Mathematical Inequalities & Applications
Volume 14, Number 2 (2011), 295–312

BRÉZIS–GALLOUËT–WAINGER INEQUALITY WITH A DOUBLE LOGARITHMIC TERM ON A BOUNDED DOMAIN AND ITS SHARP CONSTANTS

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(*Communicated by J. Pečarić*)

Abstract. The Brézis-Gallouët-Wainger inequality gives an estimate of the L^∞-norm by the critical Sobolev norm with the aid of the logarithmic dependency of a higher order Sobolev norm. We investigate the Brézis-Gallouët-Wainger inequality on a bounded domain with the first order critical Sobolev space, and give the best constant in the inequality in some special cases. Furthermore, since the inequality does not hold with the sharp constant, we add a double logarithmic term and give the sharp constant for its coefficient. A part of our results is mainly based on an investigation of the inequality with the higher-order Sobolev norm replaced by the Hölder seminorm.

1. Introduction and main results

In this paper, we consider a Brézis-Gallouët-Wainger inequality with a double logarithmic term. First we recall the Sobolev embedding theorem in the critical case. For $1 < p < \infty$, it is well-known that the embedding $W^{n/p,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ holds for any $p \leqslant q < \infty$, and does not hold for $q = \infty$, that is, one cannot estimate the L^∞-norm by the $W^{n/p,p}$-norm. However, the Brézis-Gallouët-Wainger inequality states that the L^∞-norm can be estimated by the $W^{n/p,p}$-norm with the partial aid of the $W^{s,r}$-norm with $s > n/r$ and $1 \leqslant r \leqslant \infty$. Precisely,

$$\|u\|_{L^\infty(\mathbb{R}^n)}^{p/(p-1)} \leqslant C(1 + \log(1 + \|u\|_{W^{s,r}(\mathbb{R}^n)})) \tag{1.1}$$

holds for all $u \in W^{n/p,p}(\mathbb{R}^n) \cap W^{s,r}(\mathbb{R}^n)$ under the normalization $\|u\|_{W^{n/p,p}(\mathbb{R}^n)} = 1$. Note that the embedding $W^{s,r}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ holds for s and r specified as above. Originally, Brézis-Gallouët [2] proved (1.1) for the case $n = p = r = s = 2$. Later on, Brézis-Wainger [3] obtained (1.1) for the general case, and remarked that the power $p/(p-1)$ in (1.1) is optimal in the sense that one cannot replace it by any larger power. However, it seems that little is known about the sharp constant in (1.1).

We make the following replacements in the inequality (1.1). First we replace the domain $\mathbb{R}^n$ by Ω, which is a bounded domain in $\mathbb{R}^n$, and we consider $W_0^{n/p,p}(\Omega)$

Mathematics subject classification (2010): 46E35.
Keywords and phrases: Sobolev embedding theorem, Brézis-Gallouët-Wainger inequality, sharp constants, double logarithmic term.

Paper MIA-14-24

instead of $W^{n/p,p}(\mathbb{R}^n)$. Then (1.1) holds for all $u \in W_0^{n/p,p}(\Omega) \cap W^{s,r}(\Omega)$ under the normalization $\|u\|_{W^{n/p,p}(\Omega)} = 1$. We restrict our attention to the case $p = n \geqslant 2$, and investigate the sharp constant in this inequality under the normalization $\|\nabla u\|_{L^n(\Omega)} = 1$ instead of $\|u\|_{W^{1,n}(\Omega)} = 1$, using an equivalent norm. We assume that $s = m$ is a positive integer due to a technical reason, and take $r = n/(m-\alpha)$ with $0 < \alpha < 1$. Then we formulate the problem as follows.

PROBLEM A. *For a given constant $L_1 > 0$, does there exist a constant C such that*

$$\|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leqslant L_1 \log(1 + \|u\|_{W^{m,n/(m-\alpha)}(\Omega)}) + C \tag{1.2}$$

holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$?

In the introduction, let $m \in \{1,2,\dots,n\}$ and $0 < \alpha < 1$. We can show that $L_1 = \Lambda_1/\alpha$ is the sharp constant in (1.2). Here, we define

$$\Lambda_1 = \frac{1}{\omega_{n-1}^{1/(n-1)}}$$

and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$. We can show the following, which solves Problem A@. See Definition 2.2 below for the definition of the *strong local Lipschitz condition* for a domain Ω.

THEOREM 1.1. *Let $n \geqslant 2$, $m \in \{1,2,\dots,n\}$, $0 < \alpha < 1$ and Ω be a bounded domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition.*

(i) *If*

$$L_1 > \frac{\Lambda_1}{\alpha},$$

then there exists a constant C such that the inequality (1.2) *holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$.*

(ii) *If*

$$L_1 \leqslant \frac{\Lambda_1}{\alpha},$$

then for any constant C, the inequality (1.2) *does not hold for some $u \in W_0^{1,n}(\Omega) \cap W_0^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$.*

Furthermore, we formulate the problem more precisely as follows.

PROBLEM B. *For given constants $L_1 > 0$ and $L_2 \in \mathbb{R}$, does there exist a constant C such that*

$$\begin{aligned}\|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leqslant{}& L_1 \log(1 + \|u\|_{W^{m,n/(m-\alpha)}(\Omega)}) \\ &+ L_2 \log(1 + \log(1 + \|u\|_{W^{m,n/(m-\alpha)}(\Omega)})) + C\end{aligned} \tag{1.3}$$

holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ *with* $\|\nabla u\|_{L^n(\Omega)} = 1$*?*

The double logarithmic term in the right hand side of (1.3) is essentially meaningful only if $L_1 = \Lambda_1/\alpha$ because of Theorem 1.1. Then we can also show that $L_2 = \Lambda_2/\alpha$ is the sharp constant in (1.3) in the critical case $L_1 = \Lambda_1/\alpha$, and Problem B can be solved completely. Here, we define

$$\Lambda_2 = \frac{\Lambda_1}{n} = \frac{1}{n\omega_{n-1}^{1/(n-1)}}.$$

THEOREM 1.2. *Let* $n \geqslant 2$, $m \in \{1, 2, \ldots, n\}$, $0 < \alpha < 1$ *and* Ω *be a bounded domain in* $\mathbb{R}^n$ *satisfying the strong local Lipschitz condition.*

(i) *If*

$$\text{(I) } L_1 > \frac{\Lambda_1}{\alpha} \text{ and } L_2 \in \mathbb{R} \quad \text{or} \quad \text{(II) } L_1 = \frac{\Lambda_1}{\alpha} \text{ and } L_2 \geqslant \frac{\Lambda_2}{\alpha},$$

then there exists a constant C *such that the inequality* (1.3) *holds for all* $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ *with* $\|\nabla u\|_{L^n(\Omega)} = 1$.

(ii) *If*

$$\text{(III) } L_1 < \frac{\Lambda_1}{\alpha} \text{ and } L_2 \in \mathbb{R} \quad \text{or} \quad \text{(IV) } L_1 = \frac{\Lambda_1}{\alpha} \text{ and } L_2 < \frac{\Lambda_2}{\alpha},$$

then for any constant C*, the inequality* (1.3) *does not hold for some* $u \in W_0^{1,n}(\Omega) \cap W_0^{m,n/(m-\alpha)}(\Omega)$ *with* $\|\nabla u\|_{L^n(\Omega)} = 1$.

