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ON HILBERT'S INTEGRAL INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper it is shown that a new improvement on Hilbert's integral inequality can be established by introducing a weight function of the form $\left(\frac{1}{1+\sqrt{x}}-\frac{1}{1+x}\right)$ (with $x \geqslant 0$). As applications, some refinements on Widder's inequality and Hardy-Littlewood's inequality are given.

1. Introduction and Lemmas

Let $f(x),\ g(x) \in L^2(0,+\infty)$. It is all known that the inequality of the form

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leqslant \pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}. \tag{1.1}$$

is called Hilbert's integral inequality, where the coefficient π is the best possible.

In view of the importance of the Hilbert inequality in theory and applications (see [1]-[2]), it has been absorbing much interest of analysts. Recently, various improvements and extensions of (1.1) appear in a great deal of papers, such as Gao and Hsu enumerated more than 40 research articles in the paper [3]. In particular, Hsu and Guo introduced firstly the weight function to give an improvement of Hardy-Hilbert's inequality (see [4]). Afterward, Gao and Yang et al applied the weight function method to obtain a lot of graceful refinements of (1.1), such as [5]-[8] etc.

The aim of this paper is to apply weight function method and classical real analysis to give a new improvement of (1.1), and to simplify the corresponding result of the paper [8], and then to consider its some applications.

In order to prove our assertion, we need the following lemmas.

LEMMA 1.1. *If* $c(x)=\frac{1}{1+x}$, $x\in[0,+\infty)$, *then*

$$\int_0^\infty \frac{c(xt^2)}{1+t^2}dt=\frac{\pi}{2(1+\sqrt{x})}.$$

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LEMMA 1.2. *Let* $(x,y) \in (0,+\infty) \times (0,+\infty), F(x,y) = 1 - \frac{1}{1+x} + \frac{1}{1+y}$,

$$A_1 = \int_0^\infty \int_0^\infty \frac{f^2(x)}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}} F(x,y)dxdy \quad and\ A_2 = \int_0^\infty \int_0^\infty \frac{f^2(y)}{x+y}\left(\frac{y}{x}\right)^{\frac{1}{2}} F(x,y)dxdy \tag{1.2}$$

Then

$$A_1A_2 = \pi^2 \left\{ \Big(\int_0^\infty f^2(x)dx\Big)^2 - \Big(\int_0^\infty \omega(x) f^2(x)dx\Big)^2 \right\} \tag{1.3}$$

where the weight function $\omega(x)$ *is defined by*

$$\omega(x) = \frac{1}{1+\sqrt{x}} - \frac{1}{1+x} \tag{1.4}$$

Proof. Let $c(t) = \frac{1}{1+t}$, $t \in (0,+\infty)$. Then

$$\begin{aligned}
A_1 &= \int_0^\infty \left\{ \int_0^\infty \frac{1}{x(1+\frac{y}{x})}\left(\frac{x}{y}\right)^{\frac{1}{2}} F(x,y)dy \right\} f^2(x)dx \\
&= \int_0^\infty \left\{ \int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}} \Big(1 - c(x) + c(xu)\Big)du \right\} f^2(x)dx \\
&= \int_0^\infty \left\{ \pi + \int_0^\infty \tfrac{c(xu)}{1+u}\left(\tfrac{1}{u}\right)^{\frac{1}{2}} du - \pi\, c(x) \right\} f^2(x)dx \\
&= \int_0^\infty \left\{ \pi + \int_0^\infty \frac{2c\,(xt^2)}{1\,+\,t^2}dt - \pi\, c(x) \right\} f^2(x)dx \\
&= \pi \left\{ \int_0^\infty f^2(x)dx + \int_0^\infty \omega(x) f^2(x)dx \right\}
\end{aligned}$$

where $\omega(x)$ is a function defined by (1.4).

