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CERTAIN INEQUALITIES FOR CLASSES OF ANALYTIC FUNCTIONS WITH VARYING ARGUMENT OF COEFFICIENTS

J. DZIOK

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Abstract. In this paper we introduce new classes of analytic functions with varying argument of coefficients defined by subordination. Several properties like the coefficients inequalities, distortion bounds, subordination theorems and integral means inequalities are investigated. Some consequences of our main results for new or well-known classes of functions are also pointed out.

1. Introduction

Let $\widetilde{\mathscr{A}}$ denote the class of functions which are *analytic* in $\mathscr{U} = \mathscr{U}(1)$, where

$$\mathscr{U}(r) = \{z \in \mathbf{C} : |z| < r\}.$$

and let $\mathscr{A}$ denote the class of functions $f \in \widetilde{\mathscr{A}}$ normalized by $f(0) = f'(0) - 1 = 0$. Each function $f \in \mathscr{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathscr{U}). \tag{1}$$

Also, by $\mathscr{T}_\eta$ $(\eta \in \mathbb{R})$ we denote the class of functions $f \in \mathscr{A}$ of the form (1) for which all of non-vanishing coefficients satisfy the condition

$$\arg(a_n) = \pi + (1-n)\eta \quad (n = 2, 3, \ldots). \tag{2}$$

For $\eta = 0$ we obtain the class $\mathscr{T}_0$ of functions with negative coefficients.

Moreover, we define

$$\mathscr{T} := \bigcup_{\eta \in \mathbb{R}} \mathscr{T}_\eta.$$

The class $\mathscr{T}$ was introduced by Silverman [16] (see also [22]). It is called the class of functions with varying argument of coefficients.

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We say that a function $f \in \widetilde{\mathscr{A}}$ is *subordinate* to a function $F \in \widetilde{\mathscr{A}}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function $\omega \in \widetilde{\mathscr{A}}$, $|\omega(z)| \leqslant |z| \quad (z \in \mathscr{U})$, such that

$$f(z) = F(\omega(z)) \quad (z \in \mathscr{U}).$$

In particular, if F is univalent in $\mathscr{U}$, we have the following equivalence

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0) \text{ and } f(\mathscr{U}) \subset F(\mathscr{U}).$$

For functions $f, g \in \widetilde{\mathscr{A}}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote *the Hadamard product* (or *convolution*) of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathscr{U}).$$

Let A, B be real parameters, $-1 \leqslant A < B \leqslant 1$, and let $\varphi, \phi \in \mathscr{A}$ be functions of the form

$$\varphi(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n, \quad \phi(z) = z + \sum_{n=2}^{\infty} \beta_n z^n \quad (z \in \mathscr{U}), \tag{3}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are real and

$$0 \leqslant \alpha_n < \beta_n \quad (n = 2, 3, \ldots).$$

Moreover, let us put

$$d_n := (1 + B)\beta_n - (1 + A)\alpha_n \quad (n = 2, 3, \ldots). \tag{4}$$

By $\mathscr{W}(\phi, \varphi; A, B)$ we denote the class of functions $f \in \mathscr{A}$ such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{5}$$

In particular, the classes

$$\mathscr{S}^* := \mathscr{W}\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 1, -1\right),$$

$$\mathscr{S}^c := \mathscr{W}\left(\frac{z(z+1)}{(1-z)^3}, \frac{z}{(1-z)^2}; 1, -1\right),$$

are the well-known classes of starlike functions and convex functions, respectively.

Now, we define the classes of functions with varying argument of coefficients related to the class $\mathscr{W}(\phi,\varphi;A,B)$. Let us denote

$$\mathscr{T}\mathscr{W}(\phi,\varphi;A,B) := \mathscr{T}\cap\mathscr{W}(\phi,\varphi;A,B),$$
$$\mathscr{T}\mathscr{W}_\eta(\phi,\varphi;A,B) := \mathscr{T}_\eta\cap\mathscr{W}(\phi,\varphi;A,B).$$

The object of the present paper is to investigate the coefficients estimates, distortion properties subordination theorems and integral means inequalities for the class $\mathscr{T}\mathscr{W}_\eta(\phi,\varphi;A,B)$. Some remarks depicting consequences of the main results are also mentioned.

2. Coefficients inequalities

We first mention a sufficient condition for a function to belong to the class $\mathscr{W}(\phi,\varphi;A,B)$.