Now we give some remarks on our results.

REMARK 1.3. The power $n/(n-1)$ on the left hand side of (1.2) is optimal in the sense that $q = n/(n-1)$ is the largest power for which

$$\|u\|_{L^\infty(\Omega)}^q \leqslant L_1 \log(1 + \|u\|_{W^{m,n/(m-\alpha)}(\Omega)}) + C \tag{1.4}$$

can hold for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$. Indeed, if $q > n/(n-1)$, then for any $L_1 > 0$ and any constant C, (1.4) does not hold for some $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$, which is shown by carrying out a similar calculation to the proof of Theorem 1.2 (ii); we omit the details. On the contrary, if $1 \leqslant q < n/(n-1)$, then for any $L_1 > 0$, there exists a constant C such that (1.4) holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$. This fact follows from the embedding $W^{m,n/(m-\alpha)}(\Omega) \hookrightarrow \dot{C}^\alpha(\bar{\Omega})$ and the same assertion concerning the Brézis-Gallouët-Wainger type inequality in the Hölder space, which is shown in [5, Remark 3.5].

REMARK 1.4. Let us consider the best constant C for the inequality (1.2). For fixed L_1 such that (1.2) holds, i.e., for $L_1 > \Lambda_1/\alpha$, we introduce the notion of the best constant as follows. We call

$$C(L_1) = \sup\{F[u; L_1]; u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega), \|\nabla u\|_{L^n(\Omega)} = 1\}$$

the *best constant* for (1.2), where $F[u;L_1]$ is defined by

$$F[u;L_1] = \|u\|_{L^\infty(\Omega)}^{n/(n-1)} - L_1 \log(1 + \|u\|_{W^{m,n/(m-\alpha)}(\Omega)}).$$

In the case that Ω is an open ball $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$, we can show that

$$C(L_1) \to \infty \text{ as } L_1 \searrow \Lambda_1/\alpha,\ C(L_1) \to -\infty \text{ as } L_1 \to \infty.$$

However, we know little about their limiting behaviors.

REMARK 1.5. It is essentially meaningless to consider an inequality with any weaker term. More precisely, we can prove the following facts. In each part, the former fact follows from the embedding and the same assertion in the Hölder space, which is shown in [5, Remark 3.6], and the latter fact is shown by carrying out a similar calculation to the proof of Theorem 1.2 (ii).

(i) We choose a continuous function $\Phi\colon [0,\infty) \to [0,\infty)$ such that

$$\Phi(s) \to \infty,\ \frac{\Phi(s)}{\log(1+\log(1+s))} \to 0 \text{ as } s \to \infty,$$

and consider the inequality

$$\begin{aligned} \|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leqslant L_1 &\log(1 + \|u\|_{W^{m,n/(m-\alpha)}(\Omega)}) \\ &+ L_2 \log(1 + \log(1 + \|u\|_{W^{m,n/(m-\alpha)}(\Omega)})) \\ &+ L\Phi(\|u\|_{W^{m,n/(m-\alpha)}(\Omega)}) + C \end{aligned}$$

for $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$. For the completeness of the argument, we assume in addition that

$$\max\{\Phi(st), \Phi(s+t)\} \leqslant \Phi(s) + \Phi(t) + c \text{ for } s,t \geqslant 0$$

with some constant $c \geqslant 0$, the functions

$$(0,\infty) \ni s \mapsto \frac{\Phi(s)}{s^{n/(n-1)}} \in (0,\infty) \text{ and } (0,\infty) \ni s \mapsto \frac{\Phi \circ \Phi(s)}{s^{n/(n-1)}} \in (0,\infty)$$

are both decreasing. Then this inequality holds if and only if one of the following holds:

(I) $L_1 > \Lambda_1/\alpha$ and $L_2, L \in \mathbb{R}$;

(II-1) $L_1 = \Lambda_1/\alpha$, $L_2 > \Lambda_2/\alpha$ and $L \in \mathbb{R}$;

(II-2) $L_1 = \Lambda_1/\alpha$, $L_2 = \Lambda_2/\alpha$ and $L \geqslant 0$.

(ii) Let $N \geqslant 3$ and consider the N-ple logarithmic inequality

$$\|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leqslant \sum_{j=1}^{N} L_j \underbrace{\log(1+\log(1+\cdots+\log(1+}_{j \text{ times}}\|u\|_{W^{m,n/(m-\alpha)}(\Omega)})\cdots)) + C$$

for $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$. Then this inequality holds if and only if one of the following holds:

(I) $L_1 > \Lambda_1/\alpha$ and $L_2,\dots,L_N \in \mathbb{R}$;

(II-1) $L_1 = \Lambda_1/\alpha$, $L_2 > \Lambda_2/\alpha$ and $L_3,\dots,L_N \in \mathbb{R}$;

(II-2′) $L_1 = \Lambda_1/\alpha$, $L_2 = \Lambda_2/\alpha$, $L_3 = \cdots = L_{m-1} = 0$, $L_m > 0$ for some $3 \leqslant m \leqslant N$ and $L_{m+1},\dots,L_N \in \mathbb{R}$;

(II-2″) $L_1 = \Lambda_1/\alpha$, $L_2 = \Lambda_2/\alpha$ and $L_3 = \cdots = L_N = 0$.

Since Theorem 1.1 is completely contained in Theorem 1.2, we may prove only the latter. The proof of Theorem 1.2 (i) is mainly based on [5, Theorem 1.2], which gives the sharp constants in the Brézis-Gallouët-Wainger type inequality with the Hölder seminorm instead of the higher-order Sobolev norm, combining with the embedding theorem [1, Theorem 4.12]. We have to assume that Ω is bounded because of the assumption of [5, Theorem 1.2]. In order to prove Theorem 1.2 (ii), we concretely construct a sequence of functions $\{u_j\}_{j=1}^{\infty} \subset W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u_j\|_{L^n(\Omega)} = 1$ so that

$$\|u_j\|_{L^\infty(\Omega)}^{n/(n-1)} - L_1 \log(1 + \|u_j\|_{W^{m,n/(m-\alpha)}(\Omega)}) \\ - L_2 \log(1 + \log(1 + \|u_j\|_{W^{m,n/(m-\alpha)}(\Omega)})) \to \infty \text{ as } j \to \infty.$$

The organization of this paper is as follows. In Section 2, we introduce some notation of function spaces and state an embedding theorem under the strong local Lipschitz condition. In Section 3, we shall give the proof of the main theorem by using [5, Theorem 1.2], [1, Theorem 4.12] and a key lemma, which gives a sequence $\{u_j\}_{j=1}^{\infty}$ as above for given constants L_1 and L_2 under the assumption (III) or (IV). The key lemma will be proved in Section 4.

2. Preliminaries

First we introduce some function spaces. Throughout this paper, let the dimension $n \geqslant 2$, and Ω be a bounded domain in $\mathbb{R}^n$. We denote by B_R the open ball in $\mathbb{R}^n$ centered at the origin with the radius $R > 0$, i.e., $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$.

