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MULTIPLICATIVE SOBOLEV INEQUALITIES OF THE LADYZHENSKAYA TYPE

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Abstract. We present a set of multiplicative Sobolev inequalities that yield less restrictive embedding and Trudinger-type results. The proofs are elementary and deduced only from a multiplicative inequality of Gagliardo-Nirenberg and Hölder's inequality with no recourse to any potential estimates.

1. Introduction

As for the Sobolev inequality $\|u\|_{p_*} \leqslant C(n,p)\|\nabla u\|_p$ for $u \in W^{1,p}(\mathbb{R}^n)$, if we apply it to the functions $u_t(x) = t^{-n}u(x/t)$ for $t > 0$, then it is plain to see that the condition $1/p_* = 1/p - 1/n$ and $1 \leqslant p < n$ arises necessarily. A more flexible Sobolev inequality may be obtainable in the form

$$\|u\|_q \leqslant C(n,p)\,\|u\|_p^{\alpha}\,\|\nabla u\|_p^{\beta} \quad (\alpha, \beta > 0) \tag{1}$$

for the necessary condition $1 - 1/q = (\alpha+\beta)(1-1/p) + \beta/n$ would be much less restrictive. For instance, when $\alpha + \beta = 1$, it follows from interpolating two end-points p, p_* that (1) holds for all q with $0 \leqslant 1/p - 1/q \leqslant 1/n$.

In this note we consider Sobolev inequalities of type

$$\|u\|_q \leqslant C(n,p)\,\|u\|_p^{\alpha} \prod_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|_p^{\frac{\beta}{n}} \quad (\alpha, \beta > 0), \tag{2}$$

which may be regarded as a sharper version of (1) in view of

$$\prod_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|_p^{\frac{1}{n}} \leqslant \left(\frac{1}{n}\sum_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|_p^{p}\right)^{1/p} \leqslant n^{-1/p}\|\nabla u\|_p .$$

Perhaps best known is an inequality of Gagliardo-Nirenberg ([3], [4])

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\,dx \leqslant \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \prod_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|_1^{\frac{1}{n-1}} . \tag{3}$$

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Multiplicative inequalities of type (2) have been also proven to be useful in the study of non-linear differential equations of mathematical physics. For example, O. A. Ladyzhenskaya [4] has used the following inequalities in studying boundary value problems for Navier-Stokes equations:

$$\|u\|_4^4 \leqslant \|u\|_2^2 \prod_{j=1}^{2} \left\| \frac{\partial u}{\partial x_j} \right\|_2 \quad \text{for all} \quad u \in W^{1,2}(\mathbb{R}^2), \tag{4}$$

$$\|u\|_4^4 \leqslant \|u\|_2 \prod_{j=1}^{3} \left\| \frac{\partial u}{\partial x_j} \right\|_2 \quad \text{and} \quad \|u\|_6^3 \leqslant \frac{9}{2} \prod_{j=1}^{3} \left\| \frac{\partial u}{\partial x_j} \right\|_2 \tag{5}$$

for all $u \in W^{1,2}(\mathbb{R}^3)$. It would be interesting to extend the results of this kind to all dimensions and $1 \leqslant p < \infty$.

2. Inequalities of Ladyzhenskaya Type

To begin with, we observe the following which come in a straightforward manner from the Gagliardo-Nirenberg inequality (3) and Hölder's inequality.

THEOREM A. *Let* $u \in W^{1,p}(\mathbb{R}^n)$.

(a) *For* $1 \leqslant p < \infty$ *and* $n \geqslant 1$,

$$\|u\|_{\frac{pn}{n-1}} \leqslant \left(\frac{p}{2}\right)^{\frac{1}{p}} \|u\|_p^{1-\frac{1}{p}} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_p^{\frac{1}{pn}}. \tag{6}$$

(b) *For* $1 \leqslant p < n$ *and* $n \geqslant 2$, *if* $p_* = np/(n-p)$, *then*

$$\|u\|_{p_*} \leqslant \frac{p\,(n-1)}{2\,(n-p)} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_p^{\frac{1}{n}}. \tag{7}$$