THEOREM 1. *Let $\{d_n\}$ be defined by (4), $-1\leqslant A<B\leqslant 1$. If a function f of the form (1) satisfies the inequality*

$$\sum_{n=2}^{\infty} d_n|a_n|\leqslant B-A, \tag{6}$$

then f belongs to the class $\mathscr{W}(\phi,\varphi;A,B)$.

Proof. A function f of the form (1) belongs to the class $\mathscr{W}(\phi,\varphi;A,B)$ if and only if there exists a function ω, $|\omega(z)|\leqslant|z|$ $(z\in\mathscr{U})$, such that

$$\frac{(\phi*f)(z)}{(\varphi*f)(z)}=\frac{1+A\omega(z)}{1+B\omega(z)}\quad(z\in\mathscr{U}),$$

or equivalently

$$\left|\frac{(\phi*f)(z)-(\varphi*f)(z)}{B(\phi*f)(z)-A(\varphi*f)(z)}\right|<1\quad(z\in\mathscr{U}). \tag{7}$$

Thus, it is sufficient to prove that

$$|(\phi*f)(z)-(\varphi*f)(z)|-|B(\phi*f)(z)-A(\varphi*f)(z)|<0\quad(z\in\mathscr{U}\setminus\{0\}).$$

Indeed, letting $|z|=r$ $(0\leqslant r<1)$ we have

$$\begin{aligned}
&|(\phi*f)(z)-(\varphi*f)(z)|-|B(\phi*f)(z)-A(\varphi*f)(z)|\\
&\quad=\left|\sum_{n=2}^{\infty}(\beta_n-\alpha_n)a_nz^n\right|-\left|(B-A)z-\sum_{n=2}^{\infty}(B\beta_n-A\alpha_n)a_nz^n\right|\\
&\quad\leqslant r\left(\sum_{n=2}^{\infty}(\beta_n-\alpha_n)|a_n|r^{n-1}-(B-A)+\sum_{n=2}^{\infty}(B\beta_n-A\alpha_n)|a_n|r^{n-1}\right)\\
&\quad<\sum_{n=2}^{\infty}d_n|a_n|r^{n-1}-(B-A)\leqslant 0,
\end{aligned}$$

whence $f\in\mathscr{W}(\phi,\varphi;A,B)$. □

THEOREM 2. *Let f be a function of the form (1), with (2). Then f belongs to the class $\mathcal{TW}_{\eta}(\phi,\varphi;A,B)$ if and only if the inequality (6) holds true.*

Proof. In view of Theorem 1 we need only show that each function f from the class $\mathcal{TW}_{\eta}(\phi,\varphi;A,B)$ satisfies the coefficient inequality (6). Let $f\in\mathcal{TW}_{\eta}(\phi,\varphi;A,B)$. Then, by (7) and (1), we have

$$\left|\frac{\sum_{n=2}^{\infty}(\beta_n-\alpha_n)\,a_n z^{n-1}}{B-A-\sum_{n=2}^{\infty}(B\beta_n-A\alpha_n)\,a_n z^{n-1}}\right|<1 \quad (z\in\mathcal{U}).$$

Therefore, putting $z=re^{i\eta}$ $(0\leqslant r<1)$, and applying (2) we obtain

$$\frac{\sum_{n=2}^{\infty}(\beta_n-\alpha_n)\,|a_n|\,r^{n-1}}{B-A-\sum_{n=2}^{\infty}(B\beta_n-A\alpha_n)\,|a_n|\,r^{n-1}}<1.$$

It is clear that the denominator of the left hand said cannot vanish for $r\in\langle 0,1)$. Moreover, it is positive for $r=0$, and in consequence for $r\in\langle 0,1)$. Thus, we have

$$\sum_{n=2}^{\infty}\left[(1+B)\,\beta_n-(1+A)\,\alpha_n\right]|a_n|\,r^{n-1}<B-A,$$

which, upon letting $r\to 1^{-}$, readily yields the assertion (6). □

Since the condition (6) is independent of η, Theorem 2 yields the following theorem.

THEOREM 3. *Let f be a function of the form (1), with (2). Then f belongs to the class $\mathcal{TW}(\phi,\varphi;A,B)$ if and only if the condition (6) holds true.*

From Theorems 2 and 3 we obtain coefficients estimates for the classes $\mathcal{TW}_{\eta}(\phi,\varphi;A,B)$ and $\mathcal{TW}(\phi,\varphi;A,B)$, respectively.

