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A FAMILY OF NECESSARY STABILITY INEQUALITIES VIA QUADRATIC FORMS

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Abstract. Inequalities relating three coefficients of the even or odd part of a Hurwitz-stable polynomial have been established recently via the Newton-MacLaurin inequalities, and via optimization techniques for multivariate functions on the positive orthant. From the theory of quadratic forms we derive a family of strict inequalities which includes and generalizes the known inequalities. For polynomials of higher degree quantifiable improvements are obtained. Benefit of these inequalities is low-cost instability testing for polynomials with varying coefficients.

1. Introduction

Borobia and Dormido [1] established recently an inequality relating three coefficients of the even part of a Hurwitz-stable polynomial. This was generalized by Yang [7, 8]. We repeat Yang's statement [8], but call the reader's attention to a necessary correction of the case $n = 2N$, see Corollary 1 below.

Yang's inequalities: Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be Hurwitz-stable with positive coefficients. Designate by N the integer $\lfloor n/2 \rfloor$. For $1 \leqslant r$, and $2r+3 \leqslant n$ it is claimed that

$$\left(\frac{a_{2r+1}}{\binom{N}{r}}\right)^2 \geqslant \frac{a_{2r-1}}{\binom{N}{r-1}} \frac{a_{2r+3}}{\binom{N}{r+1}};$$

for $1 \leqslant r$, and $2r+2 \leqslant n$ it is claimed that

$$\left(\frac{a_{2r}}{\binom{N}{r}}\right)^2 \geqslant \frac{a_{2r-2}}{\binom{N}{r-1}} \frac{a_{2r+2}}{\binom{N}{r+1}}.$$

Borobia and Dormido [1] derived their result for coefficients of even indices from discussion of a multivariate function on the positive orthant (as later Yang in [7]), while Yang proceeded in [8] via the Newton-MacLaurin inequalities for the real roots of the odd/even part. In this note we derive two series of inequalities connecting $3,4,5,\ldots$ coefficients which generalize the estimates by Borobia, Dormido and Yang. We obtain definite improvements on the latter estimates if more than 3 coefficients are involved. Combining both sets of the new inequalities the improvement can be quantified.

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2. Hurwitz-stability, Quadratic forms and Cauchy Indices

A polynomial is Hurwitz-stable *iff* all its roots lie in the open half-plane $\Re z < 0$. This implies that all coefficients of a real Hurwitz-stable polynomial have the same definite sign. Hurwitz-stability of a real polynomial $P(x) = h(x^2) + xg(x^2) = \sum a_i x^i$ is equivalent by the Hermite-Biehler theorem to the strict interlacing of the roots of h and g on the half-ray of negative reals (cf., e.g., [6, 2, 4]). A coefficient characterisation of this interlacing can be found in Hurwitz' conditions [3] (if $a_0 > 0$)

$$a_1 > 0, \quad \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \quad \ldots . \tag{1}$$

Suppose that $N = \deg h \geqslant \deg g$, and expand $g(z)/h(z)$ in a Laurent series at Infinity as

$$g(z)/h(z) = s_0 + \frac{s_1}{z} + \frac{s_2}{z^2} + \frac{s_3}{z^3} + \ldots . \tag{2}$$

If the zeros of h and g lie exclusively on the real axis, and strictly interlace, we observe an alternating asymptotic behaviour at the pole places of g/h. Thus, the sign change at poles may be used to count zeros.

DEFINITION 1. The *Cauchy index* $I_a^b R$ of a real rational function R with respect to an interval $[a,b]$ is the difference $N^+ - N^-$, where N^+ is the number of transitions from $-\infty$ to $+\infty$ at a pole of $R(x)$, and N^- is the number of transitions from $+\infty$ to $-\infty$ at a pole, where x steadily increases from a to b.

To compute the Cauchy index, Hurwitz [3] constructed a quadratic form using h and g. Quadratic forms have been considered by Jacobi, Hermite, Sylvester, Hurwitz as well as others in the course of studying root-distribution problems. (A good source for results and proofs as well as pointers to the literature is [2]; see also the research survey [4].)