(c) *For* $p = n$ *and* $n \geqslant 2$, *if* ℓ *is an integer with* $\ell \geqslant n-1$, *then*

$$\|u\|_{\frac{\ell n}{n-1}}^{\ell} \leqslant \frac{\ell!}{(n-1)!\,2^{\ell-n+1}} \|u\|_n^{n-1} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_n^{\frac{\ell-n+1}{n}}. \tag{8}$$

Proof. By the inequality (3) and Hölder's inequality, we have

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{\gamma n}{n-1}} dx \leqslant \left(\frac{\gamma}{2}\right)^{\frac{n}{n-1}} \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{(\gamma-1)p}{p-1}} dx \right)^{\frac{n(p-1)}{(n-1)p}} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_p^{\frac{1}{n-1}} \tag{9}$$

for any $\gamma > 0$. The inequalities (6) and (7) follow from (9) with the choice of $\gamma = p$ and $\gamma n/(n-1) = (\gamma-1)p/(p-1)$, respectively. If we let

$$I_\ell = \int_{\mathbb{R}^n} |u(x)|^{\frac{\ell n}{n-1}} dx \quad (\ell \geqslant n-1),$$

then the inequality (9), applied to $\gamma = \ell$, $p = n$, yields

$$I_\ell \leqslant \left(\frac{\ell}{2}\right)^{\frac{n}{n-1}} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_n^{\frac{1}{n-1}} I_{\ell-1} \quad (\ell \geqslant n), \quad I_{n-1} = \|u\|_n^n$$

and the inequalities of (c) follow from this recursive relation. □

In the special case $p = (n-1)/(n-2)$, it is possible to obtain a slight improvement in the bound of (7) and an additional estimate.

THEOREM B. *Let $u \in W^{1,p}(\mathbb{R}^n)$ for $p = (n-1)/(n-2)$ with $n \geqslant 3$. Then*

$$\|u\|_{p_*} \leqslant \frac{p_*}{2p} \left[\frac{(n-1)^2}{n(n-2)}\right]^{\frac{1}{n}} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_p^{\frac{1}{n}}, \tag{10}$$

$$\|u\|_{p^2} \leqslant \left(\frac{p}{2}\right)^{\frac{n}{p(n-1)}} \|u\|_p^{\frac{1}{(n-1)^2}} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_p^{\frac{1}{(n-1)^2}}. \tag{11}$$

Proof. Let $x = (\hat{x}, x_n)$ and $\gamma > 0$. Applying (3) to the $\hat{x}$-variable function $u^\gamma(\cdot, x_n)$ and then integrating with respect to dx_n, we get

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |u(\hat{x}, x_n)|^{\gamma\frac{n-1}{n-2}} \, d\hat{x}\right] dx_n \leqslant \left(\frac{\gamma}{2}\right)^{\frac{n-1}{n-2}} \left(\int_{\mathbb{R}^{n-1}} \left[\max_{x_n} |u(\hat{x}, x_n)|^{(\gamma-1)(n-1)}\right] d\hat{x}\right)^{\frac{1}{n-2}}$$
$$\cdot \int_{\mathbb{R}} \prod_{j=1}^{n-1} \left(\int_{\mathbb{R}^{n-1}} \left|\frac{\partial u}{\partial x_j}\right|^p d\hat{x}\right)^{\frac{1}{n-1}} dx_n .$$

By using the estimate

$$\max_{x_n} |u(\hat{x}, x_n)|^{(\gamma-1)(n-1)} \leqslant \frac{(\gamma-1)(n-1)}{2} \int_{\mathbb{R}} |u(x)|^{(\gamma-1)(n-1)-1} \left|\frac{\partial u}{\partial x_n}\right| dx_n ,$$

a successive application of Hölder's inequality yields

$$\int_{\mathbb{R}^n} |u(x)|^{\gamma\frac{n-1}{n-2}} \, dx \leqslant \left[\frac{\gamma^{n-1}(\gamma-1)(n-1)}{2^n}\right]^{\frac{1}{n-1}}$$
$$\cdot \left(\int_{\mathbb{R}^n} |u(x)|^{[(\gamma-1)(n-1)-1](n-1)} dx\right)^{\frac{1}{(n-1)(n-2)}} \prod_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|_p^{\frac{1}{n}}. \tag{12}$$

Now the inequalities (10), (11) follow from (12) upon choosing γ so that $\gamma = [(\gamma-1)(n-1)-1](n-2)$ and $[(\gamma-1)(n-1)-1](n-1) = p$, respectively, and simplifying algebra. □

REMARKS.