COROLLARY 1. *If a function f of the form (1) belongs to the class $\mathcal{TW}_{\eta}(\phi,\varphi;A,B)$, then*

$$|a_n|\leqslant\frac{B-A}{d_n}\quad (n=2,3,\ldots), \tag{8}$$

where d_n is defined by (4). The result is sharp. The functions $f_{n,\eta}$ of the form

$$f_{n,\eta}(z)=z-\frac{B-A}{d_n}e^{i(1-n)\eta}z^n\quad (z\in\mathcal{U};\ n=2,3,\ldots) \tag{9}$$

are the extremal functions.

COROLLARY 2. *If a function f of the form (1) belongs to the class $\mathcal{TW}(\phi,\varphi;A,B)$, then the coefficients estimates (8) holds true. The result is sharp. The functions $f_{n,\eta}$ of the form (9) $(\eta\in\mathbb{R})$ are the extremal functions.*

3. Distortion bounds

From Theorem 2 we have the following lemma.

LEMMA 1. *Let a function f of the form* (1) *belong to the class $\mathcal{TW}_\eta(\phi,\varphi;A,B)$. If the sequence $\{d_n\}$ defined by* (4) *satisfies the inequality*

$$d_2 \leqslant d_n \quad (n = 2,3,\ldots), \tag{10}$$

then

$$\sum_{n=2}^{\infty} a_n \leqslant \frac{B-A}{d_2}.$$

Moreover, if

$$nd_2 \leqslant 2d_n \quad (n = 2,3,\ldots), \tag{11}$$

then

$$\sum_{n=2}^{\infty} na_n \leqslant \frac{2(B-A)}{d_2}.$$

THEOREM 4. *Let a function f belong to the class $\mathcal{TW}_\eta(\phi,\varphi;A,B)$. If the sequence $\{d_n\}$ defined by* (4) *satisfies* (10)*, then*

$$r - \frac{B-A}{d_2} r^2 \leqslant |f(z)| \leqslant r + \frac{B-A}{d_2} r^2 \quad (|z| = r < 1). \tag{12}$$

Moreover, if (11) *holds, then*

$$1 - \frac{2(B-A)}{d_2} r \leqslant \left|f'(z)\right| \leqslant 1 + \frac{2(B-A)}{d_2} r \quad (|z| = r < 1). \tag{13}$$

The result is sharp, with the extremal function $f_{2,\eta}$ of the form (9)*.*

Proof. Let a function f of the form (1) belong to the class $\mathcal{TW}_\eta(\phi,\varphi;A,B)$, $|z| = r < 1$. Since

$$\begin{aligned} |f(z)| &= \left|z + \sum_{n=2}^{\infty} a_n z^n\right| \leqslant r + \sum_{n=2}^{\infty} |a_n| r^n \\ &= r + r^2 \sum_{n=2}^{\infty} |a_n| r^{n-2} \leqslant r + r^2 \sum_{n=2}^{\infty} |a_n| \end{aligned}$$

and

$$\begin{aligned} |f(z)| &= \left|z + \sum_{n=2}^{\infty} a_n z^n\right| \geqslant r - \sum_{n=2}^{\infty} |a_n| r^n \\ &= r - r^2 \sum_{n=2}^{\infty} |a_n| r^{n-2} \geqslant r - r^2 \sum_{n=2}^{\infty} |a_n|, \end{aligned}$$

then by Lemma 1 we have (12). Analogously we prove (13). □

Theorem 4 implies the following corollary.

COROLLARY 3. *Let a function f belong to the class $\mathcal{TW}(\phi,\varphi;A,B)$. If the sequence $\{d_n\}$ defined by (4) satisfies (10), then the assertion (12) holds true. Moreover, if we assume (11), then then the assertion (12) holds true. The result is sharp, with the extremal functions $f_{2,\eta}\quad(\eta\in\mathbb{R})$ of the form (9).*

4. Subordination results

Before stating and proving our subordination theorems for the classes $\mathcal{TW}_\eta(\phi,\varphi;A,B)$ and $\mathcal{TW}(\phi,\varphi;A,B)$ we need the following definition and lemma:

DEFINITION 1. A sequence $\{b_n\}$ of complex numbers is said to be a subordinating factor sequence if for each function f of the form (1) from the class $\mathcal{S}^c$ we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad (a_1=1). \tag{14}$$

LEMMA 2. [23] *The sequence $\{b_n\}$ is a subordinating factor sequence if and only if*

$$\mathrm{Re}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\}>0 \ \ (z\in\mathcal{U}). \tag{15}$$

THEOREM 5. *Let the sequence $\{d_n\}$, defined by (4), satisfy the inequality (10). If $g\in\mathcal{S}^c$ and $f\in\mathcal{TW}_\eta(\phi,\varphi;A,B)$, then*

$$\varepsilon(f*g)(z)\prec g(z) \tag{16}$$

and

$$\mathrm{Re}\, f(z)>-\frac{1}{2\varepsilon} \ \ (z\in\mathcal{U}), \tag{17}$$

where

$$\varepsilon=\frac{d_2}{2\,(B-A+d_2)}. \tag{18}$$

The constant factor ε cannot be replaced by a larger number.