With the s_i in (2) we construct the matrix $S = (\sigma_{i,j})_0^{N-1}$ with entries: $\sigma_{i,j} := s_{i+j+1}, i \leqslant j; \sigma_{i,j} := \overline{s_{i+j+1}}, i > j$. The matrix S is Hermitian, and if g and h are real it is a Hankel matrix. In the real case, the quadratic form $\sum_0^{N-1} s_{i+j+1} x_i x_j$ is usually called the *associated quadratic form* of g/h.

The Cauchy index of a proper rational function can be computed as the signature of the associated bezoutiants as was noted in the earliest works on quadratic forms by Sylvester and Hermite (see, e.g., [2], Ch.16.11, Satz 9, p. 560 or [4], Sec. 2.2, Th.X (and its footnote), p. 280). This result was extended by Hurwitz (cf. the last section of Hurwitz' work [3] or, e.g., [4], Sec. 3.2, pp. 292/293) who used it to derive a special characterisation for root-location inside the left half-plane.

THEOREM 1. (Hurwitz) *Consider real polynomials* g, h *such that* $\deg g \leqslant \deg h = N$ *with positive coefficients and associated quadratic form determined by the matrix*

$S = (s_{i+j+1})_{i,j=0}^{N-1}$. *The polynomials have only simple, strictly interlacing roots on the negative real axis if and only if*

$$I_{-\infty}^{+\infty} g/h = signature(\sum_{i,j=0}^{N-1} s_{i+j+1} x_i x_j) = N,$$
$$I_{-\infty}^{+\infty} zg(z)/h(z) = signature(\sum_{i,j=0}^{N-1} s_{i+j+2} x_i x_j) = -N.$$

A result by Jacobi and Frobenius (cf., e.g. [4], Sec. 1.1, pp. 267/268, or, [2], Chap. 10.3, Satz 2, eq. (33), p. 313) allows to compute the signature via the sign variation of the consecutive principal minors. We may re-formulate Hurwitz' result as follows.

THEOREM 2. *Consider two real polynomials h and g $(\deg h \geqslant \deg g)$ with positive coefficients and the expansion at Infinity*

$$g(z)/h(z) = s_0 + \frac{s_1}{z} + \frac{s_2}{z^2} + \frac{s_3}{z^3} + \dots .$$

The roots of the polynomials g and h are all real, simple, and strictly interlace each other on the negative real axis if and only if the determinants

$$H_k^{(0)} := |s_{i+j+1}|, \qquad (3)$$
$$H_k^{(1)} := (-1)^{k+1} |s_{i+j+2}|, \quad 0 \leqslant i, j \leqslant k.$$

are positive for all $k \leqslant \deg h - 1$.

Positivity of the determinants in (3) is a natural source of coefficient inequalities as detailed in the next section.

3. Necessary conditions for stable polynomials

We make use of two operations on the class of Hurwitz-stable polynomials: i) differentiation (which yields a Hurwitz-stable polynomial by the Gauss-Lucas theorem [5]), ii) taking of reciprocals. The reciprocal polynomial P^* of a *real* polynomial P is defined by $P^*(x) := x^{\deg P} P(1/x)$, and has as its roots the reciprocals of the roots of P. Using these two operations we reduce to low-degree Hurwitz-stable polynomials.

Let us consider the real Hurwitz-stable polynomial

$$P(x) = \sum_{i=0}^{n} a_i x^i = h(x^2) + xg(x^2)$$

of degree n with positive coefficients. Put $N := \lfloor n/2 \rfloor$. To treat coefficient criteria for even and odd coefficients simultaneously we employ the index shift $\tau \in \{0,1\}$. An even-degree polynomial, $n = 2N$, has N odd coefficients and $N+1$ even coefficients. To index the coefficients $a_{2k+\tau}$ via k we use $\mu := N - \tau \cdot (2N + 1 - n)$, and count from 0 to μ. For even n and $\tau = 1$ we have $\mu = N - 1$, whereas in all other cases $\mu = N$.