(i) Each inequality of Theorem A or B serves as an end-point of interpolation with the other end-point p. For instance, if $1 \leqslant p < \infty$ and $p \leqslant q \leqslant pn/(n-1)$, then

$$\|u\|_q \leqslant \left(\frac{p}{2}\right)^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_p^{1-n\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|_p^{\frac{1}{p}-\frac{1}{q}} . \tag{13}$$

(ii) When $n=3$, $p=2$, the inequalities (10), (11) coincide with the Ladyzhenskaya inequalities (4). When $n=p=2$, (6) reduces to (3).

(iii) Given any domain $\Omega \subset \mathbb{R}^n$, both theorems remain valid on $W_0^{1,p}(\Omega)$ with replacing $L^p(\mathbb{R}^n)$ norms by $L^p(\Omega)$ norms, where $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the usual Sobolev norm. Indeed, it is plain to observe that (3) and Theorems A, B hold for each $C_0^\infty(\Omega)$-function because all integrals may be written as integrals over $\mathbb{R}^n$ and the results follow by simple limiting arguments. As a consequence, $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds if (a) $1 \leqslant p < \infty, p \leqslant q \leqslant pn/(n-1)$ or (b) $1 \leqslant p < n, p \leqslant q \leqslant p_*$ or (c) $p=n, q \geqslant n$. Of particular interest is the case (a) with $p > n$ for which the standard result is the embedding $W_0^{1,p}(\Omega) \hookrightarrow C^{1-n/p}(\overline{\Omega})$ due to C. Morrey (see [3], pp. 164). Moreover, these results can be extended to $W^{1,p}(\Omega)$ provided that Ω satisfies certain smoothness condition on the boundary such as the Lipschitz condition or the uniform cone condition (refer to [3], pp. 158, for instance).

3. Inequalities of Trudinger Type

In the special case $p=n$, while $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \geqslant n$, there have been a great deal of literature concerning the substitutes of (ii), Theorem A, or the best bounds since the fundamental work of N. Trudinger ([3], [7]) which states that

$$\int_\Omega \exp\left(\frac{c_1\,|u(x)|}{\|\nabla u\|_n}\right)^{\frac{n}{n-1}} dx \leqslant c_2\,|\Omega|,$$

where the constants c_1, c_2 depend only on n. The key ingredient comes from the Hardy-Littlewood-Sobolev inequality for the Riesz potentials. Using more refined estimates for the Bessel potentials, R. Strichartz [6] extended Trudinger's result to $L^p_{n/p}(\Omega)$ spaces and T. Ozawa [5] extended Strichartz's result to $\mathbb{R}^n$. In the present case, as we are interested in multiplicative inequalities of type (2), an approach by either Riesz or Bessel potentials would not be of much help. We are only able to observe the following which resembles Ozawa's result but with explicit constants and a bit different style.

THEOREM C. *For* $u \in W_0^{1,n}(\Omega)$ *with* $n \geqslant 2$, *let* $0 < \alpha < 2$ *and*

$$\Phi(t) = \sum_{k=n-1}^{\infty} \frac{t^k}{k!}, \quad \Psi(t) = t\,\Phi(t), \quad \lambda = \prod_{j=1}^{n} \left\|\frac{\partial u}{\partial x_j}\right\|_n^{\frac{1}{n}} .$$

Then there exists a constant $C(n,\alpha)$ *such that*

$$\int_\Omega \Psi\left(\frac{\alpha|u(x)|}{\lambda}\right) dx \leqslant \frac{2\,\alpha^n}{(n-1)!(2-\alpha)} \int_\Omega \left(\frac{|u(x)|}{\lambda}\right)^n dx, \tag{14}$$

$$\int_\Omega \left[\Phi\left(\frac{\alpha|u(x)|}{\lambda}\right)\right]^{\frac{n}{n-1}} dx \leqslant C(n,\alpha) \int_\Omega \left(\frac{|u(x)|}{\lambda}\right)^n dx. \tag{15}$$

Proof. The estimates (8) and Hölder's inequality give

$$\int_{\mathbb{R}^n} |u|^{n+k}\, dx \leqslant \frac{(n+k-1)!}{(n-1)!\,2^k} \|u\|_n^n \left(\prod_{j=1}^n \left\|\frac{\partial u}{\partial x_j}\right\|_n^{\frac{1}{n-1}}\right)^k \tag{16}$$

for $k = 0,1,2,\cdots$. The desired inequality (14) follows from this estimates upon summing the resulting power series.

