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WILLMORE LAGRANGIAN SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACE

SHICHANG SHU AND SANYANG LIU

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Abstract. Let M be an n-dimensional compact Willmore Lagrangian submanifold in a complex projective space CP^n and let S and H be the squared norm of the second fundamental form and the mean curvature of M. Denote by $\rho^2 = S - nH^2$ the non-negative function on M, K and Q the functions which assign to each point of M the infimum of the sectional curvature and Ricci curvature at the point. We prove some integral inequalities of Simons' type for n-dimensional compact Willmore Lagrangian submanifolds in CP^n in terms of ρ^2, K, Q and H and obtain some characterization theorems.

1. Introduction

Let N^{n+p} be an oriented smooth Riemannian manifold of dimension $(n+p)$ and let $x: M \mapsto N^{n+p}$ be an n-dimensional compact submanifold of N^{n+p}. Denote by $h^{\alpha}_{ij}, S, \vec{H}$ and H the second fundamental form, the squared norm of the second fundamental form, the mean curvature vector and the mean curvature of M. We denote by $W(x)$ the Willmore functional on M (see [1], [15], [18]), that is, $W(x) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv$. From [1], [15] and [18], we know that $W(x)$ is an invariant under Moebius (or conformal) transformations of N^{n+p}. The Willmore submanifold was defined by Li [11] and Hu-Li [8], [9], a submanifold is called a Willmore submanifold if it is a extremal submanifold to the Willmore functional. When $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are the Willmore surfaces. In [11] (also see [15], [6]), Li obtained a Willmore equation for Willmore functional in terms of Euclidean geometry. It is very important for study of rigidity and geometry of Willmore submanifold in N^{n+p}.

Let CP^n be a complete connected n-dimensional Kaehler manifold with constant holomorphic sectional curvature 4, we call it the n-dimensional complex projective space. Let $x: M \mapsto CP^n$ be an immersion of an n-dimensional Riemannian manifold M into CP^n. M is called a Lagrangian submanifold if the complex structure J of CP^n carries each tangent space of M into its corresponding normal space. We note that in recent years, due to their backgrounds in mathematical physics, special Lagrangian submanifolds have been extensively studied (see[7], [14] and [17]). Moreover, Hu-Li [8]

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obtained an important Willmore equation and showed that a Lagrangian submanifold is Willmore submanifold if and only if it satisfies the Willmore equation.

For Willmore Lagrangian submanifolds, we have the following examples:

EXAMPLE 1.1. ([8]). Let $RP^n(1)$ be the n-dimensional real projective space with constant sectional curvature 1. $RP^n(1)$ can be isometrically immersed into CP^n as a totally geodesic Lagrangian submanifold of CP^n. From proposition 6.2 in [8], we know that $RP^n(1)$ is a (compact) Willmore submanifold of CP^n.

EXAMPLE 1.2. ([14], [8]). The Clifford torus $T^n \subset CP^n$.

Consider the isometric embedding of (n+1)-torus

$$T^{n+1} : S^1\left(\frac{1}{\sqrt{n+1}}\right) \times \cdots \times S^1\left(\frac{1}{\sqrt{n+1}}\right) \mapsto S^{2n+1}(1) \mapsto C^{n+1},$$

this embedding is Lagrangian in C^{n+1} and it is minimal in $S^{2n+1}(1)$. Since the standard action by S^1 on C^{n+1} restricts to both the above torus T^{n+1} and $S^{2n+1}(1)$, we take the quotients of these. The induced quotient metric on CP^n as the quotient $S^{2n+1}(1)/S^1$ has holomorphic sectional curvature 4. The torus $T^n := T^{n+1}/S^1$ in the CP^n is both Lagrangian and minimal. Since T^n is flat, it follows from proposition 6.2 in [8] that T^n is a Willmore Lagrangian submanifold in CP^n.

It is well known that in the theory of minimal Lagrangian submanifolds in CP^n, Chen and Ogiue [2], Ludden, Okumura and Yano [13], Shen [16] and Li [10] had obtained some important rigidity theorems in terms of the squared norm of the second fundamental form and section curvature of the minimal Lagrangian submanifolds. In this paper, we shall establish the rigidity theorems of Willmore Lagrangian submanifolds in CP^n in terms of the scalar curvature, the Ricci curvature, the sectional curvature and the mean curvature of the Willmore Lagrangian submanifolds.

2. Preliminaries

Let $x : M \mapsto CP^n$ be an n-dimensional Lagrangian submanifold in CP^n. We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{1^*} = Je_1, \cdots, e_{n^*} = Je_n$ in CP^n, such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M, where J is the complex structure of CP^n. Let $\omega_1, \cdots, \omega_{2n}$ is the field of dual frames. We make the following convention on the range of indices:

$$i, j, k, \cdots = 1, \cdots, n; 1^* = n+1, \cdots, n^* = 2n; A, B, C, \cdots = 1, \cdots, n, 1^*, \cdots, n^*.$$

Then the structure equations of CP^n are

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}, \tag{2.3}$$

where J_{AB} are the components of the complex structure $J=\sum J_{AB}e_A\bigotimes e_B$ of CP^n. Let θ_A,θ_{AB} be the restriction of ω_A,ω_{AB} to M. Then $\theta_{i^*}=0$. Taking its exterior derivative and making use of (2.1) and the Cartan lemma, we obtain that

$$\theta_{ik^*}=\sum_j h_{ij}^{k^*}\theta_j,\ \ h_{ij}^{k^*}=h_{ji}^{k^*}. \tag{2.4}$$

Since $x: M\mapsto CP^n$ is Lagrangian, we have for any i,j

$$\langle Je_i,e_j\rangle=0,\ \ \langle e_{i^*},Je_j\rangle=\delta_{ij}. \tag{2.5}$$