We describe a standard notation of multi-indices for the sake of completeness. Let $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. For a multi-index $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{Z}_+^n$, we define

$$x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}, \quad \left(\frac{\partial}{\partial x}\right)^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_n^{\nu_n}},$$

where $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$. For multi-indices $\nu, \mu \in \mathbb{Z}_+^n$, we write $\mu \leqslant \nu$ if $\mu_k \leqslant \nu_k$ for all $k \in \{1, 2, \dots, n\}$. In what follows, we denote

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \ \|u\|_{m,p} = \|u\|_{W^{m,p}(\Omega)} = \sum_{j=0}^{m} \|\nabla^j u\|_p, \ \|\nabla^j u\|_p = \|\,|\nabla^j u|\,\|_p$$

for $1 \leqslant p \leqslant \infty$ and $m \in \mathbb{N}$, for simplicity. Here, for $j \in \mathbb{N}$, we define the j-th order derivative of the function u as

$$\nabla^j u = \left(\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_j}} u\right)_{1 \leqslant i_1, i_2, \dots, i_j \leqslant n},$$

$$|\nabla^j u| = \left(\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_j=1}^{n} \left| \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_j}} u \right|^2 \right)^{1/2}.$$

We note that the norm of $W_0^{1,p}(\Omega)$ is equivalent to $\|\nabla u\|_p$ if Ω is bounded and $1 \leqslant p < \infty$, because of the Poincaré inequality.

First we note that the inequality (1.3) holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_n = 1$ if and only if there exists a constant C such that $F^{m,\alpha}[u; L_1, L_2] \leqslant C$ holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega) \setminus \{0\}$, where

$$\begin{aligned} F^{m,\alpha}[u; L_1, L_2] = \left(\frac{\|u\|_\infty}{\|\nabla u\|_n} \right)^{n/(n-1)} &- L_1 \log \left(1 + \frac{\|u\|_{m,n/(m-\alpha)}}{\|\nabla u\|_n} \right) \\ &- L_2 \log \left(1 + \log \left(1 + \frac{\|u\|_{m,n/(m-\alpha)}}{\|\nabla u\|_n} \right) \right) \\ &\text{for } u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega) \setminus \{0\}. \end{aligned}$$

The following proposition shows that Theorem 1.2 remains valid without changing the assumptions concerning L_1 and L_2 if we replace the definition of the Sobolev norm defined as above with any equivalent norm.

PROPOSITION 2.1. *Let $m \in \mathbb{N}$, $0 < \alpha \leqslant m$ and Ω be a bounded domain in $\mathbb{R}^n$. Assume that $\|\cdot\|$ is a norm on $W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ which is equivalent to $\|\cdot\|_{m,n/(m-\alpha)}$. Then for given constants $L_1 > 0$ and $L_2 \in \mathbb{R}$, the inequality* (1.3) *with some constant C holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_n = 1$ if and only if the inequality*

$$\|u\|_\infty^{n/(n-1)} \leqslant L_1 \log(1 + \|u\|) + L_2 \log(1 + \log(1 + \|u\|)) + \tilde{C} \tag{2.1}$$

with some constant $\tilde{C}$ holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_n = 1$.

Proof. We have only to show that the inequality (1.3) implies (2.1). Note that there exists a constant $c \geqslant 1$ such that

$$\frac{1}{c}\|u\| \leqslant \|u\|_{m,n/(m-\alpha)} \leqslant c\|u\| \quad \text{for } u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega).$$

Since

$$\max\{\log(1+st), \log(1+s+t)\} \leqslant \log(1+s) + \log(1+t) \quad \text{for } s, t \geqslant 0, \tag{2.2}$$

we have that

$$\begin{aligned} \log(1 + \|u\|_{m,n/(m-\alpha)}) &\leqslant \log(1 + c\|u\|) \leqslant \log(1+c) + \log(1 + \|u\|) \\ \log(1 + \log(1 + \|u\|_{m,n/(m-\alpha)})) &\leqslant \log(1 + \log(1+c) + \log(1 + \|u\|)), \\ &\leqslant \log(1 + \log(1+c)) + \log(1 + \log(1 + \|u\|)) \end{aligned}$$

and

$$\begin{aligned}\log(1+\log(1+\|u\|)) &\leqslant \log(1+\log(1+c\|u\|_{m,n/(m-\alpha)})) \\ &\leqslant \log(1+\log(1+c)) + \log(1+\log(1+\|u\|_{m,n/(m-\alpha)}))\end{aligned}$$

hold for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$. By using these inequalities, we can easily show that the inequality (1.3) implies (2.1) with $\tilde{C} = C + L_1 \log(1+c) + |L_2| \log(1+\log(1+c))$. □

For $0 < \alpha \leqslant 1$, $\dot{C}^\alpha(\Omega)$ denotes the subspace of the homogeneous Hölder space of order α endowed with the seminorm

$$\|u\|_{(\alpha)} = \|u\|_{\dot{C}^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

and $C^{0,\alpha}(\bar{\Omega})$ denotes the Hölder space of order α endowed with the norm

$$\|u\|_{(0,\alpha)} = \|u\|_{C^{0,\alpha}(\bar{\Omega})} = \|u\|_\infty + \|u\|_{(\alpha)}.$$

As is mentioned in the introduction, the embedding $W^{m,n/(m-\alpha)}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ holds under our assumption, which can be found in [1, Theorem 4.12].

DEFINITION 2.2. We say that a bounded domain Ω satisfies the *strong local Lipschitz condition* if Ω has a locally Lipschitz boundary, that is, each point x on the boundary of Ω has a neighborhood U_x whose intersection with the boundary of Ω is the graph of a Lipschitz continuous function.

The definition for a general domain is more complicated; see [1] for details.

LEMMA 2.3. ([1, Theorem 4.12]) *Let $n \geqslant 1$ and Ω be a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition.*

(i) *If $m \in \{1,2,\ldots,n\}$ and $0 < \alpha < 1$, then the embedding $W^{m,n/(m-\alpha)}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ holds, that is, there exists a constant C_α such that*

$$\|u\|_\infty + \|u\|_{(\alpha)} \leqslant C_\alpha \|u\|_{m,n/(m-\alpha)} \text{ for } u \in W^{m,n/(m-\alpha)}(\Omega).$$

(ii) *The embedding $W^{n+1,1}(\Omega) \hookrightarrow C^{0,1}(\bar{\Omega})$ holds, that is, there exists a constant C_1 such that*

$$\|u\|_\infty + \|u\|_{(1)} \leqslant C_1 \|u\|_{n+1,1} \text{ for } u \in W^{n+1,1}(\Omega).$$

3. Proof of the main results

In this section, we shall give the proof of Theorem 1.2. The key point is to consider the sharp constants in a slightly modified inequality

$$\|u\|_\infty^{n/(n-1)} \leqslant L_1 \log(1+\|u\|_{(\alpha)}) + L_2 \log(1+\log(1+\|u\|_{(\alpha)})) + C \tag{3.1}$$

for $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$. The following result in our previous paper is essential for the proof of Theorem 1.2.

