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SOME GENERALIZATIONS OF MARONI'S INEQUALITY ON TIME SCALES

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Abstract. The familiar Opial's inequality is of great interest in differential and difference equations, and is also potentially useful in other areas of mathematical sciences. In an earlier work, we systematically investigated several general Opial-type inequalities on time scales. Here, in this sequel, we propose to present some extensions and generalizations of Maroni's inequality to hold true on time.

1. Introduction and Preliminaries

Almost five decades ago, Opial [14] established an integral inequality which we recall here as Theorem A below (see also a sequel by Olech [13] for a *simpler* proof under *weaker* conditions as well as for the explicit *extremal function*).

THEOREM A. *Let* $f \in C^1[0,a]$ $(a>0)$ *with*

$$f(0)=f(a)=0 \qquad \textit{and} \qquad f(x)>0 \qquad (0<x<a).$$

Then

$$\int_0^a \left|f(x)f'(x)\right| dx \leqq \frac{a}{4}\int_0^a \left|f'(x)\right|^2 dx, \tag{1}$$

where the constant factor $\frac{a}{4}$ *is the best possible. Equality holds true in* (1) *if and only if*

$$f(x)=\begin{cases} cx & \left(0 \leqq x \leqq \frac{a}{2}\right) \\ c(a-x) & \left(\frac{a}{2} \leqq x \leqq a\right), \end{cases}$$

where c *is a positive constant.*

The inequality (1) is well-known in the literature as Opial's inequality which indeed is of great interest in differential and difference equations, and is also potentially useful in other areas of mathematical sciences. For some recent results which generalize, improve and extend this classical inequality (1), see (for example) [1] to [5], [8] to [12], [15], [16], and [21] (see also the edited volume [17]). In particular, Beesack [5] generalized Opial's inequality (1) as follows.

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THEOREM B. *Let*

$$-\infty \leqq a < \tau \leqq \infty.$$

Suppose also that the function $p(x)$ *is positive and continuous on* (a,τ). *If the function* $f(x)$ *is absolutely continuous on* $[a,\tau]$ *with*

$$f(x) = \int_a^x f'(s)\,ds \qquad (x \in [a,\tau]),$$

then

$$\int_a^\tau |f(x) f'(x)|\,dx \leqq \frac{1}{2}\left(\int_a^\tau [p(x)]^{-1}\,dx\right)\left(\int_a^\tau p(x)\,[f'(x)]^2\,dx\right). \tag{2}$$

Subsequently, Maroni [12] *further* generalized Beesack's result (see Theorem B) in a form given by Theorem C below (see also [8] for more general results).

THEOREM C. *Let the function* $p(x)$ *be nonnegative and continuous on* $[a,\tau]$ *with*

$$\int_a^\tau [p(x)]^{1-\mu}\,dx < \infty \qquad (\mu \geqq 1).$$

Also let the function $f(x)$ *be absolutely continuous on* $[a,\tau]$ *and suppose that*

$$f(a) = f(\tau) = 0.$$

Then the following inequality holds true:

$$\int_a^\tau |f(x) f'(x)|\,dx \leqq \frac{1}{2}\left(\int_a^\tau [p(x)]^{1-\mu}\,dx\right)^{\frac{2}{\mu}}\left(\int_a^\tau p(x)\,|f'(x)|^\nu\,dx\right)^{\frac{2}{\nu}}$$
$$\left(\frac{1}{\mu}+\frac{1}{\nu}=1\right). \tag{3}$$

Equality in (3) *holds true only if*

$$f(x) = \begin{cases} \mathfrak{A}\int_a^x [p(t)]^{1-\mu}\,dt & (a \leqq x < c) \\ \mathfrak{B}\int_x^\tau [p(t)]^{1-\mu}\,dt & (c < x \leqq \tau), \end{cases}$$

where $\mathfrak{A}$ *and* $\mathfrak{B}$ *are constants and the numbers* $\mathfrak{K}$ *and* τ *are so constrained that*

$$\mathfrak{K} := \left(\int_a^c [p(x)]^{1-\mu}\,dx\right)^{\frac{2}{\mu}} = \left(\int_c^\tau [p(x)]^{1-\mu}\,dx\right)^{\frac{2}{\mu}}.$$

Agarwal *et al.* [4] extended Theorems A and B to hold true on time scales and obtained the following results.