Similarly, we have

$$A_2 = \pi \left\{ \int_0^\infty f^2(x)dx - \int_0^\infty \omega(x) f^2(x)dx \right\}.$$

It follows that the equality (1.3) holds. □

LEMMA 1.3. *Let* $H(x,y)=\frac{f(x)f(y)}{x+y}$, $F(x,y)=1-c(x)+c(y)$, *where* $c(x)=\frac{1}{1+x}$. *Then*

$$\int_0^\infty\int_0^\infty H(x,y)dxdy = \int_0^\infty\int_0^\infty H(x,y)F(x,y)dxdy.$$

Proof. It is obvious that

$$\int_0^\infty\int_0^\infty H(x,y)F(x,y)dxdy = \int_0^\infty\int_0^\infty H(x,y)dxdy - \int_0^\infty\int_0^\infty H(x,y)c(x)dxdy + \int_0^\infty\int_0^\infty H(x,y)c(y)dxdy$$

We only need to show that $\int_0^\infty\int_0^\infty H(x,y)c(x)dxdy = \int_0^\infty\int_0^\infty H(x,y)c\Big(y\Big)dxdy$.

Let $\varphi(x)=\int_0^\infty \frac{f(t)}{x+t}dt$. Then

$$\begin{aligned}\int_0^\infty\int_0^\infty H(x,y)c(x)dxdy &= \int_0^\infty\Big(\int_0^\infty \frac{f(y)}{x+y}dy\Big)f(x)c(x)dx\\ &= \int_0^\infty\Big(\int_0^\infty \frac{f(t)}{x+t}dt\Big)f(x)c(x)dx = \int_0^\infty \varphi(x)f(x)c(x)dx\\ &= \int_0^\infty \varphi(y)f(y)c(y)dy = \int_0^\infty\Big(\int_0^\infty \frac{f(t)}{y+t}dt\Big)f(y)c(y)dy\\ &= \int_0^\infty\Big(\int_0^\infty \frac{f(x)}{y+x}dx\Big)f(y)c(y)dy = \int_0^\infty\int_0^\infty H(x,y)c(y)dxdy. \qquad \square\end{aligned}$$

2. Main Results

In this section we shall prove our assertions with the help of the above lemmas.

THEOREM 2.1. *Let* $f(x)$ *be a real function. If* $0<\int_0^\infty f^2(x)dx<+\infty$, *then*

$$\Big(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}dxdy\Big)^2 < \pi^2\left\{\Big(\int_0^\infty f^2(x)dx\Big)^2 - \Big(\int_0^\infty \omega(x)f^2(x)dx\Big)^2\right\} \tag{2.1}$$

where the weight function $\omega(x)$ *is defined by (*1.4*).*

Proof. Let $F(x,y) = 1 - \frac{1}{1+x} + \frac{1}{1+y}$. It is obvious that $F(x,y) \geqslant 0$. By using Lemma 1.3, and then we apply Hardy's technique and Schwarz's inequality to estimate the left-hand side of (2.1) as follows:

$$\Big(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{x+y}dxdy\Big)^2 = \Big(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{x+y}F(x,y)dxdy\Big)^2$$

$$= \left(\int_0^\infty\int_0^\infty \left\{\frac{f(x)}{(x\,+\,y)^{1/2}}\left(\frac{x}{y}\right)^{\frac{1}{4}}\Big(F(x,y)\Big)^{\frac{1}{2}}\right\}\left\{\frac{f(y)}{(x\,+\,y)^{1/2}}\left(\frac{y}{x}\right)^{\frac{1}{4}}\Big(F(x,y)\Big)^{\frac{1}{2}}\right\}dxdy\right)^2$$

$$\leqslant \int_0^\infty\int_0^\infty \frac{f^2(x)}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}}F(x,y)dxdy\int_0^\infty\int_0^\infty \frac{f^2(y)}{x+y}\left(\frac{y}{x}\right)^{\frac{1}{2}}F(x,y)dxdy. \tag{2.2}$$

Since $f(x) \neq 0$, it is impossible to take equality in (2.2).

Based on (1.2) and (1.3) we obtain

$$\Big(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{x+y}dxdy\Big)^2 < A_1A_2.$$