Proof. Let a function f of the form (1) belong to the class $\mathcal{TW}_\eta(\phi,\varphi;A,B)$ and suppose that

$$g(z)=z+\sum_{n=2}^{\infty} c_n z^n \quad (z\in\mathcal{U})$$

belongs to the class $\mathcal{S}^c$. Then

$$\varepsilon(f*g)(z)=\varepsilon z+\sum_{n=2}^{\infty}(\varepsilon a_n)\,c_n z^n.$$

Thus, by Definition 1 the subordination result (16) holds true if

$$\{\varepsilon a_n\}_{n=1}^{\infty} \quad (a_1=1)$$

is a subordinating factor sequence. In view of Lemma 2, this is equivalent to the following inequality

$$\mathrm{Re}\left\{1+2\sum_{n=1}^{\infty}\varepsilon a_n z^n\right\}>0 \quad (z\in\mathscr{U}). \tag{19}$$

Thus, by (10) for $|z|=r<1$ we have

$$\begin{aligned}\mathrm{Re}\left\{1+2\sum_{n=1}^{\infty}\varepsilon a_n z^n\right\} &= \mathrm{Re}\left\{1+2\varepsilon z+\sum_{n=2}^{\infty}\frac{d_2}{B-A+d_2}a_n z^n\right\}\\ &\geqslant 1-2\varepsilon r-\frac{r}{B-A+d_2}\sum_{n=2}^{\infty}d_n|a_n|r^{n-1},\end{aligned}$$

and consequently by using Theorem 2 we obtain

$$\mathrm{Re}\left\{1+2\sum_{n=1}^{\infty}\varepsilon a_n z^n\right\}\geqslant 1-\frac{d_2}{B-A+d_2}r-\frac{B-A}{B-A+d_2}r>0.$$

This evidently proves the inequality (19) and hence the subordination result (16). The inequality (17) follows from (16) by taking

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n \quad (z\in\mathscr{U}).$$

Next we observe that the function $f_{2,\eta}$ of the form (9) belongs to the class $\mathscr{TW}_\eta(\phi,\varphi;A,B)$. It is easily verified that

$$\min\left\{\mathrm{Re}\left(\varepsilon f_{2,\eta}(z)\right)\right\}=-\frac{1}{2} \quad (z\in\mathscr{U}).$$

This shows that the constant (18) cannot be replaced by any larger one. □

Directly from Theorem 5 we obtain

THEOREM 6. *Let the sequence* $\{d_n\}$, *defined by* (4), *satisfy the inequality* (10). *If* $g\in\mathscr{S}^c$ *and* $f\in\mathscr{TW}(\phi,\varphi;A,B)$, *then conditions* (16) *and* (17) *hold true. The constant factor* ε *in* (16) *cannot be replaced by a larger number.*

5. Integral means inequalities

Due to Littlewood [10] we obtain integral means inequalities for the functions from the class $\mathscr{TW}_\eta(\phi,\varphi;A,B)$.

LEMMA 3. [10]. *Let* $f,g\in\widetilde{\mathscr{A}}$. *If* $f\prec g$, *then*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^\eta d\theta\leqslant\int_0^{2\pi}\left|g(re^{i\theta})\right|^\eta d\theta \quad (0<r<1,\ \eta>0). \tag{20}$$

Silverman [15] found that the function

$$g(z) = z - \frac{z^2}{2} \quad (z \in \mathscr{U}),$$

is often extremal over the family of functions with negative coefficients. He applied this function to resolve integral means inequality, conjectured in [17] and settled in [18], that (20) holds true for all functions f with negative coefficients. In [18] he also proved his conjecture for some subclasses of $\mathscr{T}$.

Applying Lemma 3 and Theorem 2 we prove the following result.