THEOREM D. *Let the function* $f(x)$ *given by*

$$f : [0,h] \cap \mathbb{T} \to \mathbb{R} \quad (h > 0) \quad \textit{with} \quad f(0) = 0.$$

Then

$$\int_0^h \left|[f(x) + f^\sigma(x)]\,f^\Delta(x)\right|\Delta x \leqq h \int_0^h \left|f^\Delta(x)\right|^2 \Delta x. \tag{4}$$

THEOREM E. *Let each of the functions* $p(x)$ *and* $q(x)$ *be positive and continuous on* $[0,h]\cap\mathbb{T}$ *and suppose that*

$$\int_0^h [p(x)]^{-1}\,\Delta x < \infty.$$

Also let $q(x)$ *be non-increasing on* $[0,h]\cap\mathbb{T}$. *Then, for a delta-differentiable function*

$$f:[0,h]\cap\mathbb{T}\to\mathbb{R} \quad \textit{with} \quad f(0)=0,$$

$$\int_0^h \left[q^{\sigma}(x)\left|[f(x)+f^{\sigma}(x)]\,f^{\Delta}(x)\right|\right]\Delta x \leqq \left(\int_0^h [p(x)]^{-1}\,\Delta x\right)\left(\int_0^h p(x)\,q(x)\left|f^{\Delta}(x)\right|^2\Delta x\right). \tag{5}$$

In our earlier investigation [18], we presented several general weighted Opial-type inequalities on time scales. Some of our results were intended to provide corrections and modifications of several claims made by Wong *et al.* [20]. Here, in this sequel to our earlier paper [18], we propose to provide extensions of Maroni's generalization (see, for example, Theorem C above) to hold true for some cases of time scales.

2. Definitions, Notations and Preliminaries in Time Scales Theory and Delta Differentiability

In this section, we present some definitions, notations and preliminaries concerning the *Time Scales Theory* and the concept of *Delta Differentiability*. These concepts, together with the notion of rd-continuity, were used in Section 1 above (see, for details, [4], [6] and [7]).

DEFINITION 1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, the two most popular examples being

$$\mathbb{T}=\mathbb{R} \qquad \text{and} \qquad \mathbb{T}=\mathbb{Z}:=\{0,\pm1,\pm2,\cdots\}.$$

The *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ and the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ are defined by

$$\sigma(t):=\inf\{\tau \mid \tau\in\mathbb{T} \ \text{ and } \ \tau>t\} \qquad (t\in\mathbb{T};\ t<\sup\{\mathbb{T}\})$$

and

$$\rho(t):=\sup\{\tau \mid \tau\in\mathbb{T} \ \text{ and } \ s<t\} \qquad (t\in\mathbb{T};\ t>\inf\{\mathbb{T}\}),$$

respectively, each of which is being supplemented by

$$\inf\{\varnothing\}=\sup\{\mathbb{T}\} \qquad \text{and} \qquad \sup\{\varnothing\}=\inf\{\mathbb{T}\}.$$

Furthermore, a point $t\in\mathbb{T}$ is called *right-scattered*, *right-dense*, *left-scattered* or *left-dense* if

$$\sigma(t)>t,\ \ \sigma(t)=t,\ \ \rho(t)<t \quad \text{or} \quad \rho(t)=t,$$

respectively.