THEOREM 3. *Let* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = h(x^2) + xg(x^2)$ *be Hurwitz-stable with positive coefficients. Designate by* N *the integer* $\lfloor n/2 \rfloor$. *Fix* $\tau \in \{0,1\}$, *and put* $\mu := N - \tau \cdot (2N+1-n)$. *We set* $r(x) := (1-\tau)\cdot h(x) + \tau \cdot g(x)$. *With* $a(x) := r^{(k)}(x)$, $b(x) := x^{\mu-k}a(1/x)$ *we define* $c(x) := b^{(\mu-k-l)}(x)$ *and* $R(x) := c'(x)/c(x)$ *with expansion at Infinity*

$$R(z) = \sum_{i=1}^{\infty} \frac{s_i}{z^i}.$$

If $k+l+2 \leqslant \mu$, *we have the inequalities*

$$\begin{aligned} H_k^{(0)} &:= |s_{i+j+1}| > 0, \\ H_k^{(1)} &:= (-1)^{k+1}|s_{i+j+2}| > 0, \quad 0 \leqslant i,j \leqslant k. \end{aligned} \tag{4}$$

Proof. The polynomial r has negative, simple roots by the Hermite-Biehler theorem. Thus, by Rolle's theorem, the derivative a and its reciprocal b share this property. The same holds true for the derivative c of b. Thus, Hurwitz' result as captured in Theorem 2 yields positivity of the determinants. □

What practical consequences may we draw from the preceding theorem? A large number of coefficients is best dealt with using some machine support, but for a small number of coefficients we may explicitly write out the inequalities. It turns out that we obtain Yang's inequalities as well as improvements thereof. We start with Yang's inequalities (rectified for the odd coefficients of an even-degree polynomials as necessary in view of Yang's proof [8]).

COROLLARY 1. *Let* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *be Hurwitz-stable with positive coefficients. Designate by* N *the integer* $\lfloor n/2 \rfloor$, *fix* $\tau \in \{0,1\}$, *and set* $\mu := N - \tau \cdot (2N+1-n) \geqslant 2$. *For* $0 \leqslant k$, $2k+4+\tau \leqslant n$ *it holds true that*

$$a_{2(k+1)+\tau}^2 > \frac{k+2}{k+1} \cdot \frac{\mu-k}{\mu-k-1} a_{2k+\tau} a_{2(k+2)+\tau}. \tag{5}$$

Proof. For fixed $\tau \in \{0,1\}$, and $P(x) = h(x^2) + xg(x^2)$, we set $r(x) := (1-\tau)h(x) + \tau g(x)$. The polynomial r is of degree $\mu = N - \tau \cdot (2N+1-n) \geqslant 2$, and has the explicit representation

$$r(x) = \sum_{j=0}^{\mu} a_{2j+\tau} x^j,$$

from which we compute the k-th derivative

$$a(x) := r^{(k)}(x) = k! a_{2k+\tau} + (k+1)! a_{2(k+1)+\tau} x + \frac{(k+2)!}{2!} a_{2(k+2)+\tau} x^2 + \cdots.$$

Differentiating the reciprocal $b(x) := a^*(x) = x^{\mu-k}a(1/x)$, we obtain the $(\mu-k-2)$-nd derivative, denoted by $c(x)$, as

$$c(x) := a_{2k+\tau}\frac{\gamma_k^{\mu}}{2}x^2 + a_{2(k+1)+\tau}\gamma_{k+1}^{\mu} x + a_{2(k+2)+\tau}\frac{\gamma_{k+2}^{\mu}}{2},$$

where $\gamma_k^\mu := (\mu-k)!k!$. When we consider the expansion near Infinity

$$R(z) = c'(z)/c(z) = \sum_{i=1}^{\infty} \frac{s_i}{z^i}, \tag{6}$$

we obtain the first three coefficients in (6) as

$$s_1 = 2, \quad s_2 = \frac{-2a_{2k+2+\tau}}{a_{2k+\tau}} \frac{\gamma_{k+1}^\mu}{\gamma_k^\mu}, \quad s_3 = \frac{4a_{2k+2+\tau}^2(\gamma_{k+1}^\mu)^2 - 2a_{2k+4+\tau}\gamma_{k+2}^\mu a_{2k+\tau}\gamma_k^\mu}{(a_{2k+\tau}\gamma_k^\mu)^2}.$$