LEMMA 3.1. ([5, Theorems 1.1 and 1.2]) *Let $n \geqslant 2$, $0 < \alpha \leqslant 1$ and Ω be a bounded domain in $\mathbb{R}^n$.*

(i) *If*

$$\text{(I) } L_1 > \frac{\Lambda_1}{\alpha} \text{ and } L_2 \in \mathbb{R} \quad \text{or} \quad \text{(II) } L_1 = \frac{\Lambda_1}{\alpha} \text{ and } L_2 \geqslant \frac{\Lambda_2}{\alpha},$$

then there exists a constant C such that the inequality (3.1) *holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$.*

(ii) *If*

$$\text{(III) } L_1 < \frac{\Lambda_1}{\alpha} \text{ and } L_2 \in \mathbb{R} \quad \text{or} \quad \text{(IV) } L_1 = \frac{\Lambda_1}{\alpha} \text{ and } L_2 < \frac{\Lambda_2}{\alpha},$$

then for any constant C, the inequality (3.1) *does not hold for some $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$.*

REMARK 3.2. For the proof of Lemma 3.1, in the case $\Omega = B_1$, it is essential to investigate the behavior of $F^\alpha[u_\tau^\alpha; L_1, L_2]$ as $\tau \searrow 0$, where

$$F^\alpha[u; L_1, L_2] = \left(\frac{\|u\|_\infty}{\|\nabla u\|_n} \right)^{n/(n-1)} - L_1 \log\left(1 + \frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) - L_2 \log\left(1 + \log\left(1 + \frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right)\right),$$

and for $0 < \tau \leqslant 1$, the function u_τ^α is defined by

$$u_\tau^\alpha(x) = \begin{cases} 1 - \dfrac{1}{\alpha \log(1/\tau) + 1} \left(\dfrac{|x|}{\tau} \right)^\alpha & \text{for } x \in \bar{B}_\tau, \\ \dfrac{\alpha}{\alpha \log(1/\tau) + 1} \log \dfrac{1}{|x|} & \text{for } x \in B_1 \setminus B_\tau. \end{cases} \tag{3.2}$$

The assertion (ii) follows from the behavior of $F^\alpha[u_\tau^\alpha; L_1, L_2]$. The assertion (i) can be proved by the fact that the minimizer of

$$m_\tau^\alpha = \inf\Big\{ \|\nabla u\|_n^n;\ u \in W_0^{1,n}(B_1), \quad u(x) \geqslant 1 - \frac{1}{\alpha \log(1/\tau) + 1} \left(\frac{|x|}{\tau} \right)^\alpha \text{ for a.e. } x \in B_1 \Big\}$$

is given by u_τ^α.

REMARK 3.3. In the case $n = 2$, a part of Lemma 3.1 is originally proved by Ibrahim-Majdoub-Masmoudi [4, Theorems 1.3 and 1.4]. However, they did not mention the assertion that (IV) implies the failure of the inequality (3.1).

Now we can prove the following theorem, which implies Theorem 1.2 (i).

THEOREM 3.4. *Let $n \geqslant 2$ and Ω be a bounded domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition. Assume $m \in \{1,2,\dots,n\}$ and $0<\alpha<1$, or $m=n+1$ and $\alpha=1$. If*

$$\text{(I) } L_1 > \frac{\Lambda_1}{\alpha} \text{ and } L_2 \in \mathbb{R} \quad \text{or} \quad \text{(II) } L_1 = \frac{\Lambda_1}{\alpha} \text{ and } L_2 \geqslant \frac{\Lambda_2}{\alpha},$$

then there exists a constant C such that the inequality (1.3) *holds for all $u \in W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_n = 1$.*

Proof. The assertion immediately follows from Lemma 3.1 (i) and Lemma 2.3 by using a similar argument to the proof of Proposition 2.1. □

As we mentioned above, Lemma 3.1 (i) and Lemma 2.3 immediately yield Theorem 1.2 (i). Hence, it is a natural question whether the sharp constants for the inequality (1.3) are strictly smaller than those for (3.1). However, we can show that these sharp constants coincide with those in Lemma 3.1.

REMARK 3.5. The embeddings in Lemma 2.3 are also valid for any bounded domain Ω without satisfying the strong local Lipschitz condition if we replace the spaces $W^{m,n/(m-\alpha)}(\Omega)$ and $W^{n+1,1}(\Omega)$ with $W_0^{m,n/(m-\alpha)}(\Omega)$ and $W_0^{n+1,1}(\Omega)$, respectively. This fact can be also found in [1, Theorem 4.12]. Therefore, Theorem 3.4 is also valid for an arbitrary bounded domain Ω if we make the same replacement.

To complete the proof of Theorem 1.2, we shall prove the following theorem, which does not require the strong local Lipschitz condition.

THEOREM 3.6. *Let $n \geqslant 2$, $m \in \{1,2,\dots,n\}$, $0<\alpha<1$ and Ω be a bounded domain in $\mathbb{R}^n$. If*

$$\text{(III) } L_1 < \frac{\Lambda_1}{\alpha} \text{ and } L_2 \in \mathbb{R} \quad \text{or} \quad \text{(IV) } L_1 = \frac{\Lambda_1}{\alpha} \text{ and } L_2 < \frac{\Lambda_2}{\alpha},$$

then for any constant C, the inequality (1.3) *does not hold for some $u \in W_0^{1,n}(\Omega) \cap W_0^{m,n/(m-\alpha)}(\Omega)$ with $\|\nabla u\|_n = 1$.*

For the proof of Theorem 3.6, we have to find a sequence $\{u_j\}_{j=1}^{\infty} \subset W_0^{1,n}(\Omega) \cap W^{m,n/(m-\alpha)}(\Omega) \setminus \{0\}$ such that $F^{m,\alpha}[u_j; L_1, L_2] \to \infty$ as $j \to \infty$ under the assumption (III) or (IV). In the case $\Omega = B_1$, we can choose such a sequence as follows by modifying $\{u_\tau^\alpha\}_{0<\tau\leqslant 1}$ defined by (3.2).

We choose a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ satisfying $\phi(s)=1$ for $s \leqslant 1/3$, $\phi(s)=0$ for $s \geqslant 2/3$, and introduce polynomials

$$P_\tau^{m,\alpha}(r) = \log\frac{1}{\tau} + \sum_{l=1}^{m-1}\frac{1}{l}\left(1-\frac{r}{\tau}\right)^l + \left(\frac{1}{\alpha} - \sum_{l=1}^{m-1}\frac{1}{l}\right)\left(1-\frac{r}{\tau}\right)^m$$

(we regard the summations above as zeroes when $m=1$). For $0<\tau\leqslant 1/e$ and $0<\theta<1$, define $u^{m,\alpha}_{\tau,\theta}$ and $u^{1,\alpha}_{\tau,0}$ by

$$u^{m,\alpha}_{\tau,\theta}(x)=\tilde{u}^{m,\alpha}_{\tau,\theta}(|x|)$$
$$=\begin{cases}\dfrac{\alpha}{\alpha\log(1/\tau)+1}P^{m,\alpha}_{\tau}((1+\theta)|x|) & \text{for } x\in\bar{B}_{\tau/(1+\theta)},\\ \dfrac{\alpha}{\alpha\log(1/\tau)+1}\phi\left(\dfrac{(1+\theta)|x|-1}{\theta}\right)\log\dfrac{1}{(1+\theta)|x|} & \text{for } x\in\mathbb{R}^n\setminus B_{\tau/(1+\theta)}\end{cases}$$

if $m\in\{2,3,\dots,n\}$, and

$$u^{1,\alpha}_{\tau,0}(x)=\tilde{u}^{1,\alpha}_{\tau,0}(|x|)=\begin{cases}\dfrac{\alpha}{\alpha\log(1/\tau)+1}P^{1,\alpha}_{\tau}(|x|) & \text{for } x\in\bar{B}_{\tau},\\ \dfrac{\alpha}{\alpha\log(1/\tau)+1}\log\dfrac{1}{|x|} & \text{for } x\in\bar{B}_1\setminus B_{\tau},\\ 0 & \text{for } x\in\mathbb{R}^n\setminus B_1\end{cases}$$

if $m=1$. Then we can prove the following lemma, which will be proved in Section 4.