DEFINITION 2. Let the time scale $\mathbb{T}$ have a right-scattered minimum $\mathfrak{m}$. Then we define the set $\mathbb{T}^{\kappa}$ by

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus \{\mathfrak{m}\} & (\mathfrak{m} \quad \text{exists}) \\ \mathbb{T} & (\mathfrak{m} \quad \text{does not exist}). \end{cases} \tag{6}$$

On the other hand, if the time scale $\mathbb{T}$ has a left-scattered maximum $\mathfrak{M}$, then we define the set $\mathbb{T}^{\kappa}$ by

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus \{\mathfrak{M}\} & (\mathfrak{M} \quad \text{exists}) \\ \mathbb{T} & (\mathfrak{M} \quad \text{does not exist}). \end{cases} \tag{7}$$

Moreover, the *forward graininess* $\mu : \mathbb{T} \to [0,\infty)$ is defined by

$$\mu(t) := \sigma(t) - t \quad (t \in \mathbb{T}) \quad = \begin{cases} 0 & (\mathbb{T} = \mathbb{R}) \\ 1 & (\mathbb{T} = \mathbb{Z}) \end{cases}$$

and the *backward graininess* $\nu : \mathbb{T} \to [0,\infty)$ is defined by

$$\nu(t) := t - \rho(t) \quad (t \in \mathbb{T}) \quad = \begin{cases} 0 & (\mathbb{T} = \mathbb{R}) \\ 1 & (\mathbb{T} = \mathbb{Z}) \end{cases}$$

DEFINITION 3. A mapping $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if

$$1 + \mu(t)\mathfrak{f}(t) \neq 0 \qquad (t \in \mathbb{T}).$$

Furthermore, if $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$, then the mapping $\mathfrak{f}^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined by

$$\mathfrak{f}^{\sigma}(t) := \mathfrak{f}\big(\sigma(t)\big) \qquad (t \in \mathbb{T}),$$

where $\sigma(t)$ is given in Definition 1 above.

DEFINITION 4. A mapping $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ is said to be rd-*continuous* if it satisfies each of the following conditions:

(i) f is continuous at every right-dense point or maximal point of $\mathbb{T}$;

(ii) The left-sided limit:

$$\lim_{\tau \to t-} \mathfrak{f}(\tau) = \mathfrak{f}(t-)$$

exists at every left-dense point of $\mathbb{T}$.

Just as in Equation (6) above, the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ is denoted as follows:

$$C_{\text{rd}}(\mathbb{T},\mathbb{R}) := \{\mathfrak{f} \mid \mathfrak{f} : \mathbb{T} \to \mathbb{R} \text{ and } \mathfrak{f}(t) \text{ is an rd-continuous function}\}.$$

DEFINITION 5. Assume that $\mathfrak{f}:\mathbb{T}\to\mathbb{R}$. Then we define $\mathfrak{f}^{\Delta}(t)$ to be the number (if it exists) with the property that, for any given $\varepsilon>0$, there is a neighborhood $\mathscr{N}$ of t such that

$$\left|\mathfrak{f}\big(\sigma(t)\big)-\mathfrak{f}(\tau)-\mathfrak{f}^{\Delta}(t)\,[\sigma(t)-\tau]\right| \leqq \varepsilon\,|\sigma(t)-\tau| \qquad (\tau\in\mathscr{N}).$$

In this case, we say that $\mathfrak{f}^{\Delta}(t)$ is the *delta derivative* of $\mathfrak{f}(t)$ at the point $t\in\mathbb{T}^{\kappa}$. If $\mathfrak{f}$ is delta differentiable for every $t\in\mathbb{T}^{\kappa}$, then $\mathfrak{f}$ is delta differentiable on $\mathbb{T}$ and $\mathfrak{f}^{\Delta}(t)$ is a new function defined on $\mathbb{T}^{\kappa}$.

If $\mathfrak{f}$ is delta differentiable at $t\in\mathbb{T}^{\kappa}$, then it is easily seen that

$$\mathfrak{f}^{\Delta}(t)=\begin{cases}\displaystyle\lim_{\tau\to t\ (\tau\in\mathbb{T})}\frac{\mathfrak{f}(t)-\mathfrak{f}(\tau)}{t-\tau} & \big(\mu(t)=0\big)\\[2ex] \displaystyle\frac{\mathfrak{f}\big(\sigma(t)\big)-\mathfrak{f}(t)}{\mu(t)} & \big(\mu(t)>0\big).\end{cases} \tag{8}$$

Several useful delta derivative formulas are recorded here under Lemma 1 below.