As $\mu \geqslant 2$, we obtain from Theorem 3 positivity of the Hankel determinant $H_1^{(0)}$ of order 2, i.e. the inequality $s_1 \cdot s_3 - s_2^2 > 0$. Substituting the above formulas, and multiplying by the positive term $(a_0\gamma_k^\mu)^2$ we obtain

$$4a_{2k+2+\tau}^2(\gamma_{k+1}^\mu)^2 - 4a_{2k+4+\tau}\gamma_{k+2}^\mu a_{2k+\tau}\gamma_k^\mu > 0.$$

This yields the inequalities (5). □

Thus, we have obtained Yang's inequality (rectified, as necessary, in the case of the odd part of an even-degree polynomial) in strict form. If we abbreviate the numerical factor using $\sigma_{k+1} := \sqrt{[(k+2)/(k+1)]\cdot[(\mu-k)/(\mu-k-1)]}$, we may write these inequalities as

$$a_{2(k+1)+\tau}^2 > \sigma_{k+1}^2 a_{2k+\tau} a_{2(k+2)+\tau}. \tag{7}$$

We will use the double series of inequalities from Theorem 2 connecting four coefficients to reduce to a three-term inequality improving (7).

THEOREM 4. *Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be Hurwitz-stable with positive coefficients. Designate by N the integer $\lfloor n/2 \rfloor$. Fix $\tau \in \{0,1\}$, and put $\mu := N - \tau\cdot(2N+1-n)$.*

For k with $0 \leqslant k < \mu - 1$, we define

$$\sigma_{k+1} := \sqrt{[(k+2)/(k+1)]\cdot[(\mu-k)/(\mu-k-1)]}.$$

Suppose that $0 \leqslant k \leqslant \mu-3$, then $a_{2k+\tau}$ and $a_{2(k+1)+\tau}$ may be estimated as follows:

1.)

$$a_{2k+\tau} < \frac{3}{\sigma_{k+1}^2} \frac{a_{2(k+2)+\tau} a_{2(k+1)+\tau}^2}{4a_{2(k+2)+\tau}^2 - \sigma_{k+2}^2 a_{2(k+3)+\tau} a_{2(k+1)+\tau}} \tag{8}$$

$$< \frac{a_{2(k+1)+\tau}^2}{\sigma_{k+1}^2 a_{2(k+2)+\tau}}. \tag{9}$$

2.) Let

$$f_1 := \frac{a_{2(k+2)+\tau}^{3/2}}{\sigma_{k+1}(\sigma_{k+2})^2 \sqrt{a_{2k+\tau}}\, a_{2(k+3)+\tau}}.$$

Then

$$a^2_{2(k+1)+\tau} > L(f_1)\sigma^2_{k+1} a_{2k+\tau} a_{2(k+2)+\tau}, \tag{10}$$

where $L(f_1) > 1$ *is the smallest positive root of*

$$x^2 - 2\frac{(9f_1^2 - 1 + 10f_1)}{12f_1}x + 2f_1 = 0.$$

Proof. Take the even or odd part (of degree μ) of the polynomial P, reduce it via differentiation as above, to the four term polynomial

$$c(x) := \frac{\gamma_k^\mu}{3!} a_{2k+\tau} + \frac{\gamma_{k+1}^\mu}{2!} a_{2k+2+\tau} x + \frac{\gamma_{k+2}^\mu}{2!} a_{2k+4+\tau} x^2 + \frac{\gamma_{k+3}^\mu}{3!} \cdot a_{2k+6+\tau} x^3$$

with derivative

$$c'(x) := \frac{\gamma_{k+1}^\mu}{2} a_{2k+2+\tau} + \gamma_{k+2}^\mu a_{2k+4+\tau} x + \frac{\gamma_{k+3}^\mu}{2} a_{2k+6+\tau} x^2,$$

where $0 \leqslant k \leqslant \mu - 3$, and $\gamma_k^\mu := (\mu - k)!k!$. Both functions have only negative real roots according to the assumption.

Define the polynomials $h(z) := c(z)$ and $g_\varepsilon(z) := (z+\varepsilon) \cdot c'(z)$ with small $\varepsilon > 0$, take the reciprocal polynomials $g_\varepsilon^*(z) := z^n g_\varepsilon(1/z)$ and $h^*(z) := z^n h(1/z)$ resp., and consider the expansion at Infinity

$$g_\varepsilon^*(z)/h^*(z) = \tilde{s}_0 + \frac{\tilde{s}_1}{z} + \dots.$$

Application of Theorem 2 to g_ε^*/h^* for every $\varepsilon > 0$ smaller than the smallest modulus of the roots in $c(z)$ yields a sequence of inequalities which holds for $\varepsilon \to 0$, and by continuity also for $\varepsilon = 0$. To compute with these inequalities we have to write down the coefficients explicitly.