Taking exterior derivative of (2.5), we get for any i,j,k

$$h_{ij}^{k^*}=h_{jk}^{i^*}=h_{ik}^{j^*},\ \ \theta_{i^*j^*}=\theta_{ij}. \tag{2.6}$$

If we denote by R_{ijkl} the Riemannian curvature tensor of M, we obtain the Gauss equations

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_{m^*}(h_{ik}^{m^*}h_{jl}^{m^*}-h_{il}^{m^*}h_{jk}^{m^*}), \tag{2.7}$$

$$R_{ik}=(n-1)\delta_{ik}+n\sum_{m^*}H^{m^*}h_{ik}^{m^*}-\sum_{j,m^*}h_{ij}^{m^*}h_{jk}^{m^*}, \tag{2.8}$$

$$n(n-1)R=n(n-1)+n^2H^2-S, \tag{2.9}$$

where $S=\sum\limits_{i,j,k^*}(h_{ij}^{k^*})^2$, $\vec{H}=\sum\limits_{k^*}H^{k^*}e_{k^*}$, $H^{k^*}=\frac{1}{n}\sum\limits_i h_{ii}^{k^*}$, $H=|\vec{H}|$ and R is the normalized scalar curvature of M. The Codazzi equations and the Ricci identities are

$$h_{ijk}^{m^*}=h_{ikj}^{m^*}, \tag{2.10}$$

$$h_{ijkl}^{m^*}-h_{ijlk}^{m^*}=\sum_m h_{mj}^{m^*}R_{mikl}+\sum_m h_{im}^{m^*}R_{mjkl}+\sum_{k^*}h_{ij}^{k^*}R_{k^*m^*kl}. \tag{2.11}$$

The Ricci equations are

$$R_{i^*j^*kl}=\delta_{jl}\delta_{ik}-\delta_{jk}\delta_{il}+\sum_m(h_{km}^{i^*}h_{lm}^{j^*}-h_{km}^{j^*}h_{lm}^{i^*}). \tag{2.12}$$

Combining (2.6) with (2.10), we know $h_{ijl}^{k^*}$ are totally symmetric, that is, for any i,j,k,l

$$h_{ijl}^{k^*}=h_{jlk}^{i^*}=h_{lki}^{j^*}=h_{kij}^{l^*}. \tag{2.13}$$

For the fix index $m^*(n+1\leqslant m^*\leqslant 2n)$, we introduce an operator $\square^{m^*}$ due to Cheng-Yau [4] by

$$\square^{m^*}f=\sum_{i,j}(nH^{m^*}\delta_{ij}-h_{ij}^{m^*})f_{i,j}. \tag{2.14}$$

Since M is compact, the operator $\square^{m^*}$ is self-adjoint if and only if (see [4])

$$\int_M(\square^{m^*}f)g\,dv=\int_M f(\square^{m^*}g)dv, \tag{2.15}$$

where f and g are any smooth functions on M.

The following lemma will be used in order to prove our theorems.

LEMMA 2.1. ([12]). *Let $x: M \mapsto CP^n$ be an n-dimensional $(n \geqslant 2)$ Lagrangian submanifold. Then we have*

$$|\nabla h|^2 \geqslant \frac{3n^2}{n+2}|\nabla^{\perp}\vec{H}|^2, \tag{2.16}$$

where $|\nabla h|^2 = \sum_{i,j,k,m^*}(h^{m^*}_{ijk})^2$, $|\nabla^{\perp}\vec{H}|^2 = \sum_{i,m^*}(H^{m^*}_{,i})^2$.

3. Integral equalities

Define tensors

$$\tilde{h}^{m^*}_{ij} = h^{m^*}_{ij} - H^{m^*}\delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{m^*l^*} = \sum_{i,j}\tilde{h}^{m^*}_{ij}\tilde{h}^{l^*}_{ij}, \quad \sigma_{m^*l^*} = \sum_{i,j}h^{m^*}_{ij}h^{l^*}_{ij}. \tag{3.2}$$

Then the $(n \times n)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{1^*}, \cdots, e_{n^*}$. We set

$$\tilde{\sigma}_{m^*l^*} = \tilde{\sigma}_{m^*}\delta_{m^*l^*}. \tag{3.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}^{m^*}_{kk} = 0, \;\; \tilde{\sigma}_{m^*l^*} = \sigma_{m^*l^*} - nH^{m^*}H^{l^*}, \;\; \rho^2 = \sum_{m^*}\tilde{\sigma}_{m^*} = S - nH^2, \tag{3.4}$$

$$\sum_{i,j,k,m^*} h^{l^*}_{kj}h^{m^*}_{ij}h^{m^*}_{ik} = \sum_{i,j,k,m^*}\tilde{h}^{l^*}_{kj}\tilde{h}^{m^*}_{ij}\tilde{h}^{m^*}_{ik} + 2\sum_{i,j,m^*}H^{m^*}\tilde{h}^{m^*}_{ij}\tilde{h}^{l^*}_{ij} + H^{l^*}\rho^2 + nH^2H^{l^*}. \tag{3.5}$$

From (3.1), (3.4) and (3.5), we may rewrite the Willmore equation of Hu-Li [8] as follows

PROPOSITION 3.1. *A Lagrangian submanifold $x: M \mapsto CP^n$ is Willmore submanifold if and only if for $n+1 \leqslant m^*, l^* \leqslant 2n$*

$$\begin{aligned}\Box^{m^*}(\rho^{n-2}) =&(n-1)\rho^{n-2}\Delta^{\perp}H^{m^*} + 2(n-1)\sum_i(\rho^{n-2})_i H^{m^*}_{,i} \\ &+ (n-1)H^{m^*}\Delta(\rho^{n-2}) + 3(n-1)\rho^{n-2}H^{m^*} \\ &+ \rho^{n-2}\Big(\sum_{l^*}H^{l^*}\tilde{\sigma}_{m^*l^*} + \sum_{i,j,k,l^*}\tilde{h}^{m^*}_{ij}\tilde{h}^{l^*}_{ik}\tilde{h}^{l^*}_{kj}\Big).\end{aligned} \tag{3.6}$$