LEMMA 1. *The above-defined delta derivatives satisfy each of the following properties:*

$$f\big(\sigma(t)\big)=f(t)+\mu(t)f^{\Delta}(t), \tag{9}$$

$$\big(f(t)g(t)\big)^{\Delta}=f^{\Delta}(t)g(t)+f\big(\sigma(t)\big)g^{\Delta}(t) \tag{10}$$

and

$$\left(\frac{f(t)}{g(t)}\right)^{\Delta}=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g\big(\sigma(t)\big)}. \tag{11}$$

Lemma 2 below is an easy consequence of the property (9) asserted by Lemma 1.

LEMMA 2. *If $f:\mathbb{T}\to\mathbb{R}$ is* rd*-continuous at $t\in\mathbb{T}$ and t is right-scattered, then*

$$f^{\Delta}(t)=\frac{f\big(\sigma(t)\big)-f(t)}{\sigma(t)-t}. \tag{12}$$

DEFINITION 6. A function $\mathfrak{F}:\mathbb{T}\to\mathbb{R}$ is said to be an *antiderivative* of $\mathfrak{f}:\mathbb{T}\to\mathbb{R}$ if

$$\mathfrak{F}^{\Delta}(t)=\mathfrak{f}(t) \qquad (t\in\mathbb{T}^{\kappa}).$$

In this case, we define the integral of $\mathfrak{f}$ by

$$\int_s^t \mathfrak{f}(\tau)\Delta\tau=\mathfrak{F}(t)-\mathfrak{F}(s) \qquad (s,t\in\mathbb{T}) \tag{13}$$

and we say that $\mathfrak{f}$ is integrable on $\mathbb{T}$.

The results asserted by Lemma 3 below are rather immediate consequences of (10) and (13).

LEMMA 3. *Each of the following integral formulas holds true:*

$$\left(\int_a^t f(\tau)\Delta\tau\right)^{\Delta} = f(t) \tag{14}$$

and

$$\int_a^t f(\tau)g^{\Delta}(\tau)\Delta\tau = f(\tau)g(\tau)\Big|_{\tau=a}^{t} - \int_a^t f^{\Delta}(\tau)g\big(\sigma(\tau)\big)\Delta\tau \tag{15}$$

for any constant $a \in \mathbb{T}$.

3. The Main Maroni-Type Inequalities on Time Scales

We begin this section by proving a Maroni-type inequality on time scales, which is asserted by Theorem 1 below.

THEOREM 1. *For* $\tau \in [a,b] \cap \mathbb{T}$, *let* $p(x) \in C_{\mathrm{rd}}([a,\tau],\mathbb{R})$ *with*

$$\int_a^{\tau} [p(x)]^{1-\mu}\ \Delta x < \infty \qquad (\mu > 1).$$

Let the function $q(x)$ *be positive, bounded and non-increasing on* $[a,\tau] \cap \mathbb{T}$. *Suppose also that the function* $f(x)$ *is delta differentiable on* $[a,\tau] \cap \mathbb{T}$ *and* $f(a) = 0$. *Then, for* $r > 0$, *the following inequality holds true:*

$$\begin{aligned}\int_a^{\tau} q(x)\,|f(x)|^r \left|f^{\Delta}(x)\right| \Delta x \leqq \frac{1}{1+r}\left(\int_a^{\tau} [p(x)]^{1-\mu}\ \Delta x\right)^{\frac{1+r}{\mu}} \\ \cdot \left(\int_a^{\tau} p(x)\,[q(x)]^{\frac{\nu}{1+r}}\ \left|f^{\Delta}(x)\right|^{\nu} \Delta x\right)^{\frac{1+r}{\nu}} \\ \left(\frac{1}{\mu} + \frac{1}{\nu} = 1\right).\end{aligned} \tag{16}$$

Proof. Let

$$y(x) = \int_a^x [q(s)]^{\frac{1}{1+r}}\ \left|f^{\Delta}(s)\right| \Delta s. \tag{17}$$