We use auxillary terms q_i

$$q_0 := \frac{(k+1)\, a_{2(k+1)+\tau}}{(\mu - k)\, a_{2k+\tau}},$$
$$q_1 := \frac{(k+2)(k+1)\, a_{2(k+2)+\tau}}{(\mu - k)\,(\mu - k - 1)\, a_{2k+\tau}},$$
$$q_2 := \frac{(k+3)\,(k+2)\,(k+1)\, a_{2k+6+\tau}}{(\mu - k)\,(\mu - k - 1)\,(\mu - k - 2)\, a_{2k+\tau}},$$

to express the first four coefficients s_i of the expansion $R(z) = g_0^*(z)/h^*(z) = \sum_{i=1}^\infty s_i/z^i$ as

$$s_1 := 12 \cdot q_0, s_2 := 24 \cdot q_1 - 36 q_0^2,$$
$$s_3 := 12 q_2 - 108 q_0 q_1 + 108 q_0^3,$$
$$s_4 := -324 q_0^4 - 72 q_1^2 + 432 q_0^2 q_1 - 48 q_2 q_0.$$

From Theorem 2 applied to g_ε^*/h^* we obtain in the limit the non-strict inequality

$$H_1^{(0)} = s_1 \cdot s_3 - s_2^2 \geqslant 0.$$

This yields an expression involving $a_{2k+\tau}, \dots, a_{2(k+3)+\tau}$. Multiplying it by $\frac{1}{72} a_{2k+\tau}^3 (\mu - k)^3 (\mu - k - 1)^2 (\mu - k - 2)/((k+1)^2 (k+2))$ we obtain the inequality

$$\begin{aligned} &-4(\mu-k)(\mu-k-2)(k+2)a_{2k+\tau}a_{2(k+2)+\tau}^2 \\ &+3(\mu-k-1)(\mu-k-2)(k+1)a_{2(k+1)+\tau}^2 a_{2(k+2)+\tau} \\ &+(\mu-k)(\mu-1-k)(k+3)a_{2k+\tau}a_{2(k+1)+\tau}a_{2(k+3)+\tau} \geqslant 0 \end{aligned}$$

which yields the estimate (8) of Theorem 4 in non-strict form. By continuity of the roots, the set of real Hurwitz-stable polynomials is open; hence we may perturb $a_{2k+\tau}$ and thus obtain (8) in strict form. The inequality (8) implies the weaker consequence (9) by (7).

Exploiting the second set of inequalities in Th.2 for g_ε/h, and passing to the limit we obtain the inequality

$$H_1^{(1)} = s_2 \cdot s_4 - s_1^2 \geqslant 0. \tag{11}$$

The Corollary yields strict lower bounds for $a_{2\cdot j+\tau}$ of the form (7) for indices $j = k+1$ and $j = k+2$ resp. if $k < \mu - 2$. After multiplication of these lower bounds we obtain the estimate

$$a_{2k+2+\tau}a_{2k+4+\tau} > (\sigma_{k+1} \cdot \sigma_{k+2})^2 a_{2k+\tau}a_{2k+6+\tau}. \tag{12}$$

With $\phi := \dfrac{a_{2(k+2)+\tau}}{(\sigma_{k+1} \cdot \sigma_{k+2})^2 a_{2k+\tau} a_{2(k+3)+\tau}}$ we have as consequence of (7) for k and $k+1$ the inequalities

$$f_1 = \frac{a_{2(k+2)+\tau}^{3/2}}{\sigma_{k+1}(\sigma_{k+2})^2 \sqrt{a_{2k+\tau}}\, a_{2(k+3)+\tau}} \leqslant \phi a_{2(k+1)+\tau} \leqslant f_1^2.$$