Setting $f = nH^{m^*}$ in (2.14), we have

$$\Box^{m^*}(nH^{m^*}) = \sum_i(nH^{m^*})(nH^{m^*})_{i,i} - \sum_{i,j}h^{m^*}_{ij}(nH^{m^*})_{i,j}. \tag{3.7}$$

We also have

$$\frac{1}{2}\Delta(nH)^2 = \frac{1}{2}\sum_{m^*,i}[(nH^{m^*})^2]_{i,i} = n^2|\nabla^{\perp}\vec{H}|^2 + \sum_{m^*,i}(nH^{m^*})(nH^{m^*})_{i,i}. \tag{3.8}$$

Therefore, from (3.7) and (3.8), we get

$$\begin{aligned}\sum_{m^*}\Box^{m^*}(nH^{m^*}) &= \frac{1}{2}\Delta(nH)^2 - n^2|\nabla^\perp\vec{H}|^2 - \sum_{i,j,m^*} h_{ij}^{m^*}(nH^{m^*})_{i,j} \\ &= \frac{1}{2}\Delta S + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 - n^2|\nabla^\perp\vec{H}|^2 - \sum_{i,j,m^*} h_{ij}^{m^*}(nH^{m^*})_{i,j}.\end{aligned} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k,m^*}(h_{ijk}^{m^*})^2 + \sum_{i,j,m^*} h_{ij}^{m^*}\Delta h_{ij}^{m^*} \\ &= |\nabla h|^2 + \sum_{i,j,m^*} h_{ij}^{m^*}(nH^{m^*})_{i,j} + \sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk}) \\ &\quad + \sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}R_{l^*m^*jk}.\end{aligned} \tag{3.10}$$

Putting (3.10) into (3.9), we have

$$\begin{aligned}\sum_{m^*}\Box^{m^*}(nH^{m^*}) &= |\nabla h|^2 - n^2|\nabla^\perp\vec{H}|^2 + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 \\ &\quad + \sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk}) + \sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}R_{l^*m^*jk}.\end{aligned} \tag{3.11}$$

Multiplying (3.11) by ρ^{n-2} and taking integration, we have from (2.15) that

$$\begin{aligned}\sum_{m^*}\int_M (nH^{m^*})\Box^{m^*}(\rho^{n-2})dv &= \int_M \rho^{n-2}(|\nabla h|^2 - n^2|\nabla^\perp\vec{H}|^2)dv \\ &\quad + \frac{1}{2}n(n-1)\int_M \rho^{n-2}\Delta H^2 dv - \frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2 dv \\ &\quad + \int_M \rho^{n-2}\sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk})dv \\ &\quad + \int_M \rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}R_{l^*m^*jk}dv.\end{aligned} \tag{3.12}$$

Taking the Willmore equation (3.6) into (3.12) and making use of the following

$$\begin{aligned}\int_M \rho^{n-2}\sum_{m^*} H^{m^*}\triangle^\perp H^{m^*}dv &= \frac{1}{2}\int_M \rho^{n-2}\sum_{m^*}\Delta^\perp(H^{m^*})^2 dv - \int_M \rho^{n-2}\sum_{i,m^*}(H_{,i}^{m^*})^2 dv \\ &= \frac{1}{2}\int_M \rho^{n-2}\Delta H^2 dv - \int_M \rho^{n-2}|\nabla\vec{H}|^2 dv,\end{aligned}$$

$$\begin{aligned}\int_M H^2\Delta(\rho^{n-2})dv &= \sum_{m^*,i}\int_M (H^{m^*})^2(\rho^{n-2})_{i,i}dv = -\sum_{m^*,i}\int_M (\rho^{n-2})_i((H^{m^*})^2)_{,i}dv \\ &= -2\int_M \sum_{m^*} H^{m^*}\sum_i(\rho^{n-2})_i H_{,i}^{m^*}dv,\end{aligned}$$

$$-\frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2 dv = \frac{1}{2}\sum_i \int_M (\rho^2)_i(\rho^{n-2})_i dv = (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv.$$

By a direct calculation, we have

PROPOSITION 3.2. *For any n-dimensional compact Willmore Lagrangian submanifold in CP^n, the following integral equality holds*

$$\begin{aligned}&\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^{\perp}\vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \\ &\quad - 3n(n-1)\int_M \rho^{n-2}H^2 dv \\ &\quad - \int_M \rho^{n-2}\sum_{m^*,l^*} nH^{m^*}(H^{l^*}\tilde{\sigma}_{m^*l^*} + \sum_{i,j,k}\tilde{h}_{ij}^{m^*}\tilde{h}_{ik}^{l^*}\tilde{h}_{kj}^{l^*})dv \\ &\quad + \int_M \rho^{n-2}\sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk})dv \\ &\quad + \int_M \rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}R_{l^*m^*jk}dv = 0.\end{aligned} \tag{3.13}$$