Then

$$\begin{aligned}[y(x)]^r &= \left(\int_a^x \left[q^{\frac{1}{1+r}}(s)\right]^{\frac{1}{1+r}}\ \left|f^{\Delta}(s)\right| \Delta s\right)^r \\ &\geqq [q(x)]^{\frac{r}{1+r}}\ \left(\int_a^x \left|f^{\Delta}(s)\right| \Delta s\right)^r \\ &\geqq [q(x)]^{\frac{r}{1+r}}\ \big[\,|f(x)|\,\big]^r\end{aligned} \tag{18}$$

and

$$y^{\Delta}(x) = [q(x)]^{\frac{1}{1+r}}\ \left|f^{\Delta}(x)\right|. \tag{19}$$

Therefore, we have

$$\begin{aligned}
&\int_a^\tau q(x)\,|f(x)|^r \left|f^\Delta(x)\right| \Delta x \\
&\quad = \int_a^\tau [q(x)]^{\frac{1}{1+r}}\ [q(x)]^{\frac{r}{1+r}}\ |f(x)|^r \left|f^\Delta(x)\right| \Delta x \\
&\quad \leqq \int_a^\tau [y(x)]^r\ y^\Delta(x)\,\Delta x \\
&\quad \leqq \frac{1}{1+r}\ [y(\tau)]^{1+r} = \frac{1}{1+r}\left(\int_a^\tau [q(x)]^{\frac{1}{1+r}}\ \left|f^\Delta(x)\right| \Delta x\right)^{1+r}. \qquad (20)
\end{aligned}$$

In the case when $\mu > 1$, by using Hölder's inequality with indices μ and ν, we have

$$\begin{aligned}
&\int_a^\tau [q(x)]^{\frac{1}{1+r}}\ \left|f^\Delta(x)\right| \Delta x \\
&\quad = \int_a^\tau [p(x)]^{-\frac{1}{\nu}}\ [p(x)]^{\frac{1}{\nu}}\ [q(x)]^{\frac{1}{1+r}}\ \left|f^\Delta(x)\right| \Delta x \\
&\quad = \left(\int_a^\tau \left[[p(x)]^{-\frac{1}{\nu}}\right]^\mu\ \Delta x\right)^{\frac{1}{\mu}} \left(\int_a^\tau \left[[p(x)]^{\frac{1}{\nu}}\ [q(x)]^{\frac{1}{1+r}}\ \left|f^\Delta(x)\right|\right]^\nu \Delta x\right)^{\frac{1}{\nu}} \\
&\quad = \left(\int_a^\tau [p(x)]^{1-\mu}\ \Delta x\right)^{\frac{1}{\mu}} \left(\int_a^\tau p(x)\,[q(x)]^{\frac{\nu}{1+r}}\ \left|f^\Delta(x)\right|^\nu \Delta x\right)^{\frac{1}{\nu}}. \qquad (21)
\end{aligned}$$

Thus, by means of (20) and (21), we finally obtain

$$\begin{aligned}
\int_a^\tau q(x)\,|f(x)|^r \left|f^\Delta(x)\right| \Delta x \leqq \frac{1}{1+r}&\left(\int_a^\tau [p(x)]^{1-\mu}\ \Delta x\right)^{\frac{1+r}{\mu}} \\
&\cdot\left(\int_a^\tau p(x)\,[q(x)]^{\frac{\nu}{1+r}}\ \left|f^\Delta(x)\right|^\nu \Delta x\right)^{\frac{1+r}{\nu}},
\end{aligned}$$

which completes our proof of Theorem 1. □

THEOREM 2. *For* $\tau \in [a,b] \cap \mathbb{T}$, *let* $p(x) \in C_{\mathrm{rd}}([\tau,b],\mathbb{R})$ *with*

$$\int_\tau^b [p(x)]^{1-\mu}\ \Delta x < \infty \qquad (\mu > 1).$$

Let the function $q(x)$ *be positive, bounded and non-decreasing on* $[\tau,b] \cap \mathbb{T}$. *Suppose also that the function* $f(x)$ *is delta differentiable on* $[\tau,b] \cap \mathbb{T}$ *and* $f(b) = 0$. *Then, for* $r > 0$, *the following inequality holds true:*