To prove (15), we choose β with $\alpha < \beta < 2$ and put $\gamma = \alpha/\beta$. By discrete Hölder's inequality,

$$\begin{aligned}\left[\Phi\left(\frac{\alpha|u(x)|}{\lambda}\right)\right]^{\frac{n}{n-1}} &\leqslant \sum_{k=n-1}^{\infty} \left[\left(\frac{\beta\,|u(x)|}{\lambda}\right)^k \frac{1}{k!}\right]^{\frac{n}{n-1}} \cdot \left(\sum_{k=n-1}^{\infty} \gamma^{kn}\right)^{\frac{1}{n-1}} \\ &\leqslant \left(\frac{\gamma^{n(n-1)}}{1-\gamma^n}\right)^{\frac{1}{n-1}} \sum_{k=n-1}^{\infty} \left[\left(\frac{\beta}{\lambda}\right)^k \frac{1}{k!}\right]^{\frac{n}{n-1}} \left|u(x)\right|^{\frac{kn}{n-1}}.\end{aligned}$$

Integrating and using the estimates (8), the inequality (15) follows at once upon summing the resulting series. A careful computation shows that we may take

$$C(n,\alpha) = \left[\frac{2}{(n-1)!}\right]^{\frac{n}{n-1}} \frac{\alpha^n}{(1-\gamma^n)^{\frac{1}{n-1}}\left(2^{\frac{n}{n-1}} - \beta^{\frac{n}{n-1}}\right)}. \qquad \square$$

4. An Extension of Gagliardo-Nirenberg Inequality

As all of our results are basically deduced from the Gagliado-Nirenberg inequality (3), it would be meaningful to consider any of its extension. In this note we extend it to the mixed partial derivatives.

LEMMA D. (A. P. Calderón [2]) *For each ordered set* $I \subset \{1,2,\cdots,n\}$ *of the form* $I = \{i_1,\cdots,i_k\}$ *with* $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, *write*

$$x_I = (x_{i_1},\cdots,x_{i_k}) \quad \textit{and} \quad dx_I = dx_{i_1}\cdots dx_{i_k}.$$

Given $1 \leqslant k \leqslant n$, *if* $(f_I(x_I))$ *is a sequence of non-negative measurable functions on* $\mathbb{R}^k$, *where* I *ranges over all ordered subsets of* $\{1,2,\cdots,n\}$ *with* $\#(I) = k$, *then*

$$\int_{\mathbb{R}^n} \left[\prod_{\#(I)=k} f_I(x_I)\right] dx \leqslant \prod_{\#(I)=k} \left[\int_{\mathbb{R}^k} (f_I(x_I))^r\, dx_I\right]^{1/r},$$

where r denotes the binomial coefficient $r=\binom{n-1}{k-1}$.

THEOREM E. *Let* $u\in W^{\ell,1}(\mathbb{R}^n)$ *with* $1\leqslant \ell\leqslant n$. *Then*

$$\|u\|_{\frac{n}{n-\ell}} \leqslant \frac{1}{2^\ell}\prod_{1\leqslant i_1<\cdots<i_\ell\leqslant n}\left\|\frac{\partial^\ell u}{\partial x_{i_1}\cdots\partial x_{i_\ell}}\right\|_1^{\frac{\ell!(n-\ell)!}{n!}} \tag{17}$$

with the usual interpretation for the case $\ell=n$.