THEOREM 7. *Let the sequence $\{d_n\}$ defined by (4) satisfy the inequality (10). If $f \in \mathscr{T}\mathscr{W}_\eta(\phi, \varphi; A, B)$ then*

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^\lambda d\theta \leqslant \int_0^{2\pi} \left| f_{2,\eta}(re^{i\theta}) \right|^\lambda d\theta \quad (0 < r < 1,\ \lambda > 0), \tag{21}$$

where $f_{2,\eta}(z)$ be defined by (9).

Proof. For function f of the form (1), the inequality (21) is equivalent to the following

$$\int_0^{2\pi} \left| 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\lambda d\theta \leqslant \int_0^{2\pi} \left| 1 + \frac{B-A}{d_2} e^{-i\eta} z \right|^\lambda d\theta.$$

By Lemma 3, it suffices to show that

$$\sum_{n=2}^{\infty} a_n z^{n-1} \prec -\frac{B-A}{d_2} e^{-i\eta} z. \tag{22}$$

Setting

$$w(z) = \sum_{n=2}^{\infty} \frac{d_2 e^{i\eta}}{A-B} a_n\, z^{n-1} \quad (z \in \mathscr{U})$$

and using (10) and Theorem 2 we obtain

$$|w(z)| = \left| \sum_{n=2}^{\infty} \frac{d_2}{A-B} a_n\, z^{n-1} \right| \leqslant |z| \sum_{n=2}^{\infty} \frac{d_n}{A-B} |a_n| \leqslant |z| \quad (z \in \mathscr{U}).$$

Since

$$\sum_{n=2}^{\infty} a_n z^{n-1} = -\frac{B-A}{d_2} e^{-i\eta} w(z) \quad (z \in \mathscr{U}),$$

by definition od subordination we have (22) and this completes the proof. □

We can write Theorem 7 in the following form:

THEOREM 8. *Let the sequence $\{d_n\}$ defined by (4) satisfy the inequality (10). If a function f of the form (1), with (2) belongs to the class $\mathscr{T}\mathscr{W}(\phi, \varphi; A, B)$, then the integral means inequality (21) holds true.*

6. Remarks

Choosing the functions ϕ and φ in the condition (4) we can define new classes of functions. In particular, the class

$$\mathscr{W}_n(\varphi;A,B) := \mathscr{W}\left(z\varphi'(z), \sum_{k=0}^{n-1}\varphi\left(x^k z\right);A,B\right) \quad (x^n = 1)$$

contains functions $f \in A,$ such that

$$\frac{z(\varphi * f)'(z)}{\sum_{k=0}^{n-1}(\varphi * f)(x^k z)} \prec \frac{1+Az}{1+Bz}.$$

It is related to the well-known class of starlike functions with n-symmetric points. Moreover, putting $n = 1$ we obtain the class $\mathscr{W}(\varphi;A,B) = \mathscr{W}_1(\varphi;A,B)$ defined by the following condition

$$\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} \prec \frac{1+Az}{1+Bz}.$$

The class is related to the class of starlike functions.

Let λ be a convex parameter. A function $f \in A$ belongs to the class

$$\mathscr{V}_\lambda(\varphi;A,B) := \mathscr{W}\left(\lambda\frac{\varphi(z)}{z} + (1-\lambda)\varphi'(z), z;A,B\right)$$

if it satisfies the condition

$$\lambda\frac{(\varphi * f)(z)}{z} + (1-\lambda)(\varphi * f)'(z) \prec \frac{1+Az}{1+Bz}.$$

Moreover, a function $f \in A$ belongs to the class

$$\mathscr{U}_\lambda(\varphi;A,B) := \mathscr{W}\left(\lambda\frac{\varphi(z)}{z} + (1-\lambda)\varphi'(z);A,B\right)$$

if it satisfies the condition

$$\frac{z(\varphi * f)'(z) + (1-\lambda)z^2(\varphi * f)''(z)}{\lambda(\varphi * f)(z) + (1-\lambda)z(\varphi * f)'(z)} \prec \frac{1+Az}{1+Bz}. \tag{23}$$

The classes $\mathscr{W}_n(\varphi;A,B), \mathscr{U}_\lambda(\varphi;A,B)$ and $\mathscr{V}_\lambda(\varphi;A,B)$ generalize well-known important classes, which were investigated in earlier works, see for example [1]–[21]. Most of these classes were defined by using linear operators and special functions.

If we apply the results presented in this paper to the classes discussed above, we can obtain a lot of partial results. Some of these results where obtained in earlier works.

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J. Dziok
Institute of Mathematics
University of Rzeszów
ul. Rejtana 16A
PL-35-310 Rzeszów
Poland
e-mail: jdziok@univ.rzeszow.pl