From (2.8), (2.12) and (3.1), we have

$$\begin{aligned}\sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}R_{l^*m^*jk} &= \sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}) \\ &\quad + \sum_{m^*,l^*}\sum_{i,j,k,p} h_{ij}^{m^*}h_{ki}^{l^*}(h_{jp}^{l^*}h_{pk}^{m^*} - h_{kp}^{l^*}h_{pj}^{m^*}) \\ &= \rho^2 - n(n-1)H^2 - \frac{1}{2}\sum_{m^*,l^*,j,k}(\sum_p h_{jp}^{l^*}h_{pk}^{m^*} - \sum_p h_{jp}^{m^*}h_{pk}^{l^*})^2 \\ &= \rho^2 - n(n-1)H^2 - \frac{1}{2}\sum_{m^*,l^*,j,k}(\sum_p \tilde{h}_{jp}^{l^*}\tilde{h}_{pk}^{m^*} - \sum_p \tilde{h}_{jp}^{m^*}\tilde{h}_{pk}^{l^*})^2 \\ &= \rho^2 - n(n-1)H^2 - \frac{1}{2}\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}),\end{aligned} \tag{3.14}$$

where $\tilde{A}_{m^*} := (\tilde{h}_{ij}^{m^*}) = (h_{ij}^{m^*} - H^{m^*}\delta_{ij})$ and $N(A)$ denotes the square of the norm of matrix $A = (a_{ij})$.

From (2.6), (2.7), (3.2), (3.4), (3.5) and (3.14), by a simple and direct calculation, we have

$$\begin{aligned}\sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk}) &= n\rho^2 - \sum_{m^*,l^*}\sum_{i,j,k,l} h_{ij}^{m^*}h_{ij}^{l^*}h_{lk}^{m^*}h_{lk}^{l^*} \\ &\quad + n\sum_{m^*,l^*}\sum_{i,j,k} H^{l^*}h_{kj}^{l^*}h_{ij}^{m^*}h_{ik}^{m^*} + \sum_{m^*,l^*}\sum_{i,j,k,l} h_{ij}^{m^*}h_{ki}^{l^*}(h_{jl}^{l^*}h_{lk}^{m^*} - h_{kl}^{l^*}h_{lj}^{m^*})\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
=&n\rho^2-\sum_{m^*,l^*}\sigma^2_{m^*l^*}+n\sum_{m^*,l^*}\sum_{i,j,k}H^{l^*}\tilde{h}^{l^*}_{kj}\tilde{h}^{m^*}_{ij}\tilde{h}^{m^*}_{ik}+2n\sum_{m^*,l^*}\sum_{i,j}H^{m^*}H^{l^*}\tilde{h}^{m^*}_{ij}\tilde{h}^{l^*}_{ij}\\
&+n\sum_{l^*}(H^{l^*})^2\rho^2+n^2H^2\sum_{l^*}(H^{l^*})^2-\frac{1}{2}\sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})\\
=&n\rho^2-\sum_{m^*,l^*}\tilde{\sigma}^2_{m^*l^*}+nH^2\rho^2+n\sum_{m^*,l^*}\sum_{i,j,k}H^{l^*}\tilde{h}^{l^*}_{kj}\tilde{h}^{m^*}_{ij}\tilde{h}^{m^*}_{ik}\\
&-\frac{1}{2}\sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*}).
\end{aligned}
$$

Putting (3.14) and (3.15) into (3.13), we have

PROPOSITION 3.3. *For any n-dimensional compact Willmore Lagrangian submanifold in CP^n, the following integral equality holds*

$$
\begin{aligned}
&\int_M\rho^{n-2}(|\nabla h|^2-n|\nabla^{\perp}\vec{H}|^2)dv+(n-2)\int_M\rho^{n-2}|\nabla\rho|^2dv \\
&\quad-4n(n-1)\int_M\rho^{n-2}H^2dv+n\int_M\rho^{n-2}(H^2\rho^2-\sum_{m^*,l^*}H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})dv\\
&\quad+(n+1)\int_M\rho^n dv-\int_M\rho^{n-2}\sum_{m^*,l^*}(N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})+\tilde{\sigma}^2_{m^*l^*})dv=0.
\end{aligned}
\tag{3.16}
$$

4. Rigidity theorems

We shall prove the following rigidity theorems in terms of ρ, the sectional curvature and the Ricci curvature.

THEOREM 4.1. *Let M be an n-dimensional $(n\geqslant 2)$ compact Willmore Lagrangian submanifold in CP^n. Then the following integral inequality holds*

$$
\int_M\rho^{n-2}\left\{\left(\frac{1}{n}-2\right)\rho^4+(n+1)\rho^2-4n(n-1)H^2\right\}dv\leqslant 0.
$$

In particular, if

$$
\left(2-\frac{1}{n}\right)\rho^4-(n+1)\rho^2+4n(n-1)H^2\leqslant 0, \tag{4.1}
$$

then (i) $n=2$, M is totally geodesic or $M=S^1\times S^1$; (ii) $n>2$, M is totally umbilical.

Proof. From the well-known algebraic lemma 1 in [5], (3.2) and (3.3), we have

$$
\begin{aligned}
&-\sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})-\sum_{m^*,l^*}\tilde{\sigma}^2_{m^*l^*}\\
&\quad\geqslant-\sum_{m^*}\tilde{\sigma}^2_{m^*}-2\sum_{m^*\neq l^*}\tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*}=-2(\sum_{m^*}\tilde{\sigma}_{m^*})^2+\sum_{m^*}\tilde{\sigma}^2_{m^*}\\
&\quad\geqslant-2\rho^4+\frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2=-(2-\frac{1}{n})\rho^4,
\end{aligned}
\tag{4.2}
$$

We also have

$$\sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*} = \sum_{m^*}(H^{m^*})^2\tilde{\sigma}_{m^*} \leqslant \sum_{m^*}(H^{m^*})^2\sum_{l^*}\tilde{\sigma}_{l^*} = H^2\rho^2. \tag{4.3}$$