$$\begin{aligned}
\int_\tau^b q(x)\,|f(x)|^r \left|f^\Delta(x)\right| \Delta x \leqq \frac{1}{1+r}&\left(\int_\tau^b [p(x)]^{1-\mu}\ \Delta x\right)^{\frac{1+r}{\mu}} \\
&\cdot\left(\int_\tau^b p(x)\,[q(x)]^{\frac{\nu}{1+r}}\ \left|f^\Delta(x)\right|^\nu \Delta x\right)^{\frac{1+r}{\nu}} \quad \left(\frac{1}{\mu}+\frac{1}{\nu}=1\right). \qquad (22)
\end{aligned}$$

Proof. Let

$$y(x) = \int_x^b [q(s)]^{\frac{1}{1+r}} \left|f^{\Delta}(s)\right| \Delta s. \tag{23}$$

Then

$$\begin{aligned}[y(x)]^r &= \left(\int_x^b [q(s)]^{\frac{1}{1+r}} \left|f^{\Delta}(s)\right| \Delta s\right)^r \\ &\geqq [q(x)]^{\frac{r}{1+r}} \left(\int_x^b \left|f^{\Delta}(s)\right| \Delta s\right)^r \\ &\geqq [q(x)]^{\frac{r}{1+r}} [|f(x)|]^r \end{aligned} \tag{24}$$

and

$$y^{\Delta}(x) = -[q(x)]^{\frac{1}{1+r}} \left|f^{\Delta}(x)\right|. \tag{25}$$

We thus find that

$$\begin{aligned}&\int_\tau^b q(x)\,|f(x)|^r \left|f^{\Delta}(x)\right| \Delta x \\ &\quad = \int_\tau^b [q(x)]^{\frac{1}{1+r}} \left[q^{\frac{r}{1+r}}(x)\right]^{\frac{r}{1+r}} |f(x)|^r \left|f^{\Delta}(x)\right| \Delta x \\ &\quad \leqq -\int_\tau^b [y(x)]^r\ y^{\Delta}(x)\,\Delta x = \int_b^\tau [y(x)]^r\ y^{\Delta}(x)\,\Delta x \\ &\quad \leqq \frac{1}{1+r}\ [y(\tau)]^{1+r} = \frac{1}{1+r}\left(\int_\tau^b [q(x)]^{\frac{1}{1+r}} \left|f^{\Delta}(x)\right| \Delta x\right)^{1+r}. \end{aligned} \tag{26}$$

In the case when $\mu > 1$, by using Hölder's inequality with indices μ and ν, we have

$$\begin{aligned}&\int_\tau^b [q(x)]^{\frac{1}{1+r}} \left|f^{\Delta}(x)\right| \Delta x \\ &\quad \leqq \int_\tau^b [p(x)]^{-\frac{1}{\nu}}\ [p(x)]^{\frac{1}{\nu}}\ [q(x)]^{\frac{1}{1+r}} \left|f^{\Delta}(x)\right| \Delta x \\ &\quad = \left(\int_\tau^b \left[[p(x)]^{-\frac{1}{\nu}}\right]^{\mu} \Delta x\right)^{\frac{1}{\mu}} \left(\int_\tau^b \left[[p(x)]^{\frac{1}{\nu}}\ [q(x)]^{\frac{1}{1+r}} \left|f^{\Delta}(x)\right|\right]^{\nu} \Delta x\right)^{\frac{1}{\nu}} \\ &\quad = \left(\int_\tau^b [p(x)]^{1-\mu}\ \Delta x\right)^{\frac{1}{\mu}} \left(\int_\tau^b p(x)\,[q(x)]^{\frac{\nu}{1+r}} \left|f^{\Delta}(x)\right|^{\nu} \Delta x\right)^{\frac{1}{\nu}}. \end{aligned} \tag{27}$$