By Lemma 1.2, the inequality (2.1) is valid. □

THEOREM 2.2. *Let* $f(x)$ *and* $g(x)$ *be two real functions. If* $0 < \int_0^\infty f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty g^2(x)dx < +\infty$*, then*

$$\Big(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy\Big)^4 < \pi^4\left\{\Big(\int_0^\infty f^2(x)dx\Big)^2 - \Big(\int_0^\infty \omega(x)f^2(x)dx\Big)^2\right\}$$
$$\times\left\{\Big(\int_0^\infty g^2(x)dx\Big)^2 - \Big(\int_0^\infty \omega(x)g^2(x)dx\Big)^2\right\} \tag{2.3}$$

where the weight function $\omega(x)$ *is defined by (*1.4*).*

Proof. By Schwarz's inequality we have

$$\begin{aligned}\Big(\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy\Big)^4 &= \left\{\Big(\int_0^1\Big(\int_0^\infty t^{x-\frac{1}{2}}f(x)dx\int_0^\infty t^{y-\frac{1}{2}}g(y)dy\Big)dt\Big)^2\right\}^2\\ &\leqslant \left\{\int_0^1\Big(\int_0^\infty t^{x-\frac{1}{2}}f(x)dx\Big)^2dt\right\}^2\left\{\int_0^1\Big(\int_0^\infty t^{y-\frac{1}{2}}g(y)dy\Big)^2dt\right\}^2 \qquad (2.4)\\ &= \left\{\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{x+y}dxdy\right\}^2\left\{\int_0^\infty\int_0^\infty \frac{g(x)\,g(y)}{x+y}dxdy\right\}^2. \qquad (2.5)\end{aligned}$$

Since $f(x)g(x)\neq 0$, it is impossible to take equality in (2.4). It follows from (2.1) and (2.5) that the inequality (2.3) is valid. Theorem is proved. □

REMARK. The weight function $\omega(x)$ defined by (1.4) can be properly chosen in accordance with our requirement. If we select the function $c(t)=\sin^2\sqrt{t}\ \ (t\geqslant 0)$, then $F(x,y)=1-\sin^2\sqrt{x}+\sin^2\sqrt{y}$. It is easy to deduce that

$$\begin{aligned}\omega(x) &= \frac{2}{\pi}\int_0^\infty \frac{c(x\,t^2)}{1+t^2}dt - c(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin^2\sqrt{x}\,t}{1+t^2}dt - \sin^2\sqrt{x}\\ &= \frac{1}{2}\Big(\cos 2\sqrt{x} - e^{-2\sqrt{x}}\Big).\end{aligned}$$

Here we use the following integral (see [9], [10]):

$$\int_0^\infty \frac{(\sin\sqrt{x}\,t)^2}{1+t^2}dt = \frac{\pi}{4}\Big(1-e^{-2\sqrt{x}}\Big).$$

3. Applications

In this section we will give some refinements of Widder's inequality and Hardy-Littlewood's inequality with the help of Theorem 2.1.

Let $a_n\geqslant 0(n=0,1,2,\cdots)$, $A(x)=\sum\limits_{n=0}^\infty a_nx^n$, $A^*(x)=\sum\limits_{n=0}^\infty \frac{a_nx^n}{n!}$. If $A(x)\neq 0$, then

$$\int_0^1 A^2(x)dx < \pi\int_0^\infty \Big(e^{-x}A^*(x)\Big)^2dx \qquad (3.1)$$

This is Widder's inequality (see [11]).

We shall give a refinement of (3.1), below.

THEOREM 3.1. *With the assumptions as the above-mentioned, then*

$$\Big(\int_0^1 A^2(x)dx\Big)^2 < \pi^2\left\{\Big(\int_0^\infty \Big(e^{-x}A^*(x)\Big)^2 dx\Big)^2 - \Big(\int_0^\infty \omega(x)\Big(e^{-x}A^*(x)\Big)^2 dx\Big)^2\right\} \tag{3.2}$$

where $\omega(x)$ *is defined by (*1.4*).*

Proof. At first we have the following relation:

$$\int_0^\infty e^{-t}A^*(tx)dt = \int_0^\infty e^{-t}\sum_{n=0}^\infty \frac{a_n(xt)^n}{n!}dt = \sum_{n=0}^\infty \frac{a_n x^n}{n!}\int_0^\infty t^n e^{-t}dt = \sum_{n=0}^\infty a_n x^n = A(x).$$

Let $tx = s$. Then we have

$$\int_0^1 A^2(x)dx = \int_0^1 \left\{\int_0^\infty e^{-t}A^*(tx)dt\right\}^2 dx = \int_0^1 \Big(\int_0^\infty e^{-\frac{s}{x}}A^*(s)ds\Big)^2 \frac{1}{x^2}dx$$
$$= \int_1^\infty \Big(\int_0^\infty e^{-sy}A^*(s)ds\Big)^2 dy.$$

Let $u = y-1$. Then

$$\int_0^1 A^2(x)dx = \int_0^\infty \Big(\int_0^\infty e^{-su-s}A^*(s)ds\Big)^2 du$$
$$= \int_0^\infty \Big(\int_0^\infty e^{-su}f(s)ds\Big)^2 du$$
$$= \int_0^\infty\int_0^\infty \frac{f(s)f(t)}{s+t}dsdt \tag{3.3}$$

where $f(x) = e^{-x}A^*(x)$.