By making use of lemma 2.1, (3.16), (4.2) and (4.3), we have

$$\begin{aligned} 0 \geqslant & \int_M \rho^{n-2}\left(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^{\perp}\vec{H}|^2\right)dv + \int_M \rho^{n-2}\left(\frac{3n^2}{n+2} - n\right)|\nabla^{\perp}\vec{H}|^2 dv \\ & - 4n(n-1)\int_M \rho^{n-2}H^2 dv + (n+1)\int_M \rho^n dv - \int_M \rho^{n-2}\left(2 - \frac{1}{n}\right)\rho^4 dv \\ \geqslant & \int_M \rho^{n-2}\{\left(\frac{1}{n} - 2\right)\rho^4 + (n+1)\rho^2 - 4n(n-1)H^2\}dv. \end{aligned} \tag{4.4}$$

(i) If $n = 2$, from (4.1) and (4.4), we have $\frac{3}{2}\rho^4 - 3\rho^2 + 8H^2 = 0$ on M. If $\rho^2 = 0$, then $H = 0$ on M, we infer that $S = 0$ and M is totally geodesic. If $\rho^2 \neq 0$, from $\frac{3}{2}\rho^4 - 3\rho^2 + 8H^2 = 0$, we know that the equalities in (4.4) hold. Therefore, we have

$$N(\tilde{A}_3\tilde{A}_4 - \tilde{A}_4\tilde{A}_3) = 2N(\tilde{A}_3)N(\tilde{A}_4), \;\; 2(\tilde{\sigma}_3^2 + \tilde{\sigma}_4^2) = (\tilde{\sigma}_3 + \tilde{\sigma}_4)^2. \tag{4.5}$$

Thus, we have

$$\tilde{\sigma}_3 = \tilde{\sigma}_4. \tag{4.6}$$

For $m^*, l^* = 3, 4$, we also have

$$\sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*} = H^2\rho^2. \tag{4.7}$$

From lemma 1 in [5], we know that at most two of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*}), m^* = 3, 4$, are different from zero. If all of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*})$ are zero, then it is in contradiction with M is not totally umbilical. If only one of them, say $\tilde{A}_{m^*}$, is different from zero, then it is in contradiction with (4.6). Therefore, we may assume that

$$\tilde{A}_3 = \lambda\tilde{A}, \;\; \tilde{A}_4 = \mu\tilde{B}, \;\; \lambda, \mu \neq 0,$$

where $\tilde{A}$ and $\tilde{B}$ are defined by lemma 1 in [5].

From (4.7), we have

$$\lambda^2(H^3)^2 + \mu^2(H^4)^2 = (\lambda^2 + \mu^2)((H^3)^2 + (H^4)^2).$$

Since λ, $\mu \neq 0$, we infer that $H^3 = H^4 = 0$, that is, $\vec{H} = 0$ and M is a minimal Lagrangian submanifold in CP^2, we have $\rho^2 = 2$ and $S = 2$ on M. From the theorem of Ludden, Okumura and Yano [13], we know that $M = S^1 \times S^1$.

(ii) If $n > 2$, from (4.1) and (4.4), we have $\rho = 0$ on M, that is, M is totally umbilical, or $(\frac{1}{n} - 2)\rho^4 + (n+1)\rho^2 - 4n(n-1)H^2 = 0$. In the latter case, if $\rho^2 = 0$ on M, we have M is totally umbilical. If $\rho^2 \neq 0$, we know that the equalities in (4.4) hold. By the same assertion as in the proof of (i), we know that M is a minimal Lagrangian submanifold in CP^n and $S=(n+1)/(2-\frac{1}{n})$ on M. From the theorems of Chen and Ogiue [2], Ludden, Okumura and Yano [13], we know that $n=2$ and $M=S^1\times S^1$. This is in contradiction with $n>2$. This completes the proof of theorem 4.1. □

From (3.13), (3.14), (3.15) and (3.16), we know that for any real number a, the following integral equality holds

$$\begin{aligned}&\int_M \rho^{n-2}(|\nabla h|^2-n|\nabla^{\perp}\vec{H}|^2)dv+(n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \\ &-4n(n-1)\int_M \rho^{n-2}H^2dv+n\int_M \rho^{n-2}(H^2\rho^2-\sum_{m^*,l^*}H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})dv \\ &-(a+1)n\int_M H^2\rho^n dv+(1+a)\int_M \rho^{n-2}\sum_{m^*}\sum_{i,j,k,l}h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk}+h_{li}^{m^*}R_{lkjk})dv \\ &-(1+a)n\int_M \rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k}H^{m^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{ik}^{l^*}\tilde{h}_{kj}^{l^*}dv-(an-1)\int_M \rho^n dv \\ &+a\int_M \rho^{n-2}\sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 dv-\frac{1-a}{2}\int_M \rho^{n-2}\sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})dv=0.\end{aligned} \tag{4.8}$$

Denote by K the function which assigns to each point of M the infimum of the sectional curvature at that point. We have

THEOREM 4.2. *Let M be an n-dimensional $(n\geqslant 2)$ compact Willmore Lagrangian submanifold in CP^n. Then the following integral inequality holds*

$$\int_M \rho^{n-2}\left\{((2n-1)\left(K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-H^2\right)-(n-2))\rho^2-4n(n-1)H^2\right\}dv\leqslant 0.$$

In particular, if

$$\left((2n-1)\left(K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-H^2\right)-(n-2)\right)\rho^2-4n(n-1)H^2\geqslant 0, \tag{4.9}$$

then (i) $n=2$, M is totally geodesic or $M=S^1\times S^1$; (ii) $n>2$, M is totally umbilical.