Therefore, by applying (26) and (27), we finally have

$$\begin{aligned}\int_\tau^b q(x)\,|f(x)|^r \left|f^{\Delta}(x)\right| \Delta x \leqq{}& \frac{1}{1+r}\left(\int_\tau^b [p(x)]^{1-\mu}\ \Delta x\right)^{\frac{1+r}{\mu}} \\ &\cdot \left(\int_\tau^b p(x)\,[q(x)]^{\frac{\nu}{1+r}} \left|f^{\Delta}(x)\right|^{\nu} \Delta x\right)^{\frac{1+r}{\nu}}. \end{aligned}$$

The proof of Theorem 2 is thus completed. □

THEOREM 3. *For* $r > 0$ *and* $\tau \in [a,b]_T$, *let* $p(x) \in C_{\mathrm{rd}}([a,b],\mathbb{R})$ *with*

$$\int_a^\tau [p(x)]^{1-\mu}\,\Delta x < \infty \quad \text{and} \quad \int_\tau^b [p(x)]^{1-\mu}\,\Delta x < \infty \qquad (\mu \geqq 1+r).$$

Suppose that the function $q(x)$ *is positive, bounded and non-increasing on* $[a,\tau]\cap\mathbb{T}$ *and non-decreasing on* $[\tau,b]\cap\mathbb{T}$. *Also let the function* $f(x)$ *be delta differentiable on* $[a,b]\cap\mathbb{T}$ *and*

$$f(a) = f(b) = 0.$$

Then the following inequality holds true:

$$\int_a^b q(x)\,|f(x)|^r \left|f^\Delta(x)\right| \Delta x \leqq \frac{1}{1+r}\,\varkappa \left(\int_a^b p(x)\,[q(x)]^{\frac{\nu}{1+r}} \left|f^\Delta(x)\right|^\nu \Delta x\right)^{\frac{1+r}{\nu}}, \tag{28}$$

where $\varkappa$ *and* τ *are so contrained that*

$$\left(\int_a^\tau [p(x)]^{1-\mu}\,\Delta x\right)^{\frac{1+r}{\mu}} = \left(\int_\tau^b [p(x)]^{1-\mu}\,\Delta x\right)^{\frac{1+r}{\mu}} =: \varkappa.$$

Proof. It is easily observed from the hypothesis of Theorem 3 that

$$\begin{aligned}
&\int_a^b q(x)\,|f(x)|^r \left|f^\Delta(x)\right| \Delta x \\
&\quad = \int_a^\tau q(x)\,|f(x)|^r \left|f^\Delta(x)\right| \Delta x + \int_\tau^b q(x)\,|f(x)|^r \left|f^\Delta(x)\right| \Delta x \\
&\quad \leqq \frac{1}{1+r}\left(\int_a^\tau [p(x)]^{1-\mu}\,\Delta x\right)^{\frac{1+r}{\mu}} \left(\int_a^\tau p(x)\,[q(x)]^{\frac{\nu}{1+r}} \left|f^\Delta(x)\right|^\nu \Delta x\right)^{\frac{1+r}{\nu}} \\
&\qquad + \frac{1}{1+r}\left(\int_\tau^b [p(x)]^{1-\mu}\,\Delta x\right)^{\frac{1+r}{\mu}} \left(\int_\tau^b p(x)\,[q(x)]^{\frac{\nu}{1+r}} \left|f^\Delta(x)\right|^\nu \Delta x\right)^{\frac{1+r}{\nu}} \\
&\quad = \frac{1}{1+r}\,\varkappa \left(\int_a^b p(x)\,[q(x)]^{\frac{\nu}{1+r}} \left|f^\Delta(x)\right|^\nu \Delta x\right)^{\frac{1+r}{\nu}}.
\end{aligned}$$