Proof. We may assume $u\in C_0^\infty(\mathbb{R}^n)$. For $1\leqslant i_1<\cdots<i_\ell\leqslant n$, the Fundamental Theorem of Calculus shows that

$$|u(x)|\leqslant \frac{1}{2^\ell}\int_{\mathbb{R}^\ell}\left|\frac{\partial^\ell u}{\partial x_{i_1}\cdots\partial x_{i_\ell}}\right| dx_{i_1}\cdots dx_{i_\ell}. \tag{18}$$

If $\ell=n$, this inequality gives the stated result, that is,

$$\|u\|_\infty\leqslant \frac{1}{2^n}\left\|\frac{\partial^n u}{\partial x_1\cdots\partial x_n}\right\|_1. \tag{19}$$

In the case $1\leqslant \ell<n$, if we put $I=\{1,2,\cdots,n\}-\{i_1,\cdots,i_\ell\}$, arranged in an increasing order, and denote by $f_I(x_I)$ the function on the right side of (18), then

$$|u(x)|^{\binom{n}{\ell}\frac{1}{r}}\leqslant\left(\frac{1}{2}\right)^{\binom{n}{\ell}\frac{\ell}{r}}\prod_{\#(I)=n-\ell}\left[f_I(x_I)\right]^{\frac{1}{r}},$$

where $r=\binom{n-1}{n-\ell-1}$. Since $\binom{n}{\ell}\frac{1}{r}=n/(n-\ell)$, integrating and applying Calderón's lemma, we get

$$\begin{aligned}\int_{\mathbb{R}^n}|u(x)|^{\frac{n}{n-\ell}}dx &\leqslant \left(\frac{1}{2}\right)^{\frac{n\ell}{n-\ell}}\prod_{\#(I)=n-\ell}\left[\int_{\mathbb{R}^{n-\ell}}f_I(x_I)\,dx_I\right]^{1/r}\\ &=\left(\frac{1}{2}\right)^{\frac{n\ell}{n-\ell}}\prod_{1\leqslant i_1<\cdots<i_\ell\leqslant n}\left\|\frac{\partial^\ell u}{\partial x_{i_1}\cdots\partial x_{i_\ell}}\right\|_1^{1/r},\end{aligned}$$

which yields the desired inequality (17) after simplifying. □

REMARK. As in the case $W^{1,p}(\mathbb{R}^n)$ or $W_0^{1,p}(\Omega)$, Theorem E may be used in obtaining sharper Sobolev inequalities on $W^{k,p}(\mathbb{R}^n)$ or $W_0^{k,p}(\Omega)$ with $k\geqslant 2$. For example, if $u\in W^{2,2}(\mathbb{R}^2)$, then the inequality (19), applied to u^2, implies

$$\begin{aligned}\|u\|_\infty^2 &\leqslant \frac{1}{2}\left(\left\|\frac{\partial u}{\partial x_1}\right\|_2\left\|\frac{\partial u}{\partial x_1}\right\|_2+\|u\|_2\left\|\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_2\right)\\ &\leqslant \frac{1}{4}\|\nabla u\|_2^2+\frac{1}{2}\left\|\nabla^2 u\right\|_2\\ &\leqslant \frac{3}{4}\|u\|_{W^{2,2}(\mathbb{R}^2)}^2, \end{aligned}\tag{20}$$

which may be regarded as an alternative of the Brézis-Gallouet inequality [1]. While it is common to apply the first-order Sobolev inequalities repeatedly in deriving higher-order Sobolev inequalities in terms of $\left\|\nabla^k u\right\|_p$, such an approach would not be so effective in obtaining the results of the above kind.

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REFERENCES

[1] H. BRÉZIS AND T. GALLOUET, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal., **4** (1980), 677–681.
[2] A. P. CALDERÓN, *An inequality for integrals*, Studia Math., **57** (1976), 275–277.
[3] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-Heidelberg, 1983.
[4] O. A. LADYZHENSKAYA, *The Boundary Value Problems of Mathematical Physics*, Applied Math. Sciences **49**, Springer-Verlag, New York (1985)
[5] T. OZAWA, *On critical cases of Sobolev's inequalities*, J. Funct. Anal., **127** (1995), 259–269.
[6] R. S. STRICHARTZ, *A note on Trudinger's extension of Sobolev's inequalties*, Indiana Univ. Math. J., **21** (1972), 841–842.
[7] N. S. TRUDINGER, *On imbedding into Orlicz spaces and some applications*, J. Math. Mech., **17** (1967), 473–484.

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