Proof. For a fixed m^*, $n+1\leqslant m^*\leqslant 2n$, we take a local orthonormal frame field $\{e_1,\cdots,e_n\}$ such that $h_{ij}^{m^*}=\lambda_i^{m^*}\delta_{ij}$, then $\tilde{h}_{ij}^{m^*}=\mu_i^{m^*}\delta_{ij}$ with $\mu_i^{m^*}=\lambda_i^{m^*}-H^{m^*}$, $\sum_i \mu_i^{m^*}=0$. Thus, we have

$$\sum_{m^*,i,j,k,l}h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk}+h_{li}^{m^*}R_{lkjk})=\frac{1}{2}\sum_{m^*,i,j}(\mu_i^{m^*}-\mu_j^{m^*})^2R_{ijij}\geqslant nK\rho^2, \tag{4.10}$$

and the equality in (4.10) holds if and only if $R_{ijij}=K$ for any $i\neq j$.

Let $\sum_i(\tilde{h}_{ii}^{l^*})^2=\tau_{l^*}$. Then $\tau_{l^*}\leqslant\sum_{i,j}(\tilde{h}_{ij}^{l^*})^2=\tilde{\sigma}_{l^*}$. Since $\sum_i\tilde{h}_{ii}^{l^*}=0$, $\sum_i\mu_i^{m^*}=0$ and $\sum_i(\mu_i^{m^*})^2=\tilde{\sigma}_{m^*}$, from the algebraic lemmas in [3] (see lemma 3.3 and lemma 3.4 in

[3]), we have

$$\sum_{m^*,l^*}\sum_{i,j,k} H^{m^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{kj}^{l^*}\tilde{h}_{ik}^{l^*} = \sum_{l^*,m^*}\sum_{i,j,k} H^{l^*}\tilde{h}_{ij}^{l^*}\tilde{h}_{kj}^{m^*}\tilde{h}_{ik}^{m^*} = \sum_{m^*,l^*} H^{l^*}\sum_i \tilde{h}_{ii}^{l^*}(\mu_i^{m^*})^2 \tag{4.11}$$
$$\leqslant \frac{n-2}{\sqrt{n(n-1)}}\sum_{m^*,l^*}|H^{l^*}|\tilde{\sigma}_{m^*}\sqrt{\tau_{l^*}}$$
$$\leqslant \frac{n-2}{\sqrt{n(n-1)}}\sum_{m^*}\tilde{\sigma}_{m^*}\sum_{l^*}|H^{l^*}|\sqrt{\tilde{\sigma}_{l^*}}$$
$$\leqslant \frac{n-2}{\sqrt{n(n-1)}}\rho^2\sqrt{\sum_{l^*}(H^{l^*})^2\sum_{l^*}\tilde{\sigma}_{l^*}} = \frac{n-2}{\sqrt{n(n-1)}}H\rho^3.$$

From (3.3), we get

$$\sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 = \sum_{m^*}\tilde{\sigma}_{m^*}^2 \geqslant \frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2 = \frac{1}{n}\rho^4. \tag{4.12}$$

From lemma 1 in [5], (3.2) and (3.3), we have

$$\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) \leqslant 2\sum_{m^*\neq l^*}\tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = 2(\sum_{m^*}\tilde{\sigma}_{m^*})^2 - 2\sum_{m^*}\tilde{\sigma}_{m^*}^2 \tag{4.13}$$
$$\leqslant 2\rho^4 - 2\frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2 = 2\frac{n-1}{n}\rho^4.$$

Therefore, from (4.3), (4.8), lemma 2.1, (4.10)–(4.13), we obtain that

$$0 \geqslant \int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \tag{4.14}$$
$$- 4n(n-1)\int_M \rho^{n-2}H^2 dv + n\int_M \rho^{n-2}(H^2\rho^2 - \sum_{m^*,l^*}H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})dv$$
$$- (1+a)n\int_M H^2\rho^n dv + (1+a)\int_M \rho^{n-2}nK\rho^2 dv$$
$$- (1+a)n\int_M \rho^{n-2}\frac{n-2}{\sqrt{n(n-1)}}H\rho^3 dv - (an-1)\int_M \rho^n dv$$
$$+ a\int_M \rho^{n-2}\frac{1}{n}\rho^4 dv - (1-a)\int_M \rho^{n-2}\frac{n-1}{n}\rho^4 dv$$
$$\geqslant - 4n(n-1)\int_M \rho^{n-2}H^2 dv + (1+a)n\int_M \rho^n\left(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\right)dv$$
$$- (an-1)\int_M \rho^n dv + \left[\frac{a}{n} - (1-a)\frac{n-1}{n}\right]\int_M \rho^{n+2}dv.$$

Putting $a = \frac{n-1}{n}$, we have

$$0 \geqslant \int_M \rho^{n-2}\left\{\left((2n-1)\left(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\right) - (n-2)\right)\rho^2 - 4n(n-1)H^2\right\}dv. \tag{4.15}$$

(i) If $n=2$, from (4.9) and (4.15), we have $3(K-H^2)\rho^2-8H^2=0$. If $\rho^2=0$ on M, then $H=0$ on M, we infer that M is totally geodesic. If $\rho^2\neq 0$ on M, then we have the equality in (4.15) holds. Therefore, we know that the equalities in (4.14) hold. Thus, the equalities in lemma 1 of [5], (4.12) and (4.13) hold. Since we know that M is not totally umbilical, we have (4.5)–(4.7) hold. By making use of the same assertion as in the proof of theorem 4.1, we infer that M is a minimal Lagrangian submanifold in CP^2 with $K=0$ on M. From the theorem of Shen [16], we have $M=S^1\times S^1$.