The result asserted by Theorem 3 would now follow from the elementary inequality:

$$a^\lambda + b^\lambda \leqq (a+b)^\lambda \qquad (a,b \geqq 0;\ \lambda \geqq 1)$$

and the fact that

$$\mu \geqq 2 \implies 1 \leqq \nu \leqq 2. \qquad \square$$

THEOREM 4. *For* $\tau \in [a,b] \cap \mathbb{T}$, *let* $p(x) \in C_{\mathrm{rd}}([a,\tau],\mathbb{R})$ *with*

$$\int_a^\tau [p(x)]^{1-\mu}\ \Delta t < \infty \qquad (\mu > 1).$$

Suppose that the function $q(x)$ *is positive, bounded and non-increasing on* $[a,\tau] \cap \mathbb{T}$. *Also let the function* $f(x)$ *delta-differentiable* n *times* $(n \in \mathbb{N})$ *on* $[a,\tau] \cap \mathbb{T}$ *and*

$$f(a) = f^{\Delta}(a) = \cdots = f^{\Delta^{n-1}}(a) = 0.$$

Then, for $r > 0$, *the following inequality holds true:*

$$\int_a^\tau q(x)\,|f(x)|^r \left|f^{\Delta^n}(x)\right| \Delta x \leqq (\tau - a)^{n-1} \frac{1}{1+r} \left(\int_a^\tau [p(x)]^{1-\mu}\ \Delta x\right)^{\frac{1+r}{\mu}} \cdot \left(\int_a^\tau p(x)\,[q(x)]^{\frac{r}{1+r}} \left|f^{\Delta^n}(x)\right|^\nu \Delta x\right)^{\frac{1+r}{\nu}} \left(\frac{1}{\mu} + \frac{1}{\nu} = 1\right). \tag{29}$$

Proof. Suppose that

$$y(x) = \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} \left|f^{\Delta^n}(s)\right|\ \Delta s\, \Delta x_1 \cdots \Delta x_{n-1}, \tag{30}$$

so that we have

$$[y(x)]^r = \left(\int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} \left|f^{\Delta}(s)\right| \Delta s\, \Delta x_1 \cdots \Delta x_{n-1}\right)^r, \tag{31}$$

$$y^{\Delta^n}(x) = \left|f^{\Delta^n}(x)\right| \geqq 0, \tag{32}$$

and

$$y(x) \geqq |f(x)|$$

together with the following inequality:

$$y^{\Delta^i}(x) = \int_a^x y^{\Delta^{i+1}}(s)\,\Delta s \leqq (x-a)\, y^{\Delta^{i+1}}(x) \tag{33}$$

$$(i = 0, 1, \cdots, n-2).$$

Then, by applying Theorem 1 in conjunction with (30) to (33), we have

$$\begin{aligned}
\int_a^\tau q(x)\,|f(x)|^r \left|f^{\Delta^n}(x)\right| \Delta x
&\leqq \int_a^\tau q(x)\,[y(x)]^r\, y^{\Delta^n}(x)\,\Delta x \\
&\leqq \int_a^\tau q(x)\left[(x-a)\,y^{\Delta}(x)\right]^r y^{\Delta^n}(x)\,\Delta x \\
&\;\;\vdots \\
&\leqq \int_a^\tau q(x)\left[(x-a)^{n-1}\,y^{\Delta^{n-1}}(x)\right]^r y^{\Delta^n}(x)\,\Delta x \\
&\leqq (\tau-a)^{n-1}\int_a^\tau q(x)\left[y^{\Delta^{n-1}}(x)\right]^r y^{\Delta^n}(x)\,\Delta x \\
&\leqq (\tau-a)^{n-1}\,\frac{1}{1+r}\left(\int_a^\tau [p(x)]^{1-\mu}\;\Delta x\right)^{\frac{1+r}{\mu}} \\
&\qquad \cdot\left(\int_a^\tau p(x)\,[q(x)]^{\frac{r}{1+r}}\;\left|y^{\Delta^n}(x)\right|^{\nu}\Delta x\right)^{\frac{1+r}{\nu}} \\
&= (\tau-a)^{n-1}\,\frac{1}{1+r}\left(\int_a^\tau [p(x)]^{1-\mu}\;\Delta x\right)^{\frac{1+r}{\mu}} \\
&\qquad \cdot\left(\int_a^\tau p(x)\,[q(x)]^{\frac{r}{1+r}}\left|f^{\Delta^n}(x)\right|^{\nu}\Delta x\right)^{\frac{1+r}{\nu}}.
\end{aligned}$$

This evidently completes our proof of Theorem 4. □

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