By Theorem 2.1, the inequality (3.2) follows from (3.3) at once.

Let $f(x) \in L^2\Big(0,\ 1\Big)$ and $f(x) \neq 0$,

$$a_n = \int_0^1 x^n f(x)dx, \quad n = 0,\ 1,\ 2,\cdots$$

Hardy-Littlewood proved the following inequality (see [1]) of the form

$$\sum_{n=0}^\infty a_n^2 < \pi\int_0^1 f^2(x)dx \tag{3.4}$$

where π is the best constant that keeps (3.4) valid. In our previous paper [6], the inequality (3.4) was extended and established the following inequality:

$$\int_0^\infty f^2(x)dx < \pi \int_0^1 h^2(x)dx \tag{3.5}$$

where $f(x) = \int_0^1 t^x h(x)dx,\ x \in [0,\ +\infty)$

The inequality (3.5) is called the Hardy-Littlewood integral inequality.

Afterwards the inequality (3.5) was refined into the following form (see [7]):

$$\int_0^\infty f^2(x)dx \leqslant \pi \int_0^1 t\, h^2(t)dt. \tag{3.6}$$

We will further refine the inequality (3.6). □

THEOREM 3.2. *Let $h(t) \in L^2(0,\ 1),\ h(t) \neq 0$. Define a function by*

$$f(x) = \int_0^1 t^x |h(t)|\, dt (x \geqslant 0)$$

If $0 < \int_0^{+\infty} f^2(x)dx < +\infty$, then

$$\Big(\int_0^\infty f^2(x)dx\Big)^4 < \pi^2 \left\{ \Big(\int_0^\infty f^2(x)dx\Big)^2 - \Big(\int_0^\infty \omega(x) f^2(x)dx\Big)^2 \right\} \Big(\int_0^1 t\, h^2(t)dt\Big)^2. \tag{3.7}$$

where the weight function $\omega(x)$ is defined by (1.4).

Proof. Let us write $f^2(x)$ in form:

$$f^2(x) = \int_0^1 f(x) t^x |h(t)|\, dt.$$

Applying in turn Schwarz's inequality and Theorem 2.1, we have

$$\begin{aligned}\Big(\int_0^{+\infty} f^2(x)dx\Big)^2 &= \left\{ \int_0^\infty \Big(\int_0^1 f(x)t^x |h(t)|\, dt\Big) dx \right\}^2 \\ &= \left\{ \int_0^1 \Big(\int_0^{+\infty} f(x)t^{x-1/2}dx\Big) t^{1/2} |h(t)|\, dt \right\}^2\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^1 \Big(\int_0^{+\infty} f(x)t^{x-1/2}dx\Big)^2 dt \int_0^1 t\, h^2(t)dt \\
&= \int_0^1 \Big(\int_0^{+\infty} f\big(x\big)t^{x-1/2}dx\Big)\Big(\int_0^{+\infty} f(y)t^{y-1/2}dy\Big)dt \int_0^1 t\, h^2(t)dt \\
&= \int_0^1 \Big(\int_0^{+\infty}\int_0^{+\infty} f(x)f(y)t^{x+y-1}dxdy\Big)dt \int_0^1 t\, h^2(t)dt \\
&= \Big(\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y}dxdy\Big)\int_0^1 t\, h^2(t)dt \\
&\leqslant \pi \left\{ \Big(\int_0^{\infty} f^2(x)dx\Big)^2 - \Big(\int_0^{\infty} \omega(x) f^2(x)dx\Big)^2 \right\}^{\frac{1}{2}} \int_0^1 t\, h^2(t)dt. \qquad (3.8)
\end{aligned}
$$

where the weight function $\omega(x)$ is defined by (1.4).

Since $h(t) \neq 0,\ f^2(x) \neq 0$. It is impossible to take equality in (3.8). It follows that the inequality (3.7) is valid. □

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