(ii) If $n>2$, from (4.9) and (4.15), we have $\rho=0$, that is, M is totally umbilical, or

$$\left((2n-1)\left(K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-H^2\right)-(n-2)\right)\rho^2-4n(n-1)H^2=0.$$

In the latter case, if $\rho^2=0$ on M, we have M is totally umbilical. If $\rho^2\neq 0$ on M, then we have the equality in (4.15) holds. Therefore, we know that the equalities in (4.14) hold. By making use of the same assertion as in the proof of theorem 4.1, we infer that M is a minimal Lagrangian submanifold in CP^n with $K=\frac{n-2}{2n-1}$ on M. From the theorem of Shen [16], we have $n=2$ and $M=S^1\times S^1$. This is in contradiction with $n>2$. This completes the proof of theorem 4.2. □

Denote by Q the function which assigns to each point of M the infimum of the Ricci curvature at that point. We have

LEMMA 4.3. *For any n-dimensional Lagrangian submanifold in CP^n, the following inequality holds*

$$\sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})\leqslant 4\{(n-1)+(n-2)H\rho+H^2-Q\}\rho^2-\frac{4}{n}\rho^4. \tag{4.16}$$

Proof. From Gauss equation (2.8), (3.1) and

$$\sum_{m^*}H^{m^*}h_{ii}^{m^*}\leqslant\sqrt{\sum_{m^*}(H^{m^*})^2}\sqrt{\sum_{m^*}(h_{ii}^{m^*})^2}\leqslant H\rho,$$

we obtain that

$$\sum_{m^*\neq l^*,i}(\tilde{h}_{il}^{m^*})^2\leqslant(n-1)+(n-2)H\rho+H^2-Q-(\tilde{h}_{il}^{m^*})^2. \tag{4.17}$$

Thus, we have

$$\begin{aligned}\sum_{l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})&=\sum_{l^*\neq m^*,i,l}(\tilde{h}_{il}^{l^*})^2(\mu_i^{m^*}-\mu_l^{m^*})^2\leqslant 4\sum_{l^*\neq m^*,i,l}(\tilde{h}_{il}^{l^*})^2(\mu_l^{m^*})^2 \quad (4.18)\\ &\leqslant 4\{(n-1)+(n-2)H\rho+H^2-Q\}\sum_l(\mu_l^{m^*})^2-4\sum_l(\mu_l^{m^*})^4\\ &\leqslant 4\{(n-1)+(n-2)H\rho+H^2-Q\}\sum_l(\mu_l^{m^*})^2-\frac{4}{n}(\sum_l(\mu_l^{m^*})^2)^2.\end{aligned}$$

This completes the proof of lemma 4.3. □

THEOREM 4.4. *Let M be an n-dimensional $(n \geqslant 2)$ compact Willmore Lagrangian submanifold in CP^n. Then the following integral inequality holds*

$$\int_M \rho^{n-2}\left\{\left(\frac{4}{n}-1\right)\rho^4-4\left(\frac{3n-5}{4}+(n-2)H\rho+H^2-Q\right)\rho^2-4n(n-1)H^2\right\}dv \leqslant 0.$$

In particular, if

$$\left(\frac{4}{n}-1\right)\rho^4-4\left(\frac{3n-5}{4}+(n-2)H\rho+H^2-Q\right)\rho^2-4n(n-1)H^2 \geqslant 0, \quad (4.19)$$

then M is totally umbilical.

Proof. From (3.16), lemma 2.1, (3.3), (4.3), lemma 4.3 and

$$\sum_{m^*,l^*}\tilde{\sigma}^2_{m^*l^*} = \sum_{m^*}\tilde{\sigma}^2_{m^*} \leqslant (\sum_{m^*}\tilde{\sigma}_{m^*})^2 = \rho^4, \quad (4.20)$$

we obtain that

$$\begin{aligned} 0 \geqslant& -4n(n-1)\int_M \rho^{n-2}H^2 dv + (n+1)\int_M \rho^n dv \quad (4.21)\\ &- \int_M \rho^{n-2}\left\{4((n-1)+(n-2)H\rho+H^2-Q)\rho^2-\frac{4}{n}\rho^4\right\}dv - \int_M \rho^{n-2}\rho^4 dv\\ =& \int_M \rho^{n-2}\left\{\left(\frac{4}{n}-1\right)\rho^4-4\left(\frac{3n-5}{4}+(n-2)H\rho+H^2-Q\right)\rho^2-4n(n-1)H^2\right\}dv, \end{aligned}$$

From (4.19) and (4.21), we have $\rho = 0$, that is, M is totally umbilical, or

$$\left(\frac{4}{n}-1\right)\rho^4-4\left(\frac{3n-5}{4}+(n-2)H\rho+H^2-Q\right)\rho^2-4n(n-1)H^2 = 0.$$

In the latter case, if $\rho^2 = 0$, then M is totally umbilical; if $\rho^2 \neq 0$, we know that the equalities in (4.21) and (4.20) hold. From $\sum_{m^*}\tilde{\sigma}^2_{m^*} = (\sum_{m^*}\tilde{\sigma}_{m^*})^2$, we have $\sum_{m^*\neq l^*}\tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = 0$. This implies that $(n-1)$ of the $\tilde{\sigma}_{m^*}$ must be zero. Since $\rho^2 = \sum_{m^*,i,j}(\tilde{h}^{m^*}_{ij})^2 \neq 0$ and $\tilde{\sigma}_{m^*} = \sum_{i,j}(\tilde{h}^{m^*}_{ij})^2$, we infer that $(n-1)$ of the $\tilde{A}_{m^*} = (\tilde{h}^{m^*}_{ij})$ must be zero so that $n = 1$. This is in contradiction with $n \geqslant 2$. This completes the proof of Theorem 4.4. □

