpha\log x\\ &-\frac{\beta-\alpha^2/2}{x}-\frac{\gamma-\alpha\beta+\alpha^3/3}{2x^2}-\frac{\delta-\alpha\gamma-\beta^2/2+\alpha^2\beta-\alpha^4/4}{3x^3}\\ &-\frac{\varepsilon-\alpha\delta-\beta\gamma+\alpha^2\gamma+\alpha\beta^2+\alpha^5/5}{4x^4}+C.\end{aligned}$$

We can now take $s=\frac{1}{2}$. The coefficients calculated in Theorem 6 for $t=s=\frac{1}{2}$ are equal to

$$\alpha=0,\qquad \beta=\frac{1}{24},\qquad \gamma=0,\qquad \delta=-\frac{37}{5760},\qquad \varepsilon=0.$$

Hence

$$\log\Gamma(x+\frac{1}{2})=x\log x-x+\log\sqrt{2\pi}-\frac{1}{24x}+\frac{7}{2880x^3},$$

the additive constant is chosen according to the asymptotic expansion (4).

The fact that the coefficient with x^{-4} is equal to zero implies a very good quality of this approximation. Replacing $x+\frac{1}{2}$ by x, we obtain the final formula:

$$\log\Gamma(x)=\Big(x-\frac{1}{2}\Big)\log\Big(x-\frac{1}{2}\Big)-x+\frac{1}{2}+\log\sqrt{2\pi}-\frac{1}{24(x-\frac{1}{2})}+\frac{7}{2880(x-\frac{1}{2})^3} \quad (23)$$

which improve formula (6).

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Added in proofs. In the meantime, a new method for calculating asymptotic expansion of the gamma and Wallis function have been found by Burić and Elezović [6, 7, 8].

REFERENCES

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[2] H. Bateman, *Higher Transcendental Functions*, Volume 1, McGraw-Hill Book Co.Inc., New York, 1953.

[3] N. Batir, *Very accurate approximations for the factorial function*, J. Math. Inequal., (to appear)

[4] N. Batir, *Sharp inequalities for factorial* n, *Proyecciones*, 27(2008), no.1, 97-102.

[5] F. L. Bauer, *Remark on Stirling's formulas and on approximations for the double factorial*, Mathematical Intelligencer 28, 2 (2006), 10-21.

[6] T. Burić and N. Elezović, *Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions*, J. Comput. Appl. Math (to appear)

[7] T. Burić and N. Elezović, *New asymptotic expansions of the gamma function and improvements of Stirling's type formulas*, J. Comput. Anal. Appl., (to appear)

[8] T. Burić and N. Elezović, *New asymptotic expansions of the quotient of gamma functions*, (submitted)

[9] W. Burnside, *A rapidly convergent series for* $\log N!$, Messenger Math., **46** (1917), 157–159.

[10] N. Elezović, C. Giordano, J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl., 3 (2000), 239–252.

[11] W. Feller, *Stirling's Formula. §2.9 in An Introduction to Probability Theory and Its Applications*, Volume 1, 3rd ed., New York, 1968.

[12] G. M. Fichtengolc, *Kurs diferencialnogo i integralnogo iscislenia*, Tom II, (1966), Nauka, Moscow, (6th edition, in Russian).

[13] G. M. Fichtenholz, *Differential Und Interalrechnung*, Band 2, (1964), Deutscher Verlag Der Wissenschaften, Berlin.

[14] W. Gautschi, *Inequalities for gamma and incomplete gamma function*, J. Math. Phys., 39 (1959), 77-81.

[15] D. Kershaw, *Some extentions of W. Gautschi's inequalities for the gamma function*, Math. Comp., 41 (1983), 607–611.

[16] P. Luschny, web address http://www.luschny.de/math/factorial/approx/SimpleCases.html

[17] C. Mortici, *New approximations of the gamma function in terms of the digamma function*, Appl. Math. Letters 23 (2010) 97–100.

[18] C. Mortici, *An ultimate extremely accurate formula for approximation of the factorial function*, Arch. Math. (Basel), 93 (2009), no. 1, 37–45.

[19] C. Mortici, *Best estimates of the generalized Stirling formula*, Appl. Math. Comput., (2010), doi: 10.1016/j.amc.2009.12.013.

[20] C. Mortici, *A new method for establishing and proving accurate bounds for the Wallis ratio*, Math. Inequal. Appl., (to appear)

[21] C. P. Niculescu, L. E. Persson, *Convex Functions and Their Applications, A Contemporary Approach*, 2006, Springer.

[22] H. Robbins, *A Remark of Stirling's Formula*, Amer. Math. Monthly 62, 26–29, 1955.

[23] W. D. Smith, *The Gamma function revisited*, manuscript, 2006

[24] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, 1927, Cambridge University Press.

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