LEMMA 3.7. *Let $n\geqslant 2$, $m\in\{1,2,\dots,n\}$, $0<\alpha<1$ and $\Omega=B_1$.*

(i) *If $m\in\{2,3,\dots,n\}$, then $u^{m,\alpha}_{\tau,\theta}\in W^{1,n}_0(B_1)\cap W^{m,n/(m-\alpha)}_0(B_1)\setminus\{0\}$ for $0<\tau\leqslant 1/e$ and $0<\theta<1$. If $m=1$, then $u^{1,\alpha}_{\tau,0}\in W^{1,n}_0(B_1)\cap W^{1,n/(1-\alpha)}_0(B_1)\setminus\{0\}$ for $0<\tau\leqslant 1/e$.*

(ii) *Under the assumption* (III) *or* (IV) *of Theorem* 3.6*, it holds*

$$F^{m,\alpha}[u^{m,\alpha}_{\tau,\theta_\tau};L_1,L_2]\to\infty \ \text{ as } \tau\searrow 0$$

with $\theta_\tau=\tau^{n\alpha/((m-1)n-m+\alpha)}$ if $m\in\{2,3,\dots,n\}$, and $\theta_\tau=0$ if $m=1$.

We now prove Theorem 3.6 by using Lemma 3.7.

Proof. [Proof of Theorem 3.6] In order to examine the failure of (1.3), we may assume $L_1,L_2\geqslant 0$. Fix $z_0\in\Omega$ and $R_0\geqslant 1$ such that

$$B=\left\{x\in\mathbb{R}^n;\,|x-z_0|<\frac{1}{R_0}\right\}\subset\Omega.$$

By virtue of Lemma 3.7 (ii), there exists a family of functions $\{u^{m,\alpha}_{\tau,\theta_\tau}\}_{0<\tau\leqslant 1/e}\subset W^{1,n}_0(B_1)\cap W^{m,n/(m-\alpha)}_0(B_1)$ such that $F^{m,\alpha}[u^{m,\alpha}_{\tau,\theta_\tau};L_1,L_2]\to\infty$ as $\tau\searrow 0$. If we define

$$v_\tau(x)=u^{m,\alpha}_{\tau,\theta_\tau}(R_0(x-z_0)) \ \text{ for } x\in\mathbb{R}^n,$$

then $v_\tau\in W^{1,n}_0(B)\cap W^{m,n/(m-\alpha)}_0(B)\subset W^{1,n}_0(\Omega)\cap W^{m,n/(m-\alpha)}(\Omega)$ and

$$\|v_\tau\|_\infty=\|u^{m,\alpha}_{\tau,\theta_\tau}\|_\infty,\ \|\nabla v_\tau\|_n=\|\nabla u^{m,\alpha}_{\tau,\theta_\tau}\|_n,$$

$$\|v_\tau\|_{m,n/(m-\alpha)} \leqslant R_0^\alpha \|u^{m,\alpha}_{\tau,\theta_\tau}\|_{m,n/(m-\alpha)}$$

since

$$\|\nabla^l v_\tau\|_{n/(m-\alpha)} = R_0^{\alpha+l-m}\|\nabla^l u^{m,\alpha}_{\tau,\theta_\tau}\|_{n/(m-\alpha)} \leqslant R_0^\alpha\|\nabla^l u^{m,\alpha}_{\tau,\theta_\tau}\|_{n/(m-\alpha)}$$
$$\text{for } l \in \{0,1,\dots,m\}.$$

Because of (2.2), we have

$$\begin{aligned}
&F^{m,\alpha}[u^{m,\alpha}_{\tau,\theta_\tau};L_1,L_2] \\
&\leqslant \left(\frac{\|v_\tau\|_\infty}{\|\nabla v_\tau\|_n}\right)^{n/(n-1)} - L_1 \log\left(1+\frac{1}{R_0^\alpha}\frac{\|v_\tau\|_{m,n/(m-\alpha)}}{\|\nabla v_\tau\|_n}\right) \\
&\quad - L_2 \log\left(1+\log\left(1+\frac{1}{R_0^\alpha}\frac{\|v_\tau\|_{m,n/(m-\alpha)}}{\|\nabla v_\tau\|_n}\right)\right) \\
&\leqslant \left(\frac{\|v_\tau\|_\infty}{\|\nabla v_\tau\|_n}\right)^{n/(n-1)} - L_1 \log\left(1+\frac{\|v_\tau\|_{m,n/(m-\alpha)}}{\|\nabla v_\tau\|_n}\right) + L_1\log(1+R_0^\alpha) \\
&\quad - L_2 \log\left(1+\log\left(1+\frac{\|v_\tau\|_{m,n/(m-\alpha)}}{\|\nabla v_\tau\|_n}\right)\right) + L_2\log(1+\log(1+R_0^\alpha)) \\
&= F^{m,\alpha}[v_\tau;L_1,L_2] + L_1\log(1+R_0^\alpha) + L_2\log(1+\log(1+R_0^\alpha))
\end{aligned}$$

and it follows $F^{m,\alpha}[v_\tau;L_1,L_2] \to \infty$ as $\tau \searrow 0$. □

4. Proof of the key lemma

In this section, we shall prove Lemma 3.7. First we state the following proposition. We omit the proof because it is elementary; one can prove it by an induction on $|\nu|$. Here, $C^{k,1}([0,R])$ denotes the space of all k-times continuously differentiable functions on $[0,R]$ whose k-th derivative is Lipschitz continuous.

PROPOSITION 4.1. *Let $n \geqslant 1$, $m \in \{1,2,\dots,n\}$, $R > 0$ and $\tilde{u} \in C^{m-1,1}([0,R])$, and define $u(x) = \tilde{u}(|x|)$ for $x \in \bar{B}_R$. Then for any multi-index $\nu \in \mathbb{Z}^n_+ \setminus \{0\}$ with $|\nu| \leqslant m$, there exists a family of constants $\{\gamma_{\nu,\mu,k}\}_{\mu\leqslant\nu,k\in\{1,2,\dots,|\nu|\}} \subset \mathbb{Z}$ such that*

$$\left(\frac{\partial}{\partial x}\right)^\nu u(x) = \sum_{k=1}^{|\nu|}\sum_{\mu\leqslant\nu} \gamma_{\nu,\mu,k}\frac{\tilde{u}^{(k)}(|x|)x^\mu}{|x|^{|\nu|+|\mu|-k}} \quad \textit{for a.e. } x \in B_R \setminus \{0\}.$$

Furthermore, the right hand side is integrable on B_R and coincides with the ν-th derivative of u on B_R in the sense of distribution.

We also use the following estimates to prove Lemma 3.7.