We may express (12) as an equality using a suitable multiplier $f \geqslant 1$ for the right-hand side of (12), and may re-order to obtain

$$\frac{1}{f}\,\frac{k+1}{k+3}\,\frac{\mu-(k+2)}{\mu-k}\,\frac{a_{2(k+1)+\tau}a_{2(k+2)+\tau}}{a_{2k+\tau}} = a_{2(k+3)+\tau}. \tag{13}$$

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We note here that $a_{2(k+1)+\tau} \cdot \phi = f$. We obtain a three coefficient expression involving just $a_{2k+\tau}, a_{2k+2+\tau}$ and $a_{2k+4+\tau}$ from substitution of (13) into (11). This leads to the non-negative expression (where $0 \leqslant k \leqslant \mu - 3$)

$$Q(a_{2(k+1)+\tau}) := -\frac{216}{f}\frac{(k+2)(k+1)^5 a_{2(k+2)+\tau}}{(\mu-k-1)(\mu-k)^5 a_{2k+\tau}^5} \cdot a_{2(k+1)+\tau}^4 + (324 + \frac{360}{f} - \frac{36}{f^2})\frac{(k+2)^2(k+1)^4 a_{2(k+2)+\tau}^2 a_{2(k+1)+\tau}^2}{(\mu-k-1)^2(\mu-k)^4 a_{2k+\tau}^4} - \frac{432(k+2)^3(k+1)^3 a_{2(k+2)+\tau}^3}{(\mu-k)^3(\mu-k-1)^3 a_{2k+\tau}^3}.$$

Equate the expression to zero, solve for $a_{2(k+1)+\tau}$, and obtain the positive lower bound

$$a_{2(k+1)+\tau} \geqslant \sqrt{L(f)}\,\sigma_{k+1}\sqrt{a_{2k+\tau}a_{2(k+2)+\tau}},$$

where $L(f)$ is

$$(9f^2 - 1 + 10f - ((9f^2 - 1 + 10)^2 - 288f)^{0.5})/12f.$$

It is easily verified that $f = 1$ yields $L(f) = 1$, and that the quartic Q becomes negative if $a_{2(k+1)+\tau} \to +\infty$. Using the estimates $f_1 < f = a_{2(k+1)+\tau}\phi < f_1^2$ in place of the unknown $f > 1$, we obtain $L(f) > L(f_1)$. Hence, as claimed in (10), we have

$$a_{2(k+1)+\tau}^2 > L(f_1) \cdot \sigma_{k+1}^2 a_{2k+\tau} a_{2(k+2)+\tau}. \quad \square$$

Theorem 4 gives improved bounds compared to (5).

EXAMPLE. Consider a real, stable interval polynomial

$$P(x) = \sum_{i=0}^{6} a_i x^i \quad \text{with positive coefficients.}$$

Yang's inequalities (5) with $N := \lfloor n/2 \rfloor = 3$ for the four coefficients of even index yield

$$a_2 \geqslant \sqrt{\frac{2N}{N-1}}\sqrt{a_0 a_4}, \quad \text{and} \quad a_4 \geqslant \sqrt{\frac{3}{2}\frac{N-1}{N-2}}\sqrt{a_2 a_6}.$$

Suppose that a_2 and a_0 are allowed to vary in some finite interval. Let $a_4 = 1600$ and $a_6 = 1$. If $a_2 \leqslant 6500$, Yang's inequalities yield the estimate $a_0 \leqslant 8802.08$. The new estimate (8) of Th. 4 yields

$$a_0 \leqslant 6614.15,$$

while the true value is slightly smaller than 6609.968.

If the admissible lower bound for a_2 is unknown, and $a_0 \geqslant 5000$, then Yang's inequalities yield $a_2 \geqslant 4898.97$. The term f_1 used in Theorem 4 is approx. $f_1 = 174.186$, the new inequality yields

$$a_2 \geqslant 5653.24.$$

This is close to the lowest possible value of approx. 5653.726. Hence, before any test of Hurwitz-stability of an interval family

$$P(x) = x^6 + a_5x^5 + 1600x^4 + a_3x^3 + [5000,6500]x^2 + a_1x^1 + [5000,8000]$$

we may exclude certain coefficient values, and limit the test to

$$P(x) = x^6 + a_5x^5 + 1600x^4 + a_3x^3 + [5653,6500]x^2 + a_1x^1 + [5000,6615].$$

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