5. Some related results

Let M be an n-dimensional $(n \geqslant 2)$ compact Willmore Lagrangian submanifold in CP^n with nonnegative sectional curvature. Putting $a = 1$ in (4.8), by (4.3), (4.8), lemma 2.1, (4.10)–(4.12), we have

$$\begin{aligned}0 \geqslant & \int_M \rho^{n-2}\left(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2\right)dv + \int_M \rho^{n-2}\left(\frac{3n^2}{n+2} - n\right)|\nabla^\perp \vec{H}|^2 dv \\ & -4n(n-1)\int_M \rho^{n-2}H^2 dv - 2n\int_M \rho^n H^2 dv + 2\int_M \rho^{n-2} nK\rho^2 dv \\ & -2n\int_M \rho^{n-2}\frac{n-2}{\sqrt{n(n-1)}}H\rho^3 dv - (n-1)\int_M \rho^n dv + \int_M \rho^{n-2}\frac{1}{n}\rho^4 dv \\ \geqslant & \int_M \rho^{n-2}\left\{\frac{1}{n}\rho^4 - 2n\left(H^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho + \frac{n-1}{2n}\right)\rho^2 - 4n(n-1)H^2\right\}dv.\end{aligned} \tag{5.1}$$

Therefore, we have the following

THEOREM 5.1. *Let M be an n-dimensional $(n \geqslant 2)$ compact Willmore Lagrangian submanifold in CP^n with nonnegative sectional curvature. Then the following integral inequality holds*

$$\int_M \rho^{n-2}\left\{\frac{1}{n}\rho^4 - 2n\left(H^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho + \frac{n-1}{2n}\right)\rho^2 - 4n(n-1)H^2\right\}dv \leqslant 0.$$

In particular, if

$$\frac{1}{n}\rho^4 - 2n\left(H^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho + \frac{n-1}{2n}\right)\rho^2 - 4n(n-1)H^2 \geqslant 0, \tag{5.2}$$

then (i) $n=2$, M is totally geodesic or $M = S^1 \times S^1$; (ii) $n>2$, M is totally umbilical or M is an open part of the Clifford torus $T^n \subset CP^n$.

Proof. (i) If $n=2$, from (5.1) and (5.2), we have $\frac{1}{2}\rho^4 - 4(H^2+\frac{1}{4})\rho^2 - 8H^2 = 0$. If $\rho^2 = 0$ on M , then $H=0$ on M, that is, M is totally geodesic. If $\rho^2 \neq 0$ on M, we have the equalities in (5.1) hold. Therefore, the equalities in (4.12) and (4.3) hold. We obtain that $2(\tilde{\sigma}_3^2 + \tilde{\sigma}_4^2) = (\tilde{\sigma}_3 + \tilde{\sigma}_4)^2$, that is,

$$\tilde{\sigma}_3 = \tilde{\sigma}_4, \tag{5.3}$$

and for $m^*, l^* = 3,4$,

$$\sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*} = H^2\rho^2. \tag{5.4}$$

From (5.4), we have

$$(H^3)^2\tilde{\sigma}_3 + (H^4)^2\tilde{\sigma}_4 = ((H^3)^2 + (H^4)^2)(\tilde{\sigma}_3 + \tilde{\sigma}_4),$$

that is,

$$(H^3)^2\tilde{\sigma}_4 + (H^4)^2\tilde{\sigma}_3 = 0. \tag{5.5}$$

From (5.3) and $\rho^2 = \tilde{\sigma}_3 + \tilde{\sigma}_4 \neq 0$, we infer that $\tilde{\sigma}_3 = \tilde{\sigma}_4 \neq 0$. Thus, (5.5) implies that $H^3 = H^4 = 0$, that is, $\vec{H} = 0$ and M is a minimal Lagrangian submanifold in CP^2 with

$\rho^2 = 2$ and $S = 2$ on M. From the theorem of Ludden, Okumura and Yano [13], we know that $M = S^1 \times S^1$.

(ii) If $n > 2$, from (5.1) and (5.2), we have $\rho = 0$, that is M is totally umbilical, or

$$\frac{1}{n}\rho^4 - 2n\left(H^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho + \frac{n-1}{2n}\right)\rho^2 - 4n(n-1)H^2 = 0.$$

In the latter case, if $\rho^2 = 0$ on M, we have M is totally umbilical. If $\rho^2 \neq 0$ on M, then we have the equalities in (5.1) hold. Therefore, we have $\nabla^{\perp}\vec{H} = 0$ and $\nabla h = 0$. This implies that the second fundamental form of M is parallel. We also have the equalities in (4.12) and (4.3) hold, that is, we have

$$\tilde{\sigma}_{n+1} = \cdots = \tilde{\sigma}_{2n}. \tag{5.6}$$

$$\sum_{m^*,l^*} H^{m^*} H^{l^*} \tilde{\sigma}_{m^* l^*} = H^2\rho^2. \tag{5.7}$$

From (5.7) and (3.3), we have $\sum_{m^*}(H^{m^*})^2\tilde{\sigma}_{m^*} = H^2\rho^2$, that is, by (5.6), $H^2\tilde{\sigma}_{n+1} = nH^2\tilde{\sigma}_{n+1}$. Thus, we have $H^2(n-1)\tilde{\sigma}_{n+1} = 0$. Since $\rho^2 \neq 0$ on M, we infer that $\tilde{\sigma}_{n+1} \neq 0$, then we have $H = 0$ and M is a minimal Lagrangian submanifold in CP^n with parallel second fundamental form and $S = n(n-1)$. From the theorem of Li and Zhao [10], we have M is an open part of the Clifford torus $T^n \subset CP^n$. This completes the proof of the Theorem 5.1. □

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Shichang Shu
Department of Mathematics
Xianyang Normal University
Xianyang 712 000 Shaanxi
P. R. China
e-mail: shushichang@126.com

Sanyang Liu
Department of Applied Mathematics
Xidian University
Xi'an 710 071 Shaanxi
P. R. China
e-mail: liusanyang@263.net