PROPOSITION 4.2. *Let $n \geqslant 2$, $m \in \{1,2,\dots,n\}$ and $0 < \alpha < 1$. Let $0 < \theta < 1$ if $m \in \{2,3,\dots,n\}$, and $\theta = 0$ if $m = 1$. Then the following hold for $0 < \tau \leqslant 1/e$:*

(i) $\tilde{u}_{\tau,\theta}^{m,\alpha} \in C^{m-1,1}([0,1])$, *and* $(\tilde{u}_{\tau,\theta}^{m,\alpha})^{(l)}(1) = 0$ *for* $l \in \{0,1,\dots,m-1\}$. *In particular, for* $\nu \in \mathbb{Z}_+^n \setminus \{0\}$ *with* $|\nu| \leqslant m$, *the* ν*-th derivative of* $u_{\tau,\theta}^{m,\alpha}$ *on* B_1 *in the sense of distribution is given by*

$$\left(\frac{\partial}{\partial x}\right)^{\nu} u_{\tau,\theta}^{m,\alpha}(x) = \sum_{k=1}^{|\nu|} \sum_{\mu \leqslant \nu} \gamma_{\nu,\mu,k} \frac{(\tilde{u}_{\tau,\theta}^{m,\alpha})^{(k)}(|x|)x^{\mu}}{|x|^{|\nu|+|\mu|-k}} \quad \textit{for a.e. } x \in B_1.$$

(ii) *It holds* $\|u_{\tau,\theta}^{m,\alpha}\|_\infty \geqslant 1$.

(iii) *There exist constants* $K_{m,\alpha}, \tilde{K}_{m,\alpha} > 0$ *such that*

$$\frac{K_{m,\alpha}}{(\log(1/\tau))^{n-1}} \leqslant \|\nabla u_{\tau,\theta}^{m,\alpha}\|_n^n \leqslant \frac{1}{\Lambda_1^{n-1}} \frac{1}{(\log(1/\tau))^{n-1}} + \frac{\tilde{K}_{m,\alpha}}{(\log(1/\tau))^n}.$$

(iv) *If* $m \in \{2,3,\dots,n\}$, *then there exists a constant* $M_{m,\alpha} > 0$ *such that*

$$\begin{aligned} &\|u_{\tau,\theta}^{m,\alpha}\|_{m,n/(m-\alpha)}^{n/(m-\alpha)} \\ &\leqslant \frac{M_{m,\alpha}}{2(\log(1/\tau))^{n/(m-\alpha)}} \left(\frac{1}{\tau^{n\alpha/(m-\alpha)}} + \frac{1}{\theta^{(m-1)n/(m-\alpha)-1}}\right). \end{aligned}$$

If $m = 1$, *then there exists a constant* $M_{1,\alpha} > 0$ *such that*

$$\|u_{\tau,0}^{1,\alpha}\|_{1,n/(1-\alpha)}^{n/(1-\alpha)} \leqslant \frac{M_{1,\alpha}}{\tau^{n\alpha/(1-\alpha)}(\log(1/\tau))^{n/(1-\alpha)}}.$$

In particular, $u_{\tau,\theta}^{m,\alpha} \in W_0^{1,n}(B_1) \cap W_0^{m,n/(m-\alpha)}(B_1)$.

Proof. In what follows, we denote by $C_{m,\alpha}$ a constant depending only on m and α which may differ from line to line.

(i) First note that $P_\tau^{m,\alpha}$ is the unique polynomial of degree m satisfying

$$\frac{\alpha}{\alpha\log(1/\tau)+1} P_\tau^{m,\alpha}(0) = 1,$$
$$(P_\tau^{m,\alpha})^{(l)}(\tau) = \left(\frac{d}{dr}\right)^l \left[\log\frac{1}{r}\right]\Bigg|_{r=\tau} \quad \text{for } l \in \{1,2,\dots,m-1\}.$$

Hence, $\tilde{u}_{\tau,\theta}^{m,\alpha} \in C^{m-1,1}([0,1])$. We can easily see that $(\tilde{u}_{\tau,\theta}^{m,\alpha})^{(l)}(1) = 0$ for $l \in \{0,1,\dots,m-1\}$ since $\phi^{(l)}(1) = 0$ for $l \in \mathbb{Z}_+$.

(ii) Since $u_{\tau,\theta}^{m,\alpha}$ is continuous on $\overline{B}_1$, we see that

$$\|u_{\tau,\theta}^{m,\alpha}\|_\infty \geqslant u_{\tau,\theta}^{m,\alpha}(0) = \frac{\alpha}{\alpha\log(1/\tau)+1} P_\tau^{m,\alpha}(0) = 1.$$

(iii) First we consider the case $m \in \{2,3,\ldots,n\}$. Since

$$
\begin{aligned}
&\nabla u^{m,\alpha}_{\tau,\theta}(x)\\
&= (\tilde{u}^{m,\alpha}_{\tau,\theta})'(|x|)\frac{x}{|x|}\\
&= \begin{cases}
\dfrac{\alpha(1+\theta)}{\tau(\alpha\log(1/\tau)+1)}\left[\displaystyle\sum_{l=1}^{m-1}\frac{1}{l}\left(1-\frac{1+\theta}{\tau}|x|\right)^{l-1}\right. \\
\qquad\qquad \left. +m\left(\dfrac{1}{\alpha}-\displaystyle\sum_{l=1}^{m-1}\frac{1}{l}\right)\left(1-\frac{1+\theta}{\tau}|x|\right)^{m-2}\right]\dfrac{x}{|x|} \\
\qquad\qquad\qquad\qquad \text{for } x \in B_{\tau/(1+\theta)}, \\
\dfrac{\alpha}{\alpha\log(1/\tau)+1}\left[-\phi\left(\dfrac{(1+\theta)|x|-1}{\theta}\right)\dfrac{1}{|x|}\right. \\
\qquad\qquad \left. +\left(1+\dfrac{1}{\theta}\right)\phi'\left(\dfrac{(1+\theta)|x|-1}{\theta}\right)\log\dfrac{1}{(1+\theta)|x|}\right]\dfrac{x}{|x|} \\
\qquad\qquad\qquad\qquad \text{for } x \in B_1\setminus\bar{B}_{\tau/(1+\theta)},
\end{cases}
\end{aligned}
$$

and

$$
\frac{|x|}{\theta}\log((1+\theta)|x|) \leqslant \frac{|x|}{\theta}((1+\theta)|x|-1) \leqslant 1 \quad \text{for } x \in B_1\setminus\bar{B}_{1/(1+\theta)}, \tag{4.1}
$$

we have

$$
|\nabla u^{m,\alpha}_{\tau,\theta}(x)| = \frac{\alpha}{\alpha\log(1/\tau)+1}\frac{1}{|x|} \quad \text{for } x \in B_{1/(1+\theta)}\setminus\bar{B}_{\tau/(1+\theta)}, \tag{4.2}
$$

$$
|\nabla u^{m,\alpha}_{\tau,\theta}(x)| \leqslant \begin{cases}
\dfrac{C_{m,\alpha}}{\tau\log(1/\tau)} & \text{for } x \in B_{\tau/(1+\theta)}, \\
\dfrac{1}{\log(1/\tau)}\dfrac{1}{|x|} & \text{for } x \in B_{1/(1+\theta)}\setminus\bar{B}_{\tau/(1+\theta)}, \\
\dfrac{C_{m,\alpha}}{\log(1/\tau)}\dfrac{1}{|x|} & \text{for } x \in B_1\setminus\bar{B}_{1/(1+\theta)}.
\end{cases} \tag{4.3}
$$

Calculating the norms by using (4.2) and (4.3), we obtain the estimate from above. Since

$$
\begin{aligned}
\|\nabla u^{m,\alpha}_{\tau,\theta}\|_n^n &\geqslant \|\nabla u^{m,\alpha}_{\tau,\theta}\|^n_{L^n(B_{1/(1+\theta)}\setminus\bar{B}_{\tau/(1+\theta)})}\\
&= \frac{1}{\Lambda_1^{n-1}}\frac{\alpha^n\log(1/\tau)}{(\alpha\log(1/\tau)+1)^n}\\
&\geqslant \frac{1}{\Lambda_1^{n-1}}\left(\frac{\alpha}{\alpha+1}\right)^n\frac{1}{(\log(1/\tau))^{n-1}},
\end{aligned}
$$

we obtain the estimate from below.

In the case $m=1$, since

$$
\nabla u^{1,\alpha}_{\tau,0}(x) = (\tilde{u}^{1,\alpha}_{\tau,0})'(|x|)\frac{x}{|x|} = \begin{cases}
-\dfrac{1}{\tau(\alpha\log(1/\tau)+1)}\dfrac{x}{|x|} & \text{for } x \in B_\tau, \\
-\dfrac{\alpha}{\alpha\log(1/\tau)+1}\dfrac{x}{|x|^2} & \text{for } x \in B_1\setminus\bar{B}_\tau,
\end{cases}
$$

we have

$$|\nabla u_{\tau,0}^{1,\alpha}(x)| = \begin{cases} \dfrac{1}{\tau(\alpha\log(1/\tau)+1)} & \text{for } x \in B_\tau, \\ \dfrac{\alpha}{\alpha\log(1/\tau)+1}\dfrac{1}{|x|} & \text{for } x \in B_1 \setminus \bar{B}_\tau. \end{cases}$$

Arguing similarly as above, we deduce the desired conclusion.

(iv) First we consider the case $m \in \{2,3,\ldots,n\}$. From the definition of $u_{\tau,\theta}^{m,\alpha}$, we have

$$|u_{\tau,\theta}^{m,\alpha}(x)| \leqslant \begin{cases} C_{m,\alpha} & \text{for } x \in \bar{B}_{\tau/(1+\theta)}, \\ \dfrac{1}{\log(1/\tau)}\log\dfrac{1}{(1+\theta)|x|} & \text{for } x \in \bar{B}_{1/(1+\theta)} \setminus B_{\tau/(1+\theta)}, \\ \dfrac{1}{\log(1/\tau)}\log((1+\theta)|x|) & \text{for } x \in \bar{B}_1 \setminus B_{1/(1+\theta)}. \end{cases}$$

By using the change of variables $\rho = \log(1/(1+\theta)r)$ and $\rho = \log((1+\theta)r)$, we have

$$\begin{aligned} \int_{\tau/(1+\theta)}^{1/(1+\theta)} \left(\log\frac{1}{(1+\theta)r}\right)^{n/(m-\alpha)} r^{n-1}dr &= \frac{1}{(1+\theta)^n}\int_0^{\log(1/\tau)} \frac{\rho^{n/(m-\alpha)}}{e^{n\rho}}d\rho \\ &\leqslant \frac{\Gamma(n/(m-\alpha)+1)}{n^{n/(m-\alpha)+1}}\frac{1}{(1+\theta)^n} \\ &\leqslant C_{m,\alpha} \end{aligned}$$

and

$$\begin{aligned} \int_{1/(1+\theta)}^{1} (\log((1+\theta)r))^{n/(m-\alpha)} r^{n-1}dr &= \frac{1}{(1+\theta)^n}\int_0^{\log(1+\theta)} \rho^{n/(m-\alpha)}e^{n\rho}d\rho \\ &\leqslant (\log(1+\theta))^{n/(m-\alpha)+1} \\ &< 1. \end{aligned}$$

Then we can easily show that

$$\|u_{\tau,\theta}^{m,\alpha}\|_{n/(m-\alpha)}^{n/(m-\alpha)} \leqslant \frac{C_{m,\alpha}}{(\log(1/\tau))^{n/(m-\alpha)}}.$$

Next, applying Proposition 4.1, for a multi-index $\nu \in \mathbb{Z}_+^n \setminus \{0\}$ with $|\nu| \leqslant m$, we have

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x}\right)^{\nu} u_{\tau,\theta}^{m,\alpha}(x)\\
&= \begin{cases}
\dfrac{\alpha}{\alpha\log(1/\tau)+1}\displaystyle\sum_{k=1}^{|\nu|}\left(-\frac{1+\theta}{\tau}\right)^{k}\sum_{\mu\leqslant\nu}\gamma_{\nu,\mu,k}\frac{x^{\mu}}{|x|^{|\nu|+|\mu|-k}}\\
\quad\times\left[\displaystyle\sum_{l=k}^{m-1}\frac{(l-1)!}{(l-k)!}\left(1-\frac{1+\theta}{\tau}|x|\right)^{l-k}\right.\\
\qquad\left.+\left(\dfrac{1}{\alpha}-\displaystyle\sum_{l=1}^{m-1}\frac{1}{l}\right)\frac{m!}{(m-k)!}\left(1-\frac{1+\theta}{\tau}|x|\right)^{m-k}\right]\\
\qquad\qquad\qquad\qquad\qquad\qquad\text{for } x\in B_{\tau/(1+\theta)},\\
\dfrac{\alpha}{\alpha\log(1/\tau)+1}\displaystyle\sum_{k=1}^{|\nu|}\sum_{\mu\leqslant\nu}\gamma_{\nu,\mu,k}\frac{x^{\mu}}{|x|^{|\nu|+|\mu|-k}}\\
\quad\times\left[\displaystyle\sum_{l=1}^{k}(-1)^{l-1}\binom{k}{l}\left(1+\frac{1}{\theta}\right)^{k-l}\phi^{(k-l)}\left(\frac{(1+\theta)|x|-1}{\theta}\right)\frac{1}{|x|^{l}}\right.\\
\qquad\left.+\left(1+\dfrac{1}{\theta}\right)^{k}\phi^{(k)}\left(\frac{(1+\theta)|x|-1}{\theta}\right)\log\frac{1}{(1+\theta)|x|}\right]\\
\qquad\qquad\qquad\qquad\qquad\qquad\text{for } x\in B_1\setminus\bar{B}_{\tau/(1+\theta)}.
\end{cases}
\end{aligned}
$$

Hence, for $j \in \{1,2,\dots,m\}$, we have

$$
|\nabla^j u_{\tau,\theta}^{m,\alpha}(x)| \leqslant \begin{cases}
\dfrac{C_{m,\alpha}}{\tau\log(1/\tau)}\dfrac{1}{|x|^{j-1}} & \text{for } x\in B_{\tau/(1+\theta)},\\
\dfrac{C_{m,\alpha}}{\log(1/\tau)}\dfrac{1}{|x|^{j}} & \text{for } x\in B_{1/(1+\theta)}\setminus\bar{B}_{\tau/(1+\theta)},\\
\dfrac{C_{m,\alpha}}{\theta^{j-1}\log(1/\tau)}\dfrac{1}{|x|} & \text{for } x\in B_1\setminus\bar{B}_{1/(1+\theta)}.
\end{cases}
$$

Then we can show that

$$
\begin{aligned}
&\|\nabla^j u_{\tau,\theta}^{m,\alpha}\|_{n/(m-\alpha)}^{n/(m-\alpha)}\\
&\leqslant \begin{cases}
\dfrac{C_{m,\alpha}}{(\log(1/\tau))^{n/(m-\alpha)}}\left(1+\dfrac{1}{\theta^{n(j-1)/(m-\alpha)-1}}\right) & \text{for } j\in\{1,2,\dots,m-1\},\\
\dfrac{C_{m,\alpha}}{(\log(1/\tau))^{n/(m-\alpha)}}\left(\dfrac{1}{\tau^{n\alpha/(m-\alpha)}}+\dfrac{1}{\theta^{(m-1)n/(m-\alpha)-1}}\right) & \text{for } j=m,
\end{cases}
\end{aligned}
$$

and the assertion follows.

Arguing similarly as above, we have

$$
\|u_{\tau,0}^{1,\alpha}\|_{n/(1-\alpha)}^{n/(1-\alpha)} \leqslant \frac{C_{1,\alpha}}{(\log(1/\tau))^{n/(1-\alpha)}},
$$

$$\|\nabla u_{\tau,0}^{1,\alpha}\|_{n/(1-\alpha)}^{n/(1-\alpha)} \leqslant \frac{C_{1,\alpha}}{\tau^{n\alpha/(1-\alpha)}(\log(1/\tau))^{n/(1-\alpha)}},$$

which implies the desired conclusion in the case $m=1$. □

Finally we prove Lemma 3.7.

Proof of Lemma 3.7. (i) The assertion immediately follows from Proposition 4.2.

(ii) We may assume $L_1, L_2 \geqslant 0$. Let $0<\tau\leqslant 1/e^{1/\alpha}$ be sufficiently small so that $\tau^\alpha(\log(1/\tau))^{1/n}\leqslant 1$. We estimate $F^{m,\alpha}[u_{\tau,\theta_\tau}^{m,\alpha};L_1,L_2]$ from below. Since

$$\begin{aligned}
\frac{1}{\left(\frac{1}{s^{n-1}}+\frac{1}{t^{n-1}}\right)^{1/(n-1)}} &= s - s\frac{(s^{n-1}+t^{n-1})^{1/(n-1)}-t}{(s^{n-1}+t^{n-1})^{1/(n-1)}} \\
&\geqslant s - s\left(\left(\left(\frac{s}{t}\right)^{n-1}+1\right)^{1/(n-1)}-1\right) \\
&\geqslant s-\frac{s^n}{t^{n-1}} \quad \text{for } s,t>0,
\end{aligned}$$

we have from Proposition 4.2 (ii) and (iii) that

$$\begin{aligned}
\left(\frac{\|u_{\tau,\theta_\tau}^{m,\alpha}\|_\infty}{\|\nabla u_{\tau,\theta_\tau}^{m,\alpha}\|_n}\right)^{n/(n-1)} &\geqslant \frac{1}{\left(\frac{1}{\Lambda_1^{n-1}}\frac{1}{(\log(1/\tau))^{n-1}}+\frac{\tilde{K}_{m,\alpha}}{(\log(1/\tau))^n}\right)^{1/(n-1)}} \\
&\geqslant \Lambda_1\log\frac{1}{\tau}-\tilde{K}_{m,\alpha}\Lambda_1^n.
\end{aligned}$$

Moreover, it follows from Proposition 4.2 (iv) that

$$\|u_{\tau,\theta_\tau}^{m,\alpha}\|_{m,n/(m-\alpha)}^{n/(m-\alpha)} \leqslant \frac{M_{m,\alpha}}{\tau^{n\alpha/(m-\alpha)}(\log(1/\tau))^{n/(m-\alpha)}}.$$

Using the inequalities (2.2) and

$$\log(1+s)\leqslant \log s+\log 2 \quad \text{for } s\geqslant 1,$$

we have from Proposition 4.2 (iii) that

$$\begin{aligned}
&\log\left(1+\frac{\|u_{\tau,\theta_\tau}^{m,\alpha}\|_{m,n/(m-\alpha)}}{\|\nabla u_{\tau,\theta_\tau}^{m,\alpha}\|_n}\right) \\
&\leqslant \log\left(1+\frac{M_{m,\alpha}^{(m-\alpha)/n}}{K_{m,\alpha}^{1/n}}\frac{1}{\tau^\alpha(\log(1/\tau))^{1/n}}\right) \\
&\leqslant \log\left(1+\frac{1}{\tau^\alpha(\log(1/\tau))^{1/n}}\right)+\log\left(1+\frac{M_{m,\alpha}^{(m-\alpha)/n}}{K_{m,\alpha}^{1/n}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \log\left(\frac{1}{\tau^{\alpha}(\log(1/\tau))^{1/n}}\right)+\log 2+\log\left(1+\frac{M_{m,\alpha}^{(m-\alpha)/n}}{K_{m,\alpha}^{1/n}}\right)\\
&=\alpha\log\frac{1}{\tau}-\frac{1}{n}\log\left(\log\frac{1}{\tau}\right)+\log\left(2\left(1+\frac{M_{m,\alpha}^{(m-\alpha)/n}}{K_{m,\alpha}^{1/n}}\right)\right)\\
&=\alpha\log\frac{1}{\tau}-\frac{1}{n}\log\left(\log\frac{1}{\tau}\right)+C_{m,\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
&\log\left(1+\log\left(1+\frac{\|u_{\tau,\theta_\tau}^{m,\alpha}\|_{m,n/(m-\alpha)}}{\|\nabla u_{\tau,\theta_\tau}^{m,\alpha}\|_n}\right)\right)\\
&\leqslant \log\left(1+\alpha\log\frac{1}{\tau}-\frac{1}{n}\log\left(\log\frac{1}{\tau}\right)+C_{m,\alpha}\right)\\
&\leqslant \log\left(1+\alpha\log\frac{1}{\tau}+C_{m,\alpha}\right)\\
&\leqslant \log\left(1+\alpha\log\frac{1}{\tau}\right)+\log(1+C_{m,\alpha})\\
&\leqslant \log\left(\alpha\log\frac{1}{\tau}\right)+\log 2+\log(1+C_{m,\alpha})\\
&=\log\left(\log\frac{1}{\tau}\right)+\log(2\alpha(1+C_{m,\alpha})).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
F^{m,\alpha}[u_{\tau,\theta_\tau}^{m,\alpha};L_1,L_2] \geqslant (\Lambda_1-\alpha L_1)\log\frac{1}{\tau}+\left(\frac{L_1}{n}-L_2\right)\log\left(\log\frac{1}{\tau}\right)\\
-\tilde{K}_{m,\alpha}\Lambda_1^n-L_1C_{m,\alpha}-L_2\log(2\alpha(1+C_{m,\alpha}))\\
\to\infty \quad \text{as } \tau\searrow 0
\end{aligned}
$$

under the assumption (III) or (IV). □

Acknowledgement. The authors express gratitude to Professor Hideo Kozono for helpful advice. They also would like to thank Professor Makoto Nakamura for valuable comments. In writing this paper, the first author was supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. At present he is supported by the Global COE Program. The third author is supported by the Fujukai Foundation.

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(Received October 28